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First analytic correction beyond PFA for the electromagnetic field in sphere-plane geometry

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We consider the vacuum energy for a configuration of a sphere in front of a plane, both obeying conductor boundary condition, at small separation. For the separation becoming small we derive the first next-to-leading order of the asymptotic expansion in the separation-to-radius ratio ε . This correction is of order ε . In opposite to the scalar cases it contains also contributions proportional to logarithms in first and second order, $\varepsilon \ln \varepsilon$ and $\varepsilon (\ln \varepsilon)^2$. We compare this result with the available findings of numerical and experimental approaches.

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I. INTRODUCTION

The Proximity Force Approximation (PFA) is the most important approximation for calculation of forces between curved surfaces at small separation. It originates from a work on adhesion back in 1934 based on the simple idea to integrate the local force density known from parallel surfaces [1]. Given a sufficiently fast decrease of the force with separation this method works independently from the kind of force. In this respect it is universal. However, this method does not allow to go beyond nor does it give any information on its precision. Attempts to repeat the original idea 'plane based' or 'sphere based', turned out to give misleading results. It was only with the new method of calculating the Casimir force, the T-matrix or TGTG-formula approach that the door opened to go beyond PFA. In these approaches, the interaction energy is represented by a infinite dimensional determinant. At large and medium separation this matrix can be truncated to become finite (even low) dimensional and numerical evaluation is possible. At small separation this does not work and below

$$\varepsilon \equiv \frac{d}{R} \sim 0.1 \quad (1)$$

(d -distance, R -radius of curvature, both at closest separation) the numerical effort is unmanageable.

It must be mentioned that only $\varepsilon \lesssim 0.1$ is the experimentally interesting region for forces between macroscopic bodies. This is because of the van der Waals and the Casimir forces as being quantum effects are generically microscopically small and become measurable only when multiplied by a microscopically large interaction area. Since the forces decrease proportional to d^{-4} at large separation, only the combination of small separation together with large radius of curvature allows for measurements with appreciable precision. A typical value used in experiments on precision measurements of the Casimir force [2] is $\varepsilon \sim 10^{-3}$.

The interest in corrections beyond PFA is triggered from both, theoretical and experimental sides. The first follows from the challenge to improve a situation which lasted more than 60 years, the second from the high precision of the contemporary force measurements and from the accuracy one would like to achieve for their comparison with the theoretical predictions. It must be mentioned that this has implications far beyond the atomic or solid state physics as mean to obtain stronger constraints on new physics (*Fifth Forces*), see for example [3].

For scalar fields, analytical corrections beyond PFA were obtained as an asymptotic expansion

$$\frac{E}{E_{\text{PFA}}} = 1 + \alpha \varepsilon + \dots \quad (2)$$

of the energy for $\varepsilon \rightarrow 0$ and simple numbers were obtained for the coefficient α . In [4] this was done for the geometry of a cylinder in front of a plane for a scalar field obeying Dirichlet or Neumann boundary conditions. This includes the electromagnetic field at once since its polarizations separate in cylindrical geometry. The corrections for a sphere

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in front of a plane were obtained in [5] for a scalar field, again for both, Dirichlet and Neumann boundary conditions. However, in this case the corrections for the electromagnetic field do not follow because of the non-separation of the polarizations.

It is the aim of the present paper to fill this gap and to obtain the first beyond-PFA corrections for the electromagnetic field obeying conductor boundary conditions in the geometry of a sphere in front of a plane. Surprisingly, the asymptotic expansion in this case turns out to contain logarithms, i.e., it has the form

$$\frac{E}{E_{\text{PFA}}} = 1 + \left(\alpha + \beta \ln \varepsilon + \gamma (\ln \varepsilon)^2 \right) \varepsilon + \dots \quad (3)$$

The coefficients are calculated below and take the values $\alpha = -5.2$, $\beta = -0.0044$ and $\gamma = 8.5 \cdot 10^{-6}$. The logarithmic terms come in from contributions which are specific for the vector case.

In [6] an experimental effort was undertaken to measure the corrections beyond PFA by using several spheres whose radii varied from 10 to 150 μm at separations $d = 200 \dots 800 \text{ nm}$. The expansion was assumed to have the form of (2) and the coefficient α was found to be zero within the experimental precision. Also numerical efforts are reported (for a cylinder in front of a plane in [7] and for a sphere in [8, 9]) by pushing the truncation in the T-matrix approach to higher orders and extrapolating towards the known value at zero separation. The results show agreement with the analytical results for Dirichlet boundary conditions but not for Neumann boundary conditions; details will be discussed in the last section. For a scalar field obeying Dirichlet boundary conditions results were obtained in [10] using the independent method of world line approach. Like the extrapolation these confirm the analytical results.

It is a second aim of the present paper to discuss in detail the analytical corrections beyond PFA for all combinations of boundary conditions for the scalar field. For instance, it will become evident that and why the corrections for a sphere in front of a plane are the same with Dirichlet boundary conditions on the sphere, but Dirichlet or Neumann boundary conditions on the plane. This case is interesting since the numerical results are different for the two cases.

The paper is organized as follows. In the next section we give a representation of the vacuum energy in the sphere-plane geometry for all boundary conditions with special emphasis on the translation formulas used. In the third section we re-derive the asymptotic expansion for the scalar field and in the fourth section we derive the expansion for the electromagnetic field. In the last section we discuss the results.

Throughout the paper we use units with $\hbar = c = 1$

II. REPRESENTATION OF THE VACUUM ENERGY IN SPHERE-PLANE GEOMETRY

The representation of the vacuum energy in sphere-plane geometry was first derived in [11] for the scalar case within the multiple scattering approach. Subsequently there appeared numerous variations of the derivation; we use that in [12], Chap. 10. Thereby we highlight one essential step - the use of the translation formulas - having in mind their importance for understanding the logarithmic contributions in the electromagnetic case.

The general structure of the T-matrix representation of the vacuum interaction is

$$E = \frac{1}{2} \int_{-\infty}^{\infty} \frac{d\xi}{2\pi} \text{Tr} \ln (1 - GT). \quad (4)$$

The geometry is shown in Fig.1.

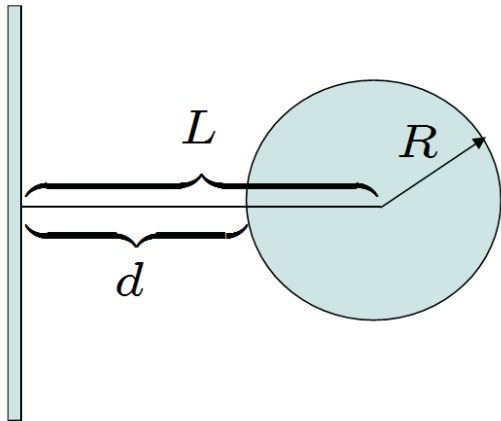


Fig.1: The configuration of a sphere in front of a plane.

In (4) the frequency was rotated towards the imaginary axis, $\omega \rightarrow i\xi$. The symbol G denotes the Greens function (or, strictly speaking, the corresponding operator) describing the propagation from the sphere to the mirror and back and T is the T-matrix operator for the scattering on the sphere. Using the basis

$$u_{k,lm}(\mathbf{r}) = j_l(kr) Y_{lm}(\Omega_r), \quad (5)$$

where the $Y_{lm}(\Omega_r)$ are the spherical harmonics and $j_l(r)$ are the spherical Bessel functions, the Greens function can be written as

$$G_\xi(\mathbf{r}, \mathbf{r}') = \frac{2}{\pi} \int_0^\infty \frac{dk k}{\xi^2 + k^2} \sum_{lm} u_{k,lm}(\mathbf{r}) u_{k,lm}^*(\mathbf{r}'). \quad (6)$$

The limits of the summations are $l \geq 0$ and $|m| \leq l$.

In (6) both spatial arguments, \mathbf{r} and \mathbf{r}' , appear to be in one and the same coordinate system with spherical coordinates (r, Ω_r) . However, in the considered geometry, we would like to expand the T-matrix operator in its own coordinate system centered in $(0, 0, d + R)$. If this system is taken for \mathbf{r} , we need for \mathbf{r}' a translation from this one to the mirror and back. This can be achieved by the *translation formula*

$$u_{k,lm}(\mathbf{r} + a\mathbf{e}_z) = \sum_{l'm'} A_{lm,l'm'}(a) u_{k,l'm'}(\mathbf{r}) \quad (7)$$

with $a = 2L$ and \mathbf{e}_z is the unit vector along the z -axis. In (7), $A_{lm,l'm'}(a)$ are the translation coefficients. These involve the Clebsch-Gordan coefficients, for details and an explicate expression see, for example, Eq.(10.125) in [12]. Applying these formulas, after some transformations, the energy (4) takes the form

$$E = \frac{1}{2} \int_{-\infty}^\infty \frac{d\xi}{2\pi} \text{Tr} \ln(1 - N), \quad (8)$$

where the trace is the orbital momentum sum,

$$\text{Tr} = \sum_{m=0}^\infty \sum_{l=|m|}^\infty. \quad (9)$$

In (8), N is an infinite dimensional matrix in the orbital momentum index, $N_{l,l'}$. Using the known relation $\text{Tr} \ln = \ln \det$ the energy can be represented as a determinant. However, we do not use this in the following.

The entries of the matrix N are, for Dirichlet boundary conditions on the sphere and with the notation $N_{l,l'}^D$ for $N_{l,l'}$,

$$N_{l,l'}^D = \sqrt{\frac{\pi}{4\xi L}} \sum_{l''=|l-l'|}^{l+l'} K_{l''+1/2}(2\xi L) H_{l''}^{l''} d_l^D(\xi R). \quad (10)$$

The function

$$d_l^D(\xi R) = \frac{I_{l+1/2}(\xi R)}{K_{l+1/2}(\xi R)} \quad (11)$$

is up to a factor the T-matrix for the scattering of a scalar field on a hard sphere in orbital momentum representation. The corresponding expression $N_{l,l'}^N$ for Neumann boundary conditions on the sphere can be obtained with

$$d_l^N(\xi R) = \frac{(I_{l+1/2}(\xi R)/\sqrt{\xi R})'}{(K_{l+1/2}(\xi R)/\sqrt{\xi R})'} \quad (12)$$

in place of $d_l^D(\xi R)$ in (10). In the above formulas I_ν and K_ν are the modified Bessel functions. We note that in opposite to eqn. (10.140) in [12] the function $I_{l+1/2}(\xi R)$ carries the index l in place of l' ; a substitution which is allowed under the trace in (8). In (10) we used the notation

$$H_{l''}^{l''} = \sqrt{(2l+1)(2l'+1)(2l''+1)} \times \begin{pmatrix} l & l' & l'' \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l & l' & l'' \\ m & -m & 0 \end{pmatrix}, \quad (13)$$

which involves the Clebsch-Gordan coefficients coming in from the translation formula. We use the notation of the $3j$ -symbols.

Eq. (8) represents the energy with Dirichlet boundary conditions on the plane. The case of Neumann boundary conditions on the plane is obtained by changing the signs in front of N . We unite all four combinations of boundary condition in

$$E^{XY} = \frac{1}{2} \int_{-\infty}^{\infty} \frac{d\xi}{2\pi} \text{Tr} \ln (1 - (-1)^x N^Y). \quad (14)$$

The first index denotes the boundary conditions on the plane, $X = D$ with $x = +1$ for Dirichlet and $X = N$ with $x = -1$ for Neumann conditions. The second index denotes the boundary conditions on the sphere with $Y = D$ for Dirichlet and $Y = N$ for Neumann conditions. In the following we write all formulas for Dirichlet boundary conditions on both and discuss the other cases at the end of the next section.

The logarithm in (8) is taken of a matrix. In the following we will use the expansion of this logarithm such that the formula for the energy takes the form

$$E = \frac{1}{2} \int_{-\infty}^{\infty} \frac{d\xi}{2\pi} \sum_{s=0}^{\infty} \frac{-1}{s+1} \times \sum_{m=0}^{\infty} \sum_{l=|m|}^{\infty} \left(\prod_{j=1}^s \sum_{l_j=|m|}^{\infty} \right) \left(\prod_{i=0}^s N_{l_i, l_{i+1}} \right) \quad (15)$$

with the formal setting $l_0 = l_{s+1} = l$. It should be mentioned that the $N_{l,l'}$ are diagonal in the azimuthal index m such that the sum over m appears only once.

The expansion (15) corresponds to a perturbation series with respect to the T-matrix of the scattering on the sphere. This series appears to converge. The convergence can be easily seen for large separations where it corresponds to the multipole expansion and also for short separations as we will see below. It is known that the vacuum energy for a transparent sphere, or for a background potential, where it makes sense to introduce a coupling constant, has the same structure as Eq.(8). The last equation, in this way, appears as a perturbative expansion with respect to the coupling constant. Examples for the first order of this expansion are considered in a number of papers, see for example [13, 14].

The T-matrix representation for the electromagnetic field has the same general structure as that for the scalar field. The main difference is in the presence of the polarizations. The expansion basis for the electromagnetic field has two components,

$$\begin{aligned} \mathbf{m}_{k,lm}(\mathbf{r}) &\equiv \mathbf{u}_{k,1lm}(\mathbf{r}) = \mathbf{L} \frac{1}{\sqrt{\mathbf{L}^2}} u_{k,lm}(\mathbf{r}), \\ \mathbf{n}_{k,lm}(\mathbf{r}) &\equiv \mathbf{u}_{k,2lm}(\mathbf{r}) = \nabla \times \mathbf{L} \frac{1}{\sqrt{-\Delta}} \frac{1}{\sqrt{\mathbf{L}^2}} u_{k,lm}(\mathbf{r}), \end{aligned} \quad (16)$$

where \mathbf{L} is the orbital momentum operator. Under a translation these functions mix and in place of (7) the translation formula is now

$$\begin{aligned} &\mathbf{m}_{k,lm}(\mathbf{r} + a\mathbf{e}_z) \\ &= \sum_{l'm'} (B_{lm,l'm'}(a) \mathbf{m}_{k,l'm'}(\mathbf{r}) + C_{lm,l'm'}(a) \mathbf{n}_{k,l'm'}(\mathbf{r})), \\ &\mathbf{n}_{k,lm}(\mathbf{r} + a\mathbf{e}_z) = \\ &= \sum_{l'm'} (C_{lm,l'm'}(a) \mathbf{m}_{k,l'm'}(\mathbf{r}) + B_{lm,l'm'}(a) \mathbf{n}_{k,l'm'}(\mathbf{r})). \end{aligned} \quad (17)$$

The Greens function (now in fact the Greens dyadic) can be expressed in the basis (16) similar to (6) and reads

$$\mathbf{G}_{\xi}(\mathbf{r}, \mathbf{r}') = \frac{2}{\pi} \int_0^{\infty} \frac{dk k}{\xi^2 + k^2} \sum_{slm} \mathbf{u}_{k,slm}(\mathbf{r}) \mathbf{u}_{k,slm}^*(\mathbf{r}'), \quad (18)$$

s taking values $s = 1, 2$ and the s-wave is excluded, $l \geq 1$. The translation coefficients $B_{lm,l'm'}(a)$ and $C_{lm,l'm'}(a)$ are known in the electromagnetic theory, as pointed out in [15] a particularly useful representation can be found in [16]. In the given geometry, the coefficient $B_{lm,l'm'}$ can be obtained by the substitution

$$H_{l'}'' \rightarrow H_{l'}'' \Lambda_{l'}'' \quad (19)$$

in $A_{lm,l'm'}$ with

$$\Lambda_{ll'}^{l''} = \frac{\frac{1}{2} [l''(l''+1) - l(l+1) - l'(l'+1)]}{\sqrt{l(l+1)l'(l'+1)}}. \quad (20)$$

The other one is given by

$$C_{lm,l'm'} = \tilde{\Lambda}_{ll'} A_{lm,l'm'} \quad (21)$$

with

$$\tilde{\Lambda}_{ll'} = \frac{2m\xi(d+R)}{\sqrt{l(l+1)l'(l'+1)}}, \quad (22)$$

where the $A_{lm,l'm'}$ are the same as in the scalar case, Eq.(7). The coefficients $\Lambda_{ll'}^{l''}$ and $\tilde{\Lambda}_{ll'}$ result from the orbital momentum operators, for instance from the normalization factors in (16).

The translation from one coordinate system to the other mixes the polarizations of the electromagnetic field. Therefore the energy has now the representation

$$E = \frac{1}{2} \int_{-\infty}^{\infty} \frac{d\xi}{2\pi} \text{Tr} \ln (1 - \mathbb{N}), \quad (23)$$

and the trace is

$$\text{Tr} = \sum_{m=0}^{\infty} \sum_{l=\max(1,|m|)}^{\infty} \text{tr}, \quad (24)$$

where tr denotes the trace over \mathbb{N} which is now a (2x2)-matrix in the polarizations,

$$\begin{aligned} \mathbb{N}_{l,l'} &\equiv \begin{pmatrix} N_{l,l'}^{(11)} & N_{l,l'}^{(12)} \\ N_{l,l'}^{(21)} & N_{l,l'}^{(22)} \end{pmatrix} \\ &= \sqrt{\frac{\pi}{4\xi L}} \sum_{l''=|l-l'|}^{l+l'} K_{l''+1/2}(2\xi L) H_{ll'}^{l''} \\ &\quad \times \begin{pmatrix} \Lambda_{l,l'}^{l''} & \tilde{\Lambda}_{l,l'} \\ \tilde{\Lambda}_{l,l'} & \Lambda_{l,l'}^{l''} \end{pmatrix} \begin{pmatrix} d_l^{\text{TE}}(\xi R) & 0 \\ 0 & -d_l^{\text{TM}}(\xi R) \end{pmatrix}. \end{aligned} \quad (25)$$

The functions d_l^{TE} and d_l^{TM} describe the scattering of the corresponding polarizations of the electromagnetic field on a conducting sphere and are similar to that of the scalar field. For the TE mode it is literally the same,

$$d_l^{\text{TE}}(\xi R) = \frac{I_{l+1/2}(\xi R)}{K_{l+1/2}(\xi R)}, \quad (26)$$

and for the TM-mode it is

$$d_l^{\text{TM}}(\xi R) = \frac{(I_{l+1/2}(\xi R)\sqrt{\xi R})'}{(K_{l+1/2}(\xi R)\sqrt{\xi R})'}. \quad (27)$$

The minus sign in front of d_l^{TM} in (25) results from the the spin of the electromagnetic field under reflection on the plane.

Representation (23) of the vacuum energy of the electromagnetic field was derived in different notations in [8, 9]. The coincidence with these formulas can be checked by comparing some first orders of the expansion for large separation.

III. THE ASYMPTOTIC EXPANSION FOR THE SCALAR FIELD

In this section we consider the asymptotic expansion for the scalar field at small separation. We follow [5] and repeat the main steps of the derivation since these appear essentially in the same form in the next section for the

electromagnetic case. At once we add a discussion on all combinations of the boundary conditions on the plane and on the sphere.

The vacuum energy is given by Eq.(15). First of all we make the substitution $\xi \rightarrow \xi/R$ to get rid of the dimensional variables. Then, as already mentioned, for decreasing separation the convergence of the integral and the sums in (15) slows down and the main contribution comes from higher and higher frequencies and orbital momenta. We know, by hindsight, the region delivering the dominating contributions. Since at small separation all summation indices involved take high values we substitute all sums by corresponding integrations. In this way we drop exponentially small contributions which is allowed aiming for an asymptotic expansion. In these integrations we make the substitutions

$$\begin{aligned} \xi &= \frac{t}{\varepsilon} \sqrt{1-\tau^2}, \quad l = \frac{t}{\varepsilon} \tau, \quad m = \sqrt{\frac{t\tau}{\varepsilon}} \mu, \\ \tilde{l}_i &= \sqrt{\frac{4t}{\varepsilon}} n_i \quad (i = 1, \dots, s), \end{aligned} \quad (28)$$

where we divided the orbital momenta by means of $l_i = l + \tilde{l}_i$ ($i = 1, \dots, s$) into the index l of the main diagonal and the off-diagonal indices \tilde{l}_i . The variable τ has the meaning of the cosine of the polar angle in the ξ, l -plane. This substitution describes the region where the main contributions come from.

In the new variables the expression for the energy reads

$$\begin{aligned} E &= -\frac{R}{4\pi d^2} \sum_{s=0}^{\infty} \frac{1}{s+1} \int_0^{\infty} dt t e^{-2t(s+1)} \\ &\times \int_0^1 \frac{d\tau \sqrt{\tau}}{\sqrt{1-\tau^2}} \int_{-\infty}^{\infty} \frac{d\mu}{\sqrt{\pi}} e^{-\mu^2(s+1)/\tau} \\ &\times \left(\prod_{j=1}^s \int_{n_0}^{\infty} \frac{dn_j}{\sqrt{\pi}} \right) \mathcal{Z}, \end{aligned} \quad (29)$$

where

$$n_0 = -\frac{\tau}{2} \sqrt{\frac{t}{\varepsilon}} + \frac{1}{2} |\mu| \sqrt{\tau} \quad (30)$$

is the lower boundary in the n_j -integrations. It follows with (28) from $l \geq |m|$. In (29),

$$\mathcal{Z} = \prod_{i=0}^s \left(\sqrt{\frac{4\pi t}{\varepsilon}} N_{l+\tilde{l}_i, l+\tilde{l}_{i+1}} \right) e^{\eta_{as}}, \quad (31)$$

collects the information from the scattering process together with the prefactors which follow from the substitution (28). In (31) we use the formal definitions $l_0 = \tilde{l}_{s+1} = 0$ and we defined

$$\eta_{as} = 2t(s+1) + \eta_1 + \mu^2 \frac{s+1}{\tau} \quad (32)$$

with $\eta_1 = \sum_{i=0}^s (n_i - n_{i+1})^2$.

Next we expand \mathcal{Z} for small ε . It turns out that it is possible to do this expansion straightforwardly by expanding all quantities entering. These are the Bessel functions and the Clebsch-Gordan coefficients in the $N_{l,l'}$, Eq.(10). Here we can follow the corresponding expansion in [5]. While for the Bessel functions the known uniform asymptotic expansion can be used, for the Clebsch-Gordan coefficients the corresponding asymptotic expansion was first derived in [5]. It rests on an integral representation of these coefficients. In this way, one arrives at an asymptotic expansion, $N_{l,l'} \rightarrow N_{l,l'}^{\text{as}}$, with

$$\begin{aligned} N_{l,l'}^{\text{as}} &= \sqrt{\frac{\varepsilon\tau}{2\pi t(1+\tau)}} e^{-2t-(n-n')^2} \int_{-\infty}^{\infty} \frac{d\eta}{\sqrt{\pi}} e^{-\eta^2+2i\eta\sqrt{2}\mu+\mu^2} \\ &\times \sum_{\nu=0}^{\nu_m} \frac{\eta^{2\nu}}{\nu!} \left(\frac{1-\tau}{1+\tau} \right)^{\nu} (1 + \sqrt{\varepsilon} f(\eta, \mu, t\tau) + \dots). \end{aligned} \quad (33)$$

The function $f(\eta, \mu, t\tau)$ and the corresponding function in the next order can be found in [5]. The integration over η results from the mentioned integral representation and the sum over ν from the summation over l'' in (10) with the substitution $l'' = l + l' - 2\nu$. By the symmetry properties of the $3j$ -symbols in (13) it follows that ν takes integer values only. The upper limit of the summation over ν is

$$\nu_m = \frac{1}{2}(l + l' - |l - l'|). \quad (34)$$

The exponential factor follows from the function $\eta(z)$ in the asymptotic expansion of the Bessel functions.

By means of the substitution (28), the upper limit of the ν summation depends on ε ,

$$\nu_m = \frac{t\tau}{\varepsilon} - \sqrt{\frac{4t}{\varepsilon}} |n - n'|. \quad (35)$$

In the considered case of a scalar field it is possible to put $\varepsilon = 0$ here. After that the sum over ν in (33) can be carried out. The integral over η is Gaussian and can be carried out too. We get

$$\begin{aligned} N_{l,l'}^{\text{as}} & \\ &= \sqrt{\frac{\varepsilon}{4\pi t}} e^{-2t - (n-n')^2 - \mu^2/\tau} \left(1 + a_{n,n'}^{(1/2)} \sqrt{\varepsilon} + a_{n,n'}^{(1)} \varepsilon + \dots \right). \end{aligned} \quad (36)$$

The functions $a^{(1/2)}(n, n')$ and $a^{(1)}(n, n')$ are given in Eq.(A.22) in [5].

Eq.(36) must be inserted into Eq.(31). The prefactors and the exponentials just cancel and the remaining dependence on ε is contained in the bracket. This fact justifies the substitution (28). When inserting $N_{l,l'}^{\text{as}}$ into (31) we get the asymptotic expansion \mathcal{Z}^{as} . After a re-expansion it takes the form

$$\begin{aligned} \mathcal{Z}^{\text{as}} &= 1 + \sum_{i=1}^s a_{n_i, n_{i+1}}^{(1/2)} \sqrt{\varepsilon} \\ &+ a^{\text{D}} \varepsilon + \dots \end{aligned} \quad (37)$$

with

$$a^{\text{D}} = \sum_{0 < i < j < s} a_{n_i, n_{i+1}}^{(1/2)} a_{n_j, n_{j+1}}^{(1/2)} + \sum_{i=1}^s a_{n_i, n_{i+1}}^{(1)}. \quad (38)$$

We mention that we used in the re-expansion the general formula

$$\prod_{i=0}^s (1 + x_i) = 1 + \sum_{i=0}^s x_i + \sum_{0 < i < j < s} x_i x_j + \dots, \quad (39)$$

which holds in the sense of an expansion for small x_i . We note that the product turned into sums. We will use this formula below several times without further notice.

Inserting this expression for \mathcal{Z}^{as} into (29) we get the asymptotic expansion of the energy. Here we have still an ε -dependence in n_0 , (43). However, we are allowed to put this $\varepsilon = 0$, i.e., to take $n_0 = -\infty$, since in doing so all integrations remain finite. After that the integrations can be carried out. These are either over simple exponentials or are Gaussian. We mention the most complicated one, which is that over the n_j . It was calculated in [4], Eq.(66),

$$\left(\prod_{j=1}^s \int_{-\infty}^{\infty} \frac{dn_j}{\sqrt{\pi}} \right) e^{-\eta_1} = \frac{1}{\sqrt{s+1}}. \quad (40)$$

The result for the energy is

$$E = -\frac{1}{16\pi} \frac{R}{d^2} \sum_{s=0}^{\infty} \frac{1}{(s+1)^4} \left(1 + \frac{1}{3} \varepsilon + \dots \right). \quad (41)$$

Carrying out the summation over s , in the leading order just the PFA emerges. In the order $\sqrt{\varepsilon}$ the integrations gave zero for symmetry reasons and in order ε we get the first correction beyond PFA.

The energy, Eq.(41), is for Dirichlet boundary conditions on both surfaces. The case of Neumann boundary conditions on the sphere can be handled in complete analogy. There are only two differences. The first one is an additional minus sign resulting from the derivative of the Bessel function K_ν in the denominator in (12). It appears in each factor $N_{l,l'}$, hence it gives a sign factor $(-1)^{s+1}$ to the sum over s . At this place we restore the notation of Eq.(14) for the different combination of the boundary conditions and accounting for all signs we get

$$\begin{aligned} E^{\text{XD}} &= \frac{1}{16\pi} \frac{R}{d^2} \sum_{s=0}^{\infty} \frac{(-1)^{1+x(s-1)}}{(s+1)^4} \left(1 + \frac{1}{3} \varepsilon + \dots \right), \\ E^{\text{XN}} &= \frac{1}{16\pi} \frac{R}{d^2} \sum_{s=0}^{\infty} \frac{(-1)^{x(s-1)+s}}{(s+1)^4} \\ &\quad \times \left(1 + \left(\frac{1}{3} - \frac{2}{3}(s+1)^2 \right) \varepsilon + \dots \right). \end{aligned} \quad (42)$$

The second difference for Neumann boundary conditions on the sphere is an additional contribution containing the factor of $(s+1)^2$ in the last line. It results from the change in the Debye polynomials due to the derivatives in (12) in the asymptotic expansion of the Bessel functions and results in a different function a^{N} in place of (38). As shown in [5], this factor is the same even if generalizing to Robin boundary conditions. For instance, it is the same for the TM mode of the electromagnetic field in the next section.

In Eq.(42) the summations result in Riemann zeta functions; we need

$$\begin{aligned} \sum_{s=0}^{\infty} \frac{1}{(s+1)^2} &= \zeta(2) = \frac{\pi^2}{6}, & \sum_{s=0}^{\infty} \frac{(-1)^s}{(s+1)^2} &= \frac{1}{2}\zeta(2), \\ \sum_{s=0}^{\infty} \frac{1}{(s+1)^4} &= \zeta(4) = \frac{\pi^4}{90}, & \sum_{s=0}^{\infty} \frac{(-1)^s}{(s+1)^4} &= \frac{7}{8}\zeta(4). \end{aligned}$$

In each case the leading order in (42) delivers the PFA. The relative corrections are

$$\begin{aligned} \frac{E^{\text{DD}}}{E_{\text{PFA}}} &= 1 + \frac{1}{3} \varepsilon + \dots, \\ \frac{E^{\text{NN}}}{E_{\text{PFA}}} &= 1 + \left(\frac{1}{3} - \frac{10}{\pi^2} \right) \varepsilon + \dots, \\ \frac{E^{\text{DN}}}{E_{\text{PFA}}} &= 1 + \left(\frac{1}{3} - \frac{5}{\pi^2} \right) \varepsilon + \dots, \\ \frac{E^{\text{ND}}}{E_{\text{PFA}}} &= 1 + \frac{1}{3} \varepsilon + \dots. \end{aligned} \quad (43)$$

We note that the corrections for Dirichlet boundary conditions on the sphere, but Dirichlet or Neumann boundary conditions on the plane (first and last line) are the same. This follows simply from the structure of the signs in (42).

IV. THE ASYMPTOTIC EXPANSION FOR THE ELECTROMAGNETIC FIELD

The asymptotic expansion for the electromagnetic field start with the same steps as in the scalar case, i.e., with the substitution of the orbital momentum sums by integrals and the substitution (28). The next step is the asymptotic expansion of the functions $N_{l,l'}^{(ss')}$ entering the matrix $\mathbb{N}_{l,l'}$ in Eq.(25). In the functions d_l^{TE} , Eq.(26), and d_l^{TM} , Eq.(27), we use again the uniform asymptotic expansion of the Bessel functions. In the factors $\Lambda_{l,l'}^{\prime\prime}$ and $\tilde{\Lambda}_{l,l'}^{\prime\prime}$, Eq. (20), we use directly the substitution (28) and write them in the form

$$\begin{aligned} \Lambda_{l,l'}^{\prime\prime} &\equiv 1 + \varepsilon \lambda_{n,n'} \\ &= 1 + \left(\frac{\mu(n)\mu(n')}{\gamma(n)\gamma(n')} - 1 \right) - \frac{2\nu\varepsilon}{\sqrt{t}} \frac{\mu(n) + \mu(n')}{\gamma(n)\gamma(n')} + \frac{\varepsilon^2}{t} \frac{\nu(2\nu-1)}{\gamma(n)\gamma(n')} \\ &\equiv 1 + A + B + C, \end{aligned} \quad (44)$$

$$\tilde{\Lambda}_{l,l'} \equiv \sqrt{\varepsilon} \tilde{\lambda}_{n,n'} = \sqrt{\varepsilon} \frac{2\sqrt{\tau t} \sqrt{1 - \tau^2} \mu(1 + \varepsilon)}{\gamma(n)\gamma(n')}, \quad (45)$$

with the notations

$$\mu(n) = \tau\sqrt{t} + 2n\sqrt{\varepsilon}, \quad \gamma(n) = \sqrt{\mu(n)(\mu(n) + \varepsilon/\sqrt{t})}. \quad (46)$$

Here $\mu(n)$ follows from l and $\gamma(n)$ from $l + 1$. We divided $\Lambda_{l,l}''$ into three parts which will be treated separately in the subsections below. All these will deliver a contribution of order ε (including $\varepsilon \ln \varepsilon$ and $\varepsilon(\ln \varepsilon)^2$) although this cannot be seen directly from Eq. (44). The same holds for $\tilde{\Lambda}_{l,l'}$, Eq. (45), after taking the trace over the polarizations.

The energy is given by Eq. (23) and its asymptotic expansion by Eq.(29) with another \mathcal{Z} which is, of course, still similar to (31). It is expressed in terms of the matrix $\mathbb{N}_{l,l'}$, Eq.(25), by

$$\mathcal{Z} = \text{tr} \prod_{i=0}^s \left(\sqrt{\frac{4\pi t}{\varepsilon}} \mathbb{N}_{l+\tilde{l}_i, l+\tilde{l}_{i+1}} \right) e^{\eta_{\text{as}}}. \quad (47)$$

For a matrix $\mathbb{N}_{l,l'}$ we get the asymptotic expansion

$$\mathbb{N}_{l,l'} = \sqrt{\frac{\varepsilon}{4\pi t}} e^{-\eta_{\text{as}}} \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \sqrt{\varepsilon} \begin{pmatrix} a_{n,n'}^{(\frac{1}{2})(\text{TE})} & \tilde{\lambda}_{n,n'} \\ \tilde{\lambda}_{n,n'} & a_{n,n'}^{(\frac{1}{2})(\text{TM})} \end{pmatrix} + \varepsilon \begin{pmatrix} a_{n,n'}^{(1)(\text{TE})} + \hat{\lambda}_{n,n'} & 0 \\ 0 & a_{n,n'}^{(1)(\text{TM})} + \hat{\lambda}_{n,n'} \end{pmatrix} + \dots \right\}. \quad (48)$$

Here the functions $a_{n,n'}$ are the same as in the scalar cases with the corresponding boundary conditions. The factors in front of the figure bracket appear in the same way as in the scalar case from the Bessel functions. We remind that in the given order of the asymptotic expansion the difference between scalar Neumann boundary conditions and those of the TM mode does not show up.

In deriving Eq. (48) we have to pay attention to the ν -dependence of $\lambda_{n,n'}$. Therefore we defined $\hat{\lambda}_{n,n'}$ by

$$\hat{\lambda}_{n,n'} = \sqrt{\frac{2\tau}{1+\tau}} \int_{-\infty}^{\infty} \frac{d\eta}{\sqrt{\pi}} e^{-\eta^2 + 2i\eta\sqrt{2}\mu + \mu^2} \sum_{\nu=0}^{\nu_m} \frac{\eta^{2\nu}}{\nu!} \left(\frac{1-\tau}{1+\tau} \right)^\nu \lambda_{n,n'}. \quad (49)$$

This definition is taken in a way that $\lambda_{n,n'} \rightarrow 1$ in (49) gives $\hat{\lambda}_{n,n'} = 1$.

Next we have to insert (48) into (47). Making a re-expansion in ε and taking care of the matrix multiplication we get

$$\mathcal{Z}^{\text{as}} = \text{tr} \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \varepsilon \left[\begin{pmatrix} a^{\text{D}} & 0 \\ 0 & a^{\text{N}} \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \left(\sum_{i=0}^s \hat{\lambda}_{n_i, n_{i+1}} + \sum_{0 < i < j < s} \tilde{\lambda}_{n_i, n_{i+1}} \tilde{\lambda}_{n_j, n_{j+1}} \right) \right] + \dots \right\} \quad (50)$$

In this formula we dropped off-diagonal contributions and those proportional to $\sqrt{\varepsilon}$ which will disappear later when carrying out the remaining integrations as already mentioned in the preceding section. In this formula, the factors resulting from the T-matrix collected in the same way as in the scalar case into the functions a^{D} and a^{N} . The contributions from the vector structures (44) and (45) do not depend on the polarization (see the structure of the translation formulas, Eq. (17)). Therefore these enter proportional to a unit matrix. Now we take the remaining trace and come to

$$\mathcal{Z}^{\text{as}} = 2 + \left(a^{\text{D}} + a^{\text{N}} + 2 \sum_{i=0}^s \hat{\lambda}_{n_i, n_{i+1}} + 2 \sum_{0 < i < j < s} \tilde{\lambda}_{n_i, n_{i+1}} \tilde{\lambda}_{n_j, n_{j+1}} \right) \varepsilon + \dots \quad (51)$$

This expression for \mathcal{Z}^{as} must be inserted for \mathcal{Z} into the energy, Eq.(29). At this place we can already read off some of the features. First of all, the factor of 2 accounts for the polarizations of the electromagnetic field. Since the remaining integrations are the same as for the scalar field we immediately get the expected result that in PFA the electromagnetic field gives twice the contribution of a scalar field. This is the leading order and it is of course independent of the boundary conditions.

In order ε , the first two factors, a^{D} and a^{N} , give the same contributions as in the scalar cases. So we have, for the electromagnetic field, in the first correction beyond PFA a contribution with Dirichlet boundary conditions on both, the sphere and the plane, and another one with Neumann boundary conditions instead. The corresponding

contributions to the relative corrections beyond PFA to the energy are given by the first two lines in Eq. (43) and must be divided by 2.

We mention the role of the minus sign in the latter case. For the scalar field there were two of them, one following from the derivative of the Bessel function in the denominator in (12). This sign has a corresponding one in the electromagnetic case in the denominator of (27). The other minus sign followed in the scalar case from the sign in the logarithm in Eq.(15). In the electromagnetic case the corresponding one is the minus sign in front of d_l^{TM} in (25).

The remaining terms in Eq.(51) just represent the additional contributions which come in from the vector character of the electromagnetic field, i.e., from its spin. There are contributions diagonal in the polarizations, resulting from $\Lambda_{l,l'}''$, Eq.(20), and off-diagonal ones resulting from $\tilde{\Lambda}_{l,l'}$, Eq.(22). It should be mentioned that in the considered first order in ε all these contributions enter additively. A mixing of these will happen in higher orders only.

In the following subsections we consider separately the contributions resulting from the three parts, A, B and C of $\Lambda_{l,l'}''$, Eq.(44) and that of $\tilde{\Lambda}_{l,l'}$, Eq.(45). Starting from part B we will meet expressions where it is not possible to make a direct expansion for small ε . For illustration we consider a simple example. Consider the integral

$$f(\varepsilon) = \int_0^1 d\tau \frac{g(\tau)}{\tau + a\varepsilon + b\varepsilon^2} \quad (52)$$

for $\varepsilon \rightarrow 0$. In case the function $g(\tau)$ has a zero, $g(0) = 0$, we can put $\varepsilon = 0$ directly under the sign of the integral. In opposite, if $g(0) \neq 0$ holds, we cannot do that since the τ -integration would diverge. The only we can do is to put $\varepsilon = 0$ where it goes with the coefficient b ,

$$f(\varepsilon) = \int_0^1 d\tau \frac{g(\tau)}{\tau + a\varepsilon} + \dots, \quad (53)$$

where the dots denote contributions of higher order in ε . The remaining integral must be treated in some other way. For example, we can integrate by parts and expand after that,

$$\begin{aligned} f(\varepsilon) &= \ln(1 + \varepsilon) g(1) - \ln \varepsilon g(0) - \int_0^1 d\tau \ln(\tau + \varepsilon) g'(\tau) + \dots, \\ &= -\ln \varepsilon g(0) - \int_0^1 d\tau \ln \tau g'(\tau) + \dots. \end{aligned} \quad (54)$$

Below such and similar situations will appear repeatedly.

A. Part A in the Λ -contribution

Part A is given by

$$A = \frac{\mu(n)\mu(n')}{\gamma(n)\gamma(n')} - 1 \quad (55)$$

with $\mu(n)$ and $\gamma(n)$ defined in Eq. (46). The contribution from this part to the energy is quite easy to calculate since it is possible to expand A directly in powers of ε ,

$$A = -\frac{1}{t\tau} \varepsilon + \dots, \quad (56)$$

without causing any divergences. We define its contribution to $\lambda_{n,n'}$ by

$$\lambda_{A;n,n'} = -\frac{1}{t\tau}. \quad (57)$$

This is a quite simple formula, for instance the dependence on n and n' dropped out. Further, since it does not depend on ν , from (49) we have $\hat{\lambda}_{A;n,n'} = \lambda_{A;n,n'}$ and its contribution to \mathcal{Z}^{as} , (51), is

$$\mathcal{Z}^{\text{as}} = 2 - 2\varepsilon \sum_{i=0}^s \frac{1}{t\tau} + \dots \quad (58)$$

Here the dots stand for all other contributions. We denote the corresponding part of the energy by ΔE_A and we come with (29) to

$$E_{\text{PFA}} + \Delta E_A = \frac{R}{4\pi d^2} \sum_{s=0}^{\infty} \frac{-2}{s+1} \int_0^{\infty} dt t \int_0^1 \frac{d\tau \sqrt{\tau}}{\sqrt{1-\tau^2}} \int_{-\infty}^{\infty} \frac{d\mu}{\sqrt{\pi}} \times \left(\prod_{j=1}^s \int_{-\infty}^{\infty} \frac{dn_j}{\sqrt{\pi}} \right) \left(1 - \varepsilon \frac{s+1}{\tau t} + \dots \right) e^{-2t(s+1) - \mu^2(s+1)/\tau - \eta_1}. \quad (59)$$

We mention that we have taken the lower integration limit in the n_j -integrations to $-\infty$ which is possible since the remaining integrations do converge. The integrations in this expression can be carried out easily and the result is

$$E_{\text{PFA}} + \Delta E_A = \frac{-2R}{4\pi d^2} \left(\frac{\zeta(4)}{4} - \varepsilon \frac{\pi\zeta(2)}{4} + \dots \right). \quad (60)$$

The correction resulting from part A is

$$\Delta E_A = \frac{R}{4\pi d^2} \frac{\pi\zeta(2)}{2} \varepsilon. \quad (61)$$

B. Part B in the Λ -contribution

Part B is given by

$$B = \frac{-2\nu\varepsilon}{\sqrt{t}} \frac{\mu(n) + \mu(n')}{\gamma(n)\gamma(n')}. \quad (62)$$

Regrettably, its contribution to the energy cannot be calculated so easy as before. First of all we have an additional dependence on ν . In addition, it is impossible to make a simple expansion in ε . This would produce a singularity in the τ -integration.

We define $\lambda_{B;n,n'} = B/\varepsilon$ which must be inserted into (49). The summation over ν is quite simple,

$$\sum_{\nu=0}^{\infty} \nu \frac{\eta^{2\nu}}{\nu!} \left(\frac{1-\tau}{1+\tau} \right)^\nu = \frac{1-\tau}{1+\tau} \eta^2 \exp\left(\frac{1-\tau}{1+\tau} \eta^2 \right). \quad (63)$$

Here we have taken $\nu_m = \infty$ since this does not cause singularities. Next we need to carry out the integration over η in (33). It is Gaussian,

$$\int_{-\infty}^{\infty} \frac{d\eta}{\sqrt{\pi}} \frac{1-\tau}{1+\tau} \eta^2 \exp\left(\frac{1-\tau}{1+\tau} \eta^2 - \eta^2 + 2i\eta\sqrt{2}\mu + \mu^2 \right) = h_B(\tau, \mu) \sqrt{\frac{1+\tau}{2\tau}} \exp\left(-\frac{\mu^2}{\tau} \right). \quad (64)$$

with

$$h_B(\tau, \mu) = \frac{1-\tau}{4\tau} \left(1 - 2\frac{1+\tau}{\tau} \mu^2 \right). \quad (65)$$

Using Eq.(64) in (49) we get

$$\hat{\lambda}_{B;n,n'} = h_B(\tau, \mu) \lambda_{B;n,n'}. \quad (66)$$

This must be inserted into \mathcal{Z}^{as} , (51), and further into the energy. With (29) it is

$$\Delta E_B = -\frac{R}{4\pi d^2} \sum_{s=0}^{\infty} \frac{-4\varepsilon}{s+1} \int_0^{\infty} dt \sqrt{t} e^{-2t(s+1)} \tilde{B}, \quad (67)$$

where we defined

$$\tilde{B} = \int_0^1 \frac{d\tau \sqrt{\tau}}{\sqrt{1-\tau^2}} \int_{-\infty}^{\infty} \frac{d\mu}{\sqrt{\pi}} e^{-\mu^2(s+1)/\tau} h_B(\tau, \mu) \left(\prod_{j=1}^s \int_{n_0}^{\infty} \frac{dn_j}{\sqrt{\pi}} \right) \sum_{i=0}^s \frac{\mu(n_i) + \mu(n_{i+1})}{\gamma(n_i)\gamma(n_{i+1})} e^{-\eta_1}. \quad (68)$$

The integration over μ is Gaussian and we get

$$\tilde{B} = \frac{1}{\sqrt{s+1}} \int_0^1 d\tau f_B(\tau) \left(\prod_{j=1}^s \int_{n_0}^{\infty} \frac{dn_j}{\sqrt{\pi}} \right) \sum_{i=0}^s \frac{\mu(n_i) + \mu(n_{i+1})}{\gamma(n_i)\gamma(n_{i+1})} e^{-n_i} \quad (69)$$

with

$$f_B(\tau) = \frac{1}{4} \sqrt{\frac{1-\tau}{1+\tau}} \left(1 - \frac{1+\tau}{s+1} \right). \quad (70)$$

Now we have still ε in a number of places in (69). Still we cannot simply expand for small ε . If we would do so, because of

$$\frac{\mu(n_i)}{\gamma(n_i)\gamma(n_{i+1})} = \frac{1}{\tau\sqrt{t}} + \dots, \quad (71)$$

where we used (46), the τ -integration would become logarithmically divergent. The only places where we can put $\varepsilon = 0$ directly are those where it goes with n in $\mu(n)$ and in $\gamma(n)$,

$$\frac{\mu(n_i)}{\gamma(n_i)\gamma(n_{i+1})} = \frac{1}{\sqrt{\tau\sqrt{t}(\tau\sqrt{t} + \varepsilon/\sqrt{t})}} + \dots \quad (72)$$

Also we can take $n_0 = -\infty$. In doing so we do not produce divergences in the integrations. After that we are left with a simpler τ -integration. Here we integrate by parts,

$$\int_0^1 d\tau \frac{f_B(\tau)}{\sqrt{\tau\sqrt{t}(\tau\sqrt{t} + \varepsilon/\sqrt{t})}} = -\frac{2}{\sqrt{t}} \int_0^1 d\tau \ln \frac{\sqrt{\tau\sqrt{t}} + \sqrt{\tau\sqrt{t} + \varepsilon/\sqrt{t}}}{\sqrt{\varepsilon/\sqrt{t}}} \frac{\partial}{\partial \tau} f_B(\tau). \quad (73)$$

The surface term is zero. In the new τ -integral it is possible to expand the logarithm for small ε . We insert the result into \tilde{B} , Eq.(69), and get

$$\tilde{B} = \frac{-4}{\sqrt{t}} \left[\left(-\frac{1}{2} \ln \varepsilon + \frac{1}{2} \ln t + \ln 2 \right) (f_B(1) - f_B(0)) + \frac{1}{2} \int_0^1 d\tau \ln \tau \frac{\partial}{\partial \tau} f_B(\tau) \right] + \dots \quad (74)$$

Since after (72) there is no more any n_j -dependence we used formula (40) and accounted also for the sum over i . The remaining integration over τ can be carried out easily,

$$\int_0^1 d\tau \ln \tau \frac{\partial}{\partial \tau} f_B(\tau) = \frac{\pi}{8} - \frac{1+s \ln 2}{4(s+1)}. \quad (75)$$

We mention that Eq.(74) is the first place where a logarithm in ε appears. Its origin is clearly seen from Eq.(71).

Next we have to insert \tilde{B} into the energy (67). The remaining t -integration is now a bit more complicated since it involves a $\ln t$. From that a logarithm $\ln(1+s)$ appears. However, simple calculations yield

$$\Delta E_B = \frac{R}{4\pi d^2} \left[(\zeta(3) - \zeta(2)) (\gamma - 2 \ln 2) + \zeta'(2) - \zeta'(3) - \frac{\pi}{2} \zeta(2) + \zeta(3) + \frac{1}{4\pi} (\zeta(3) - \zeta(2)) \ln \varepsilon \right] \varepsilon. \quad (76)$$

This is the contribution from part B in (43) to the corrections beyond PFA. It involves a logarithm in ε and it has an analytic expression.

C. Part C in the Λ -contribution

Part C is given by

$$C = \frac{\varepsilon^2 \nu(2\nu - 1)}{t \gamma(n)\gamma(n')}. \quad (77)$$

The calculation of its contribution to the energy requires most effort. First of all we observe a quadratic dependence on ν . We have to insert (77) into (33) and define the corresponding contribution to $N_{l,\nu}^{\text{as}}$ by

$$N_{C;l,\nu}^{\text{as}} = \sqrt{\frac{\varepsilon\tau}{2\pi t(1+\tau)}} e^{-2t-(n-n')^2} \int_{-\infty}^{\infty} \frac{d\eta}{\sqrt{\pi}} e^{-\eta^2+2i\eta\sqrt{2}\mu+\mu^2} \times \sum_{\nu=0}^{\nu_m} \frac{\eta^{2\nu}}{\nu!} \left(\frac{1-\tau}{1+\tau}\right)^\nu C. \quad (78)$$

Here it is impossible to take the upper limit ν_m of the summation over ν to infinity. Doing so would produce a factor τ^{-2} and making the τ -integration diverge. Therefore we must account for a finite ν_m , Eq.(34). With the substitution (28) it becomes

$$\nu_m = \frac{t\tau}{\varepsilon} + \dots, \quad (79)$$

Technically we account for it by inserting a step function into the sum and formally summing up to infinity as before. For the step function we take the integral representation

$$\Theta(\nu_m - \nu) = \int_{-\infty}^{\infty} \frac{dp}{2\pi i} \frac{\exp(ip(\frac{t\tau}{\varepsilon} - \nu))}{p - i0} \quad (80)$$

and define

$$N_{C;l,\nu}^{\text{as}} = \int_{-\infty}^{\infty} \frac{dp}{2\pi i} \frac{\exp(ip\frac{t\tau}{\varepsilon})}{p - i0} \tilde{N}_{C;l,\nu}^{\text{as}} \quad (81)$$

The summation over ν now involves the additional factor α^ν with

$$\alpha \equiv e^{-ip}. \quad (82)$$

This sum and also the integration over η can be done generalizing (63) and (64). The result is

$$\tilde{N}_{C;l,\nu}^{\text{as}} = \frac{\varepsilon^2}{t} \frac{\Lambda^{1/2}}{\gamma(n)\gamma(n')} h_C(\tau, \mu) \sqrt{\frac{\varepsilon}{4\pi t}} e^{-\eta_{\text{as}} - \mu^2/(\tau\tilde{\Lambda})} \quad (83)$$

with

$$h_C(\tau, \mu) = 6 \left(\frac{1-\tau}{4\tau} \alpha \Lambda\right)^2 \left[1 - 4\frac{1+\tau}{\tau} \Lambda \mu^2 + \frac{4}{3} \left(\frac{1+\tau}{\tau} \Lambda \mu^2\right)^2\right] + \frac{1-\tau}{4\tau} \alpha \Lambda \left[1 - 2\frac{1+\tau}{\tau} \Lambda \mu^2\right]. \quad (84)$$

Here we defined (for use only in this subsection)

$$\Lambda = \frac{2\tau}{1+\tau-\alpha(1-\tau)}, \quad \tilde{\Lambda} = \frac{2}{2-(1-\tau)(1-\alpha)}. \quad (85)$$

From (81) we can now read off the contribution from part C into $\tilde{\lambda}_{n,n'}$ and insert that into \mathcal{Z}^{as} , Eq.(50) and further into the energy (29). The corresponding contribution is

$$\Delta E_C = -\frac{R}{4\pi d^2} \sum_{s=0}^{\infty} \frac{2\varepsilon^2}{s+1} \int_0^{\infty} dt e^{-2t(s+1)} \tilde{C}, \quad (86)$$

where we defined

$$\tilde{C} = \int_0^1 \frac{d\tau \sqrt{\tau}}{\sqrt{1-\tau^2}} \int_{-\infty}^{\infty} \frac{d\mu}{\sqrt{\pi}} e^{-\mu^2(s+1)/\tau} \left(\prod_{j=1}^s \int_{n_0}^{\infty} \frac{dn_j}{\sqrt{\pi}}\right) \sum_{i=0}^s \frac{e^{-n_i}}{\gamma(n_i)\gamma(n_{i+1})} \int_{-\infty}^{\infty} \frac{dp}{2\pi i} \frac{\exp(ip\frac{t\tau}{\varepsilon})}{p - i0} h_C(\tau, \mu). \quad (87)$$

Here, again, the integration over μ is Gaussian and we come to

$$\tilde{C} = \frac{1}{\sqrt{s+1}} \int_0^1 \frac{d\tau \tau}{\sqrt{1-\tau^2}} \left(\prod_{j=1}^s \int_{n_0}^{\infty} \frac{dn_j}{\sqrt{\pi}} \right) \sum_{i=0}^s \frac{e^{-n_i}}{\gamma(n_i)\gamma(n_{i+1})} R(\tau, t) \quad (88)$$

with

$$R(\tau, t) = \int_{-\infty}^{\infty} \frac{dp}{2\pi i} \frac{\exp(ip\frac{t\tau}{\varepsilon})}{p-i0} (\Lambda\tilde{\Lambda})^{1/2} \left[6(\alpha\Lambda f_C(\tau))^2 + \alpha\Lambda f_C(\tau) \right] \quad (89)$$

and

$$f_C(\tau) = \frac{1-\tau}{4\tau} \left(1 - \frac{1+\tau}{1+s} \Lambda\tilde{\Lambda} \right). \quad (90)$$

In Eq.(88), the main contribution comes from $\tau \sim 0$. This is because the function $f_C(\tau)$ diverges like $f_C(\tau) \sim \tau^{-2}$ for $\tau \rightarrow 0$. The integration over τ is nevertheless finite for $\varepsilon \neq 0$ because of the exponential involving p . Next we want to carry out the integration over p in (89). For this we move the integration contour upwards in the complex p -plane. We have a pole at $p = 0$. Its contribution is easy to calculate and it gives just the result we would obtain with putting $\nu_m = \infty$ at the very beginning. Further there are poles in $p_0 = i \ln \frac{1+\tau}{1-\tau} + 2\pi n$ (n integer) from Λ and cuts starting from $p_c = i \ln \frac{1+\tau}{1-\tau} + \pi + 2\pi n$ resulting from $\Lambda\tilde{\Lambda}$.

For small ε , non-vanishing contributions come only from the poles in $p = 0$ and in $p = p_0$ with $n = 0$. For these we can expand $p_0 = 2i\tau + \dots$ and $\Lambda\tilde{\Lambda} = 1 + \dots$ and R takes the form

$$R(\tau, t) = \int_{-\infty}^{\infty} \frac{dp}{2\pi i} \frac{\exp(ip\frac{t\tau}{\varepsilon})}{p-i0} \left[6 \left(\alpha \frac{-p_0}{p-p_0} f_C(\tau) \right)^2 + \alpha \frac{-p_0}{p-p_0} f_C(\tau) \right] + \dots \quad (91)$$

The pole in p_0 is second order and we get

$$R(\tau, t) = 6f_C(\tau)^2 \left[1 - \left(1 + \frac{2t\tau^2}{\varepsilon} \right) e^{-2t\tau^2/\varepsilon} \right] + f_C(\tau) \left[1 - e^{-2t\tau^2/\varepsilon} \right] + \dots \quad (92)$$

We mention that the first term in both square brackets result from a ν -summation with $\nu_m = \infty$. The terms with the exponentials result from taking a finite ν_m . As a result, while $\varepsilon \neq 0$, the function $R(\tau, t)$ is finite for $\tau \rightarrow 0$ instead of diverging like τ^{-2} as the function $f_C(\tau)$ does. We emphasize that this is the justification for accounting for a finite ν_m in part C. In the other parts, because we did not have a small- τ behavior like in this one, we could take $\nu_m = \infty$ without hitting a divergence.

The expression for R , Eq.(92), must be inserted into \tilde{C} , Eq.(88). To proceed with the expansion for small ε we observe that we cannot put $\varepsilon = 0$ in R since this would return us to a situation where the τ -integration diverges at $\tau \rightarrow 0$. Instead we make in \tilde{C} the substitution $\tau \rightarrow \tau\sqrt{\varepsilon}$,

$$\tilde{C} = \frac{1}{\varepsilon} \frac{1}{\sqrt{s+1}} \int_0^{1/\sqrt{\varepsilon}} \frac{d\tau \tau}{\sqrt{1-\tau^2\varepsilon}} \left(\prod_{j=1}^s \int_{\tilde{n}_0}^{\infty} \frac{dn_j}{\sqrt{\pi}} \right) \sum_{i=0}^s \frac{e^{-n_i}}{\tilde{\gamma}(n_i)\tilde{\gamma}(n_{i+1})} R(\tau\sqrt{\varepsilon}, t) \quad (93)$$

with

$$\tilde{\gamma}(n) = \sqrt{(\tau\sqrt{t} + 2n) \left(\tau\sqrt{t} + 2n + \sqrt{\frac{\varepsilon}{t}} \right)} \quad (94)$$

and

$$\tilde{n}_0 = -\frac{1}{2}\tau\sqrt{t}. \quad (95)$$

In this way, the τ -integration produces a factor $1/\varepsilon$ which makes the contribution from part C, which initially went with a factor ε^2 , a contribution first order in ε . We mention that it resulted from the factors $\gamma(n)$ in the denominator.

There is still a dependence on ε in \tilde{C} , Eq.(93). It is twofold. First is that which goes with τ . Here we can put $\varepsilon = 0$. The same we can do in the upper integration limit. We note

$$R(\tau\sqrt{\varepsilon}, t) = \frac{3}{2\varepsilon} \left(\frac{s}{s+1} \right)^2 g(2t\tau^2) + O(1) \quad (96)$$

with

$$g(x) = \frac{1}{4} (1 - (1+x)e^{-x}) \quad (97)$$

for $\varepsilon \rightarrow 0$ and for \tilde{C} we get

$$\tilde{C} = \frac{3}{2\varepsilon} \frac{s^2}{(s+1)^{5/2}} \int_0^\infty \frac{d\tau}{\tau} g(2t\tau^2) \left(\prod_{j=1}^s \int_{\hat{n}_0}^\infty \frac{dn_j}{\sqrt{\pi}} \right) \sum_{i=0}^s \frac{e^{-\eta_1}}{\hat{\gamma}(n_i)\hat{\gamma}(n_{i+1})}. \quad (98)$$

In order to simplify the representation of \tilde{C} we make in (98) the substitution $\tau \rightarrow \tau/\sqrt{t}$. After that it takes the form

$$\tilde{C} = \sigma \int_0^\infty \frac{d\tau}{\tau} g(2\tau^2) \left(\prod_{j=1}^s \int_{\hat{n}_0}^\infty \frac{dn_j}{\sqrt{\pi}} \right) \sum_{i=0}^s \frac{e^{-\eta_1}}{\hat{\gamma}(n_i)\hat{\gamma}(n_{i+1})}, \quad (99)$$

where we introduced the notations

$$\sigma = \frac{3}{2\varepsilon} \frac{s^2}{(s+1)^{5/2}}, \quad \hat{n}_0 = -\frac{\tau}{2}, \quad \hat{\gamma}(n) = \sqrt{(\tau+2n)(\tau+2n+\sqrt{\varepsilon/t})}, \quad (100)$$

which will be used in the remaining part of this subsection.

In this way we are left with the dependence on ε in $\hat{\gamma}$. Here we can not put directly $\varepsilon = 0$. This would produce a divergence in the integrations at $\tau = -2n$ and an imaginary part would appear which is clearly not present in the energy. The way out is a partial integration in the n_i and in the n_{i+1} integrations. But before we can do that we have to pay attention to the contributions from $i = 0$ and from $i = s$ in the sum over i . From the formal setting in Eq.(31) we have to put $n_0 = n_s = 0$ in Eq.(99) and the corresponding $\hat{\gamma}$ do not depend on any n . We denote the contributions from $i = 0$ and $i = s$ (both give for symmetry reasons the same contribution) by \tilde{C}_0 and the remaining one, i.e., that for $i = 1, \dots, s-1$, by \tilde{C}_1 .

First we consider \tilde{C}_0 . Since the function $g(x) \sim x^2$ for $x \rightarrow 0$ we can take $\hat{\gamma}(0) = \tau + \dots$ and are left with

$$\tilde{C}_0 = 2\sigma \int_0^\infty \frac{d\tau}{\tau^2} g(2\tau^2) \left(\prod_{j=1}^s \int_{\hat{n}_0}^\infty \frac{dn_j}{\sqrt{\pi}} \right) \frac{e^{-\eta_1}}{\hat{\gamma}(n_1)}. \quad (101)$$

Now we integrate by parts according to

$$\begin{aligned} \int_{-\tau/2}^\infty dn \frac{e^{-\eta_1}}{\hat{\gamma}(n)} &= -\ln \left(2 \left(\frac{\varepsilon}{t} \right)^{1/4} \right) e^{-\eta_2} \\ &\quad - \int_{-\tau/2}^\infty dn \ln \left(2 \left(\sqrt{\tau+2n} + \sqrt{\tau+2n+\sqrt{\varepsilon/t}} \right) \right) \frac{\partial}{\partial n} e^{-\eta_1} \end{aligned} \quad (102)$$

with

$$\eta_2 = \eta_1|_{n=-\tau/2}. \quad (103)$$

Here we can expand for small ε ,

$$\int_{-\tau/2}^\infty dn \frac{e^{-\eta_1}}{\hat{\gamma}(n)} = -\ln \left(2 \left(\frac{\varepsilon}{t} \right)^{1/4} \right) e^{-\eta_2} - \int_{-\tau/2}^\infty dn \ln(\sqrt{\tau+2n}) \frac{\partial}{\partial n} e^{-\eta_1} + \dots \quad (104)$$

This must be inserted into Eq.(101). After that we make there the substitution $n_j \rightarrow n_j \tau/2$. This allows to change the orders of integrations,

$$\tilde{C}_0 = \frac{-2\sigma}{\sqrt{\pi}} \left(\frac{1}{4} \ln \frac{\varepsilon}{t} + \ln 2 \right) R(s) + 4\sigma Q(s), \quad (105)$$

and we introduced the notations

$$\begin{aligned} R(s) &= \left(\prod_{j=2}^s \int_{-1}^{\infty} \frac{dn_j}{\sqrt{\pi}} \right) g_1 \left(\frac{\eta_2}{4} \right) \\ Q(s) &= \left(\prod_{j=1}^s \int_{-1}^{\infty} \frac{dn_j}{\sqrt{\pi}} \right) (2n_1 - n_2) g_2 \left(\frac{\eta_1}{4} \right) \end{aligned} \quad (106)$$

with

$$\begin{aligned} g_1(x) &= \int_0^{\infty} \frac{d\tau}{\tau^2} g(2\tau^2) \left(\frac{\tau}{2} \right)^{s-1} e^{-x\tau^2}, \\ g_2(x) &= \int_0^{\infty} \frac{d\tau}{\tau^2} g(2\tau^2) \ln(4\sqrt{\tau}\sqrt{1+2n_1}) \left(\frac{\tau}{2} \right)^{s+1} e^{-x\tau^2}. \end{aligned} \quad (107)$$

The integrations over τ can be carried out explicitly, however the formulas are too voluminous as to be displayed here. The remaining integrations over the n_j can be performed only numerically. Even that is not an easy task since the integrals are s -dimensional. We could proceed only till $s = 5$ for $R(s)$ and $s = 3$ for $Q(s)$. However, since the sum over s is quite fast converging this gives at least the order of magnitude correctly.

Expression (105) must be inserted into the energy (86). Introducing the corresponding notation it is

$$\Delta E_{\tilde{C}_0} = -\frac{R}{4\pi d^2} \sum_{s=0}^{\infty} \frac{2\varepsilon^2}{s+1} \int_0^{\infty} dt e^{-2t(s+1)} \tilde{C}_0. \quad (108)$$

Here, the t -integration can be carried out easily and summing over s , as far as data are available, the result is

$$\Delta E_{\tilde{C}_0} = \frac{R}{d^2} \varepsilon (0.0020 + 0.00017 \ln \varepsilon). \quad (109)$$

Now we have to consider \tilde{C}_1 , i.e., the contributions from $i = 1, \dots, s-1$ to (99). We make the substitution $n_j \rightarrow n_j \tau/2$ such that it takes the form

$$\tilde{C}_1 = \sigma \int_0^{\infty} \frac{d\tau}{\tau^2} g(2\tau^2) \left(\frac{\tau}{2} \right)^s \left(\prod_{j=1}^s \int_{-1}^{\infty} \frac{dn_j}{\sqrt{\pi}} \right) \sum_{i=1}^{s-1} \frac{e^{-\eta_1 \tau^2/4}}{\bar{\gamma}(n_i) \bar{\gamma}(n_{i+1})} \quad (110)$$

with $\bar{\gamma}(n) = \sqrt{(1+n)(\tau(1+n) + \sqrt{\varepsilon/t})}$. For the integration by parts we adopt the following scheme,

$$\begin{aligned} \int_{-1}^{\infty} dn \frac{e^{-\eta_1 \tau^2/4}}{\bar{\gamma}(n)} &= -\frac{1}{\sqrt{\tau}} \left[\frac{1}{4} \ln \frac{\varepsilon}{\tau} - \ln(2\sqrt{\tau}) \right] e^{-\eta_1 \tau^2/4} \\ &\quad - \frac{1}{\sqrt{\tau}} \int_{-1}^{\infty} dn \ln \frac{\sqrt{\tau}\sqrt{1+n} + \sqrt{\tau(1+n) + \sqrt{\varepsilon/t}}}{\sqrt{\tau} + \sqrt{\tau + \sqrt{\varepsilon/t}}} \frac{\partial}{\partial n} e^{-\eta_1 \tau^2/4}. \end{aligned} \quad (111)$$

Here we can take $\varepsilon \rightarrow 0$ and get

$$\int_{-1}^{\infty} dn \frac{e^{-\eta_1 \tau^2/4}}{\bar{\gamma}(n)} = -\frac{1}{\sqrt{\tau}} \int_{-1}^{\infty} dn (L_1(n) + L_2(n)) e^{-\eta_1 \tau^2/4} + \dots, \quad (112)$$

where we defined

$$L_1(n) = \left(\frac{1}{4} \ln \frac{\varepsilon}{t} - \ln(2\sqrt{\tau}) \right) \delta(n+1), \quad L_2(n) = \ln \sqrt{1+n} \frac{\partial}{\partial n}. \quad (113)$$

We have to apply these formulas in (110) to n_i and to n_{i+1} ,

$$\tilde{C}_1 = \sigma \int_0^{\infty} \frac{d\tau}{\tau^2} g(2\tau^2) \left(\frac{\tau}{2} \right)^s \left(\prod_{j=1}^s \int_{-1}^{\infty} \frac{dn_j}{\sqrt{\pi}} \right) \sum_{i=1}^{s-1} (L_1(n_i) + L_2(n_i)) (L_1(n_{i+1}) + L_2(n_{i+1})) e^{-\eta_1 \tau^2/4}. \quad (114)$$

Multiplying out the two brackets,

$$(L_1(n_i) + L_2(n_i))(L_1(n_{i+1}) + L_2(n_{i+1})) = L_1(n_i)L_1(n_{i+1}) + 2L_1(n_i)L_2(n_{i+1}) + L_2(n_i)L_2(n_{i+1}) \quad (115)$$

(we used the symmetry under $i \rightarrow s-1-i$ in (114)), we split \tilde{C}_1 into three parts,

$$\tilde{C}_1 = \tilde{C}_{1A} + \tilde{C}_{1B} + \tilde{C}_{1C} \quad (116)$$

and consider them separately.

We start with \tilde{C}_{1A} and interchanging the orders of integration it is

$$\tilde{C}_{1A} = \sigma\pi^{-s/2} \left(\prod_{\substack{j=1 \\ j \neq i, i+1}}^s \int_{-1}^{\infty} dn_j \right) \sum_{i=1}^{s-1} \int_0^{\infty} \frac{d\tau}{\tau^3} g(2\tau^2) \left(\frac{\tau}{2}\right)^s \left(\frac{1}{4} \ln \frac{\varepsilon}{t} - \ln(2\sqrt{\tau})\right)^2 e^{-\eta_3 \tau^2/4} \quad (117)$$

with

$$\eta_3 = \eta_1|_{n_i=n_{i+1}=-1}. \quad (118)$$

This is the place where the logarithm of ε appears squared. Again, the integration over τ can be carried out explicitly delivering lengthy formulas. Expression (117) must be inserted into the energy (86). Introducing the corresponding notation it is

$$\Delta E_{C1A} = -\frac{2}{4\pi} \sum_{s=2}^{\infty} \frac{\varepsilon^2}{s+1} \sigma\pi^{-s/2} P(s), \quad (119)$$

where

$$\int_0^{\infty} dt e^{-2t(s+1)} \tilde{C}_{1A} = \sigma\pi^{-s/2} P(s) \quad (120)$$

collects the integrations over the n_j and over t . Again, the n -integrations must be done numerically, we were able to go up to $s=7$. As a result we get

$$\Delta E_{C1A} = \frac{R}{d^2} \varepsilon (-8.8 \cdot 10^{-7} - 2.4 \cdot 10^{-7} \ln \varepsilon - 3.6 \cdot 10^{-7} (\ln \varepsilon)^2). \quad (121)$$

Next we have to consider \tilde{C}_{1B} . In parallel to (117) it is

$$\tilde{C}_{1B} = -4\sigma\pi^{-s/2} \left(\prod_{\substack{j=1 \\ j \neq i}}^s \int_{-1}^{\infty} dn_j \right) \sum_{i=1}^{s-1} \int_0^{\infty} \frac{d\tau}{\tau^3} g(2\tau^2) \left(\frac{\tau}{2}\right)^{s+2} \left(\frac{1}{4} \ln \frac{\varepsilon}{t} - \ln(2\sqrt{\tau})\right) (2n_{i+1} - n_i - n_{i+2}) e^{-\eta_2 \tau^2/4} \quad (122)$$

with

$$\eta_2 = \eta_1|_{n_i=-1}. \quad (123)$$

Again, the integration over τ can be carried out explicitly and the integration over the n_j only numerically. Here we could go until $s=5$. The result is

$$\Delta E_{C1B} = \frac{R}{d^2} \varepsilon (-9.4 \cdot 10^{-6} + 0.000019 \ln \varepsilon). \quad (124)$$

Finally we come to \tilde{C}_{1C} . In parallel to (117) it is

$$\begin{aligned} \tilde{C}_{1C} &= \frac{\sigma}{2} \pi^{-s/2} \left(\prod_{j=1}^s \int_{-1}^{\infty} dn_j \right) \sum_{i=1}^{s-1} \int_0^{\infty} \frac{d\tau}{\tau^3} g(2\tau^2) \left(\frac{\tau}{2}\right)^{s+2} \ln(1+n_i) \ln(1+n_{i+1}) \\ &\quad \times \left(1 + \frac{\tau^2}{2} (2n_i - n_{i-1} - n_{i+1})(2n_{i+1} - n_i - n_{i+2}) \right) e^{-\eta_1 \tau^2/4}. \end{aligned} \quad (125)$$

As before, the τ -integration can be done explicitly and the n -integrations numerically. Here we went up to $s=5$. The result is

$$\Delta E_{C1C} = \frac{R}{d^2} \varepsilon (-0.000076). \quad (126)$$

D. The $\tilde{\Lambda}$ -contribution

We start from the formula (45) and insert $\tilde{\lambda}_{n,n'}$ into \mathcal{Z}^{as} , Eq.(51). From this we define with (29) the corresponding contribution to the energy,

$$\Delta E_{\tilde{\Lambda}} = \frac{-2R}{4\pi d^2} \sum_{s=0}^{\infty} \frac{\varepsilon}{s+1} \int_0^{\infty} dt t e^{-2t(s+1)} \tilde{L} \quad (127)$$

with

$$\tilde{L} = \int_0^1 \frac{d\tau \sqrt{\tau}}{\sqrt{1-\tau^2}} \int_{-\infty}^{\infty} \frac{d\mu}{\sqrt{\pi}} e^{-\mu^2(s+1)/\tau} \left(\prod_{j=1}^s \int_{n_0}^{\infty} \frac{dn_j}{\sqrt{\pi}} \right) \sum_{0 < i < j < s} \frac{4\tau t(1-\tau^2)(1-\varepsilon)^2 \mu^2}{\gamma(n_i)\gamma(n_{i+1})\gamma(n_j)\gamma(n_{j+1})} e^{-n_i}. \quad (128)$$

Integration over μ delivers

$$\tilde{L} = \frac{1}{(s+1)^{3/2}} \int_0^1 \frac{d\tau \tau}{\sqrt{1-\tau^2}} \left(\prod_{j=1}^s \int_{n_0}^{\infty} \frac{dn_j}{\sqrt{\pi}} \right) \sum_{0 < i < j < s} \frac{2\tau^2 t(1-\tau^2)(1-\varepsilon)^2}{\gamma(n_i)\gamma(n_{i+1})\gamma(n_j)\gamma(n_{j+1})} e^{-n_i}. \quad (129)$$

Now we can put $\varepsilon = 0$ where it goes with n_j and in n_0 , again keeping all integrations and summations finite. With this, we note $\gamma(n) = \sqrt{\tau t} \sqrt{\tau + \varepsilon/t} + \dots$. As a consequence, the n_j -integrations become simple and are reduced to formula (40). Also the dependence on i disappears and the sum is $\sum_{0 < i < j < s} = s(s+1)/2$. In \tilde{L} only one integration remains,

$$\tilde{L} = \frac{s}{t(s+1)} \int_0^1 d\tau \frac{\tau \sqrt{1-\tau^2}}{(\tau + \frac{\varepsilon}{t})^2} \equiv \frac{s}{t(s+1)} h\left(\frac{t}{\varepsilon}\right). \quad (130)$$

This expression must be inserted into the energy (127). Taking into account $h(t) = \ln(2t) - 2 + \dots$ for $t \rightarrow \infty$ we can carry out the t -integration,

$$\Delta E_{\tilde{\Lambda}} = \frac{-2R}{4\pi d^2} \varepsilon \sum_{s=0}^{\infty} \frac{s}{(s+1)^3} \left(-\frac{1}{2} \ln \varepsilon - 1 - \frac{\gamma}{2} - \frac{1}{2} \ln(s+1) \right) + \dots \quad (131)$$

Finally we carry out the summation and come to

$$\Delta E_{\tilde{\Lambda}} = \frac{R}{4\pi d^2} \left[(\zeta(2) - \zeta(3))(2 + \gamma) + \zeta'(3) - \zeta'(2) + \frac{1}{4\pi} (\zeta(2) - \zeta(3)) \ln \varepsilon \right] \varepsilon. \quad (132)$$

V. CONCLUSIONS

In the forgoing section we calculated separately the different parts contributing to the correction beyond PFA in the electromagnetic case. These are the 'scalar' ones (from the first two lines in (43)),

$$\frac{\Delta E_{\text{TE}} + \Delta E_{\text{TM}}}{\varepsilon E_{\text{PFA}}} = \frac{1}{3} - \frac{5}{\pi^2} \approx -0.173, \quad (133)$$

that from parts the A, (61), and B, (76), in the Lambda-contribution and that from the $\tilde{\Lambda}$ contribution, (132),

$$\frac{\Delta E_A + \Delta E_B + \Delta E_{\tilde{\Lambda}}}{\varepsilon E_{\text{PFA}}} = \frac{180}{\pi^4} [(1 + 2 \ln 2)\zeta(3) - 2(1 + \ln 2)\zeta(2)] \approx -4.99, \quad (134)$$

and that from part C in the Lambda-contribution, Eqs. (109), (121), (124), (126),

$$\frac{\Delta E_{\tilde{C}_0} + \Delta E_{C1A} + \Delta E_{C1B} + \Delta E_{C1C}}{\varepsilon E_{\text{PFA}}} = -0.045 - 0.0044 \ln \varepsilon + 8.5 \cdot 10^{-6} (\ln \varepsilon)^2. \quad (135)$$

In (134) cancellations happened, for instance the logarithmic contributions compensated each other. This contribution could be calculated analytically, like the 'scalar' one, Eq. (133). The remaining contribution (135) could be calculated

only numerically. It contains the remaining logarithm and it is numerically small. This smallness justifies the use of the quite small precision reached in subsection C. Putting all together we come to the relative correction for the electromagnetic case,

$$\frac{\Delta E^{\text{ED}}}{\varepsilon E_{\text{PFA}}} = -5.2 - 0.0044 \ln \varepsilon + 8.5 \cdot 10^{-6} (\ln \varepsilon)^2, \quad (136)$$

from which representation (3) follows. The main contribution is a constant, the logarithmic contributions are numerically very small and do not play a role at any reasonable separation.

The result (136) is quite unexpected since it is quite large. It must be mentioned that it is not in agreement with the numerical calculations performed in [8, 9], where ~ 1.4 for the constant contribution was obtained. In these calculations the fits were made without accounting for possible logarithmic contributions. However, in view of the smallness of the coefficients in (136) this should be acceptable.

The result (136) also does not support the experimental results found in [6]. However, the experiments were done with real metals. It should be a subject of future work to account for that in an analytical calculation.

In general, it must be underlined that (136) is the second term in an asymptotic expansion. Therefore it is hard to predict how small ε must be to get a good approximation in this way. In principle, it cannot be excluded that (136) gives a good approximation only for ε smaller than that which are interesting for the experiments and which are accessible by the mentioned numerical approaches.

There is still another problem with the first correction beyond PFA for Neumann boundary conditions. Already in the easier case of a cylinder in front of a plane the numerical calculations reported in [7] showed good agreement with the analytical ones in [4] only for Dirichlet boundary conditions, but not for Neumann ones. For the latter, only with a fit including a logarithmic term in the second order, $\varepsilon^2 \ln \varepsilon$, agreement was found. For a sphere in front of a plane for a scalar field, the numerical results reported in [8] are in agreement with the analytical ones, Eq.(43), only for Dirichlet conditions on both, the sphere and the plane (DD). In the other three combinations of boundary conditions the numbers are quite different. This is quite unexpected for the case of Neumann conditions on the plane but Dirichlet conditions on the sphere (ND), since the analytical results for (DD) and (ND) are the same.

It should be mentioned that the appearance of logarithms in the expansion is probably a rather common feature. This can be seen from the structure of the corrections, see, for example, Eqn.(B13) in [4]. The expansion parameter ε is always accompanied by a factor $1/t$, producing in higher orders a singularity at $t \rightarrow 0$. It is only in the order ε considered in [4] as well as in [5] that these did not show up.

In view of the agreement of the numerical results with the analytical results in the (DD) case, the disagreement in the other three cases where we have at least on one surface Neumann conditions can be viewed as a hint that the numerical approach is more difficult once Neumann conditions are involved. It could happen that an agreement can be reached for smaller ε only. We would like to point out that also in the analytical approach the calculations with Neumann conditions are a bit more delicate. The point is in the convergence of the sum over s in (29). While for Dirichlet conditions the decrease is $\sim (s+1)^{-4}$ (first line in (42)), it is only $\sim (s+1)^{-2}$ for Neumann conditions. In the electromagnetic case it is even weaker, $\sim (s+1)^{-2} \ln(s+1)$, for example in (131).

As a consequence of the mentioned disagreement it would be interesting to improve the numerical approach. The main obstacle is that very large orbital momenta must be accounted for. A way out could consist of three steps. First, one may expand the logarithm as in Eq.(15). As we know from the analytical approach this sum is converging. To get a satisfactory precision, to take a few terms should be sufficient. In the second step one would make the substitution (28) also in the numerical approach. This allows to capture the main contribution. Finally, as third step, one would need to adopt some approach of coarsening to the orbital momentum summations or their substitution by integrals. In any case, further work is necessary in this direction.

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- [1] B. Derjaguin. Untersuchungen über die Reibung und Adhäsion, IV. Theorie des Anhaftens kleiner Teilchen. *Kolloid Z.*, 69:155, 1934.
 - [2] B.W. Harris, F. Chen, and U. Mohideen. Precision measurement of the Casimir force using gold surfaces. *Phys. Rev. A*, 62:052109, 2000.
 - [3] F. Chen, U. Mohideen, and P. W. Milonni. Limits on non-Newtonian gravity and hypothetical forces from measurements of the Casimir force. *Int. J. Mod. Phys.*, A20:2222–2231, 2005.

- [4] M. Bordag. The Casimir effect for a sphere and a cylinder in front of plane and corrections to the proximity force theorem. *Phys. Rev.*, D73:125018, 2006.
- [5] M. Bordag and V. Nikolaev. Casimir force for a sphere in front of a plane beyond proximity force approximation. *J. Phys. A: Math. Gen.*, 41:164001, 2008.
- [6] D.E. Krause, R. S. Decca, D. Lopez, and E. Fischbach. Experimental investigation of the Casimir force beyond the proximity-force approximation. *Phys. Rev. Lett.*, 98:050403, 2000.
- [7] F. C. Lombardo, F. D. Mazzitelli, and P. I. Villar. Numerical evaluation of the Casimir interaction between cylinders. *Phys. Rev.*, D78:085009, 2008.
- [8] T. Emig. Fluctuation-induced quantum interactions between compact objects and a plane mirror. *J. Stat. Mech.*, 08:P04007, 2008.
- [9] P. A. Maia Neto, A. Lambrecht, and S. Reynaud. Casimir energy between a plane and a sphere in electromagnetic vacuum. *Phys. Rev. A*, 78:012115, 2008.
- [10] Holger Gies and Klaus Klingmuller. Worldline algorithms for Casimir configurations. *Phys. Rev.*, D74:045002, 2006.
- [11] Aurel Bulgac, Piotr Magjerski, and Andreas Wirzba. Scalar Casimir effect between Dirichlet spheres or a plate and a sphere. *Phys. Rev.*, D73:025007, 2006.
- [12] M. Bordag and G.L. Klimchitskaya and U. Mohideen and V.M. Mostepanenko. *Advances in the Casimir Effect*. Oxford University Press, International Series of Monographs on Physics 145, 2009.
- [13] Kimball A. Milton, Prachi Parashar, and Jef Wagner. Exact results for Casimir interactions between dielectric bodies: The weak-coupling or van der Waals Limit. *Phys. Rev. Lett.*, 101:160402, 2008.
- [14] Kimball A. Milton, Prachi Parashar, Jef Wagner, and K. V. Shajesh. Exact Casimir energies at nonzero temperature: Validity of proximity force approximation and interaction of semitransparent spheres. 2009. arXiv: 0909.0977.
- [15] T. Emig, N. Graham, R. L. Jaffe, and M. Kardar. Casimir forces between arbitrary compact objects. *Phys. Rev. Lett.*, 99:170403, 2007.
- [16] R.C. Wittmann. Spherical Wave-Operators and the Translation Formulas. *IEEE Trans. Antennas Propagat.*, 36:1078–1087, 1988.