The Ising Model on a Heavy Gravity Portfolio Applied to Default Contagion

Master’s Thesis in Financial Mathematics

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Preface

First we would like to appreciate our supervisor Eric Järpe to guide us through this thesis. You are a very dedicated teacher with great passion and patience. You have given us suggestions and advices in every meeting which are very helpful and taught us how to approach a problem. Under your guidance, we are able to learn new fields of knowledge and solve the problem in our thesis.

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Abstract
In this paper we introduce a model of default contagion in the financial market. The structure of the companies are represented by a Heavy Gravity Portfolio, where we assume there are $N$ sectors in the market and in each sector $i$, there is one big trader and $n_i$ supply companies. The supply companies in each sector are directly influenced by the big trader and the big traders are also pairwise interacting with each other. This development of the Ising model is called Heavy gravity portfolio and according to this, the relation between expectation and correlation of the default of companies are derived by means of simulations utilising the Gibbs sampler. Finally methods for maximum likelihood estimation and for a likelihood ratio test of the interaction parameter in the model are derived.
Chapter 1

Introduction

1.1 Ising model

The Ising model was named by the German student Ernst Ising who did his first work in the early 1920s [1]. Actually this model was first suggested to Ising by his Ph.D. supervisor Wilhelm Lenz who was working in Rostock University [2]. Lenz (1920) gave a sketchy idea of this model, but he never used it to do any calculations. Then his student Ising was asked to study this model and wrote his doctoral thesis about it. Ising tried to explain observed facts about ferromagnetic materials and published his results in 1925 [6].

The Ising model is a model for the non-trivial phenomenon of phase transition in physics, where a small change of the temperature or pressure parameter will cause a larger change of the whole system. The Ising model has been applied to many fields such as chemistry, molecular biology, economics, and other areas [7]. In 1936, Ising was forced to leave Germany during the Second World War [1]. The Ising model drew attention firstly when his paper was used by Heisenberg (1928). Bethe (1935) used the Ising model applied to formation of binary alloys [7]. In 1936, Peierls named this model Ising model [8]. A big step forward was taken in 1941 when Kramers and Wannier achieved the first exact result for the Ising model in a two dimensional square lattice [9].

Recently, a new type of stochastic theory, called a Markov random fields, was introduced in probability theory. The theory of Markov random fields was found by Preston (1974) and Spitzer (1971). According to the theory of Markov models one may give a general probabilistic setting of the Ising model.
1.2 Default contagion

Credit default contagion may result in tremendous financial crisis. The sub-prime crisis in the United States can be considered as a consequence of it. A default of a debtor in the real estate market leads to a price fall of the derivatives which are based on the real estate assessments, ultimately leading to the destruction of the market confidence [10].

Companies do not work in isolation and the defaults of companies depend on each other [11]. Such dependence may be caused by macro-structural reasons. It relates the sensitivity of companies to common market factors such as the price of crude oil or the market interest rate. For portfolios that are not well diversified, such common factors may cause the positive correlated impact on the assets in the portfolios [10]. Another kind of reasons are micro-structural interdependencies between individual companies [4].

Default events seem to cluster. In 2002, the telecommunication sector accounted for 56% of all corporate bankruptcies in terms of dollar debt defaulted [12]. According to Sanjiv and Darrel [3] there are three sorts of mechanisms behind default clustering. The first is that companies are exposed to correlated market risk factors. Secondly, the default of one company may be contagious and impact other companies in its sector. Finally, learning from other company’s default events may lead to new defaults.

Our paper is structured as follows. In this chapter, the background is introduced. Chapter 2 presents our models and methods to deal with the default contagion problem. In Chapter 3, we show the theoretical results and the results of the simulations. Chapter 4 concludes and indicates the possible future work after this paper.
Chapter 2
Models and Methods

In this Chapter we will present the models and methods which we use in this paper.

2.1 Ising model

The Ising model, which is one of the most useful spin models of statistical physics, has been studied on both translational symmetric lattices and fractal lattices. We call each point a site and the line segments between the sites are called bonds. The number of sites in the lattice can be finite or infinite. Each pair of sites at unit distance from each other are connected by a bond and called neighbors.

Figure 2.1: Examples of lattices. To the left is the one-dimensional structure, in the middle is the two-dimensional structure and to the right is the three-dimensional Ising model.
Chapter 2. Models and Methods

When the dimension is equal to \( d = 1 \), the output will be a straight line. Considering a sequence, \( 0, 1, 2, \ldots, n \) of points on the line. Each site has only two neighbors. In the case of the dimension \( d = 2 \), we get the lattice of squares and therefore we consider that each site has four neighbors. If the dimension \( d = 3 \), we get the lattice of cubes. In this situation, we consider that each site has six neighbors.

Following the notation of Ising we consider a random process \( X \) which is the joint state \( \{ X_0, X_1, \ldots, X_n \} \) at the sites \( i = 0, 1, \ldots, n \) in a lattice.

The sample space of this random process is \( \Omega = (\omega_0, \omega_1, \ldots, \omega_n) = \{+, -\}^n \) where each \( \omega_i = + \) if the spin at site \( i \) is "up" and \( \omega_i = - \) if the spin at site \( i \) is "down". The random process \( X \) takes its values in \( \{-1, 1\}^n \) since \( X_i(\omega) = 1 \) when \( \omega_i = + \) and \( X_i(\omega) = -1 \) when \( \omega_i = - \). The sites \( i \) and \( j \) are called neighbors if they satisfy the property \( i \sim j \) and \( j \sim i \). If an integer lattice is considered this property may for instance be that the distance is exactly 1, i.e. \( i \sim j \) whenever \( ||i - j|| = 1 \) where \( || \cdot || \) denotes the Euclidian distance.

We now introduce the concept of a Hamiltonian of a system. In a system the Hamiltonian relates to the total energy. In the Ising model, the Hamiltonian can be defined as follows

\[
H = H(X) = -JQ(X) - hR(X),
\]

\[
Q(X) = \sum_{i \sim j} X_i X_j,
\]

\[
R(X) = \sum_i X_i.
\]

Here \( h \) and \( J \) are parameters, where \( J \) is correspond to the energy associated with neighbor interactions and \( h \) reflects the effect of external interactions. The parameter \( J \) is a real number. The case \( J > 0 \) is called the attractive case since the interaction tends to keep neighboring spins aligned the same. The case \( J < 0 \) is called repulsive case since it tends to reinforce pairs in which spins are of opposite orientation. In this paper we consider to turn off the external field which means that the parameter equals zero. Thus

\[
H = H(X) = -J \sum_{i \sim j} X_i X_j
\]

where the parameter \( J = -\frac{1}{kT} \), \( k \) is Boltzmann’s constant and \( T \) is temperature. The partition function plays a fundamental role in statistical mechanics. More precisely, the probability of \( x \) on \( \Omega \) can be written as a Gibbs
distribution
\[ P(X = x) = \frac{1}{Z} e^{JQ(x)} \]  \hspace{1cm} (2.5)

where the normalizing constant \( Z \) is a partition function defined by:
\[ Z = Z(J, N) = \sum_x e^{JQ(x)}. \]  \hspace{1cm} (2.6)

### 2.2 Heavy gravity portfolio

For our financial application, we are interested in modeling how default contagion occurs in the market. A company is more likely to default when the companies which interact it default. According to the Ising model we mentioned above, each company is represented by a site in the lattice where companies influence each other according to a neighborhood structure. But with the development of technology, now companies can contact each other very easily (e.g. internet, mobile phone, tax), then the assumption in the Ising model that companies only interact with the nearest companies (e.g. four in the two-dimension lattice) is not appropriate in real life. Therefore a new interaction model called Heavy Gravity Portfolio (HGP) (see [4]) seems
preferable. We assume that there are \( N \) sectors in the market and in each sector, \( i \in \{1, 2, \ldots, N\} \), there is one big trader \( X_i \) and \( n_i \) supply companies \( X_{i,j}, j = 1, \ldots, n_i \) which are only directly influenced by the big trader. The big traders in each sector are pairwise interacting with each other (see Fig 2.2). When the company \( i \) defaults, it takes the state \( X_i = 1 \) and not default \( X_i = -1 \) according to the Ising model. We denote by \( D \) the set of all possible states of the random process \( \{X_i, X_{i,j} : i = 1, \ldots, N, j = 1, \ldots, n_i\} \). Since there are \( N \) big traders and \( \sum_{i=1}^{N} n_i \) supply companies, the state space is \( D = \{-1, 1\}^{N+\sum_{i=1}^{N} n_i} \). Finally we denote by \( \theta^D_i \) the values of all the neighbors of companies \( \{X_j, X_{ik} : j \sim i, k = 1, 2, \ldots n_i\} \) and by \( \theta^D_{ij} \) the value of the only neighbor company \( X_i \).

### 2.3 Ising model on a heavy gravity portfolio

We introduced the probability function for the general Ising model in the equations 2.3 and 2.4, and now we consider the global probability function of a state configuration \( x \in D \)

\[
p_X(x) = \frac{1}{Z_{\alpha,\beta}} \exp\{\alpha Q_1(x) + \beta Q_2(x)\},
\]

where \( Z \) is a normalizing constant

\[
Z_{\alpha,\beta} = \sum_{x \in D} \exp\{\alpha Q_1(x) + \beta Q_2(x)\},
\]

and

\[
Q_1(x) = \sum_{i=1}^{N} \sum_{j=1}^{n_i} x_i x_{i,j},
\]

\[
Q_2(x) = \sum_{i=1}^{N} \sum_{k=i+1}^{N} x_i x_k,
\]

where \( Q_1(x) \) is the summation of all the interactions between the big traders and their suppliers in each sector, and \( Q_2(x) \) is the summation of the pairwise products of the states of all the big traders. The parameters \( \alpha \) and \( \beta \) are interaction parameters between big traders and their supply companies and between big traders respectively. A very important property of the Ising model is the Markov property; that is, given the state values of all other companies, the conditional probability of a certain company is the same as
the conditional probability of the company, given only the states of all its neighbors.

**Definition 1 (Markov Random Field)**  

$x$ is a Markov Random Field with respect to the neighborhood structure, if

$$P(X_{i,j} = x_{i,j} | X^{i,j} = x^{i,j}) = P(X_{i,j} = x_{i,j} | \theta^{ij}), \quad (2.11)$$

where

$$X^{i,j} = \{X_{k,l}, X_m : (k, l) \in A \setminus (i, j)\}, \quad (2.12)$$

and

$$A = \{(i, j) : 1 \leq i \leq N, 1 \leq j \leq n_i\}. \quad (2.13)$$

The big traders interact with all the suppliers in their sector and all other big traders, so the conditional distribution of the big traders is

$$P(X_i = x_i | X^i = x^i) = P(X_i = x_i | X_{i,j} = x_{i,j}, 1 \leq j \leq n_i, X_k = x_k, k \neq i), \quad (2.14)$$

where

$$X^i = \{X_{k,l}, X_m : 1 \leq k \leq N, 1 \leq l \leq n_k, m \neq i\}. \quad (2.15)$$

The suppliers interact directly only with the big traders in their own sector, so the conditional distribution of the suppliers is

$$P(X_{i,j} = x_{i,j} | X^{i,j} = x^{i,j}) = P(X_{i,j} = x_{i,j} | X_i = x_i). \quad (2.16)$$

**Theorem 1 (Hammersley Clifford Theorem)**  

A random process $X$ is a Markov Random Field with respect to the neighborhood structure if and only if $p_X(x)$ is a Gibbs distribution with respect to that neighborhood structure.

We know by the definition of the conditional probability distribution that

$$P(A | B) = \frac{P(A \cup B)}{P(B)}, \quad (2.17)$$
where $A$ and $B$ are events. We can use this formula to calculate the conditional distributions. The conditional distribution of $X_{i,j}$ given $X_i$ is thus

$$P(X_{i,j} = x_{i,j} | X_i = x_i) = \frac{P(X_{i,j} = x_{i,j}, X_i = x_i)}{P(X_i = x_i)} = Z_{\alpha,\beta}^{-1} \exp\{\alpha x_{i,j}x_i\},$$

where

$$Z_{\alpha,\beta} = \sum_{y \in \{-1, 1\}} \exp\{\alpha y x_i\} = \exp\{\alpha x_i\} + \exp\{-\alpha x_i\} = 2 \cosh(\alpha x_i). \ (2.19)$$

The conditional distribution of $X_i$ given the states of the rest of the field is

$$P(X_i = x_i | X^i = x^i) = \frac{P(X = x)}{P(X^i = x^i)}$$

$$= \frac{1}{Z_{\alpha,\beta}} \exp\{\alpha \sum_{j=1}^{N} \sum_{k=1}^{N_j} x_j x_{j,k} + \beta \sum_{j=1}^{N-1} \sum_{k=j+1}^{N} x_j x_k\}$$

$$= \frac{1}{Z_{\alpha,\beta}} \sum_{x_i \in \{-1, 1\}} \exp\{\alpha \sum_{j=1}^{N} \sum_{k=1}^{N_j} x_j x_{j,k} + \beta \sum_{j=1}^{N-1} \sum_{k=j+1}^{N} x_j x_k\}$$

$$= \exp\left( x_i \left( \alpha \sum_{k=1}^{N_j} x_{i,k} + \beta \sum_{k \neq i} x_k \right) \right) \cdot 2 \cosh\left( \alpha \sum_{k=1}^{N_j} x_{i,k} + \beta \sum_{k \neq i} x_k \right).$$

### 2.4 The sufficient statistic

In some cases the distribution of a random variable can be characterized by some parameter and sufficient statistics can capture all the information about the parameters from a sample of the random variable. In this case, if we do not know the exact distribution or the distribution is too complex to derive analytically, we can turn to study the sufficient statistic instead.

**Definition 2 (Sufficient statistic)** Suppose that a realization of a random variable $X$ is $x = (x_1, \ldots, x_n)$ and that the distribution of $X$ is $p_X(x)$. A statistic $S$ is said to be sufficient for the parameter $\phi$ if the conditional density $p_{X|S}(x|s;\phi)$ does not depend on $\phi$. 
The sufficient statistic $S$ reflects all the information of $\phi$ available from a sample $X_1, X_2, \ldots, X_m$ of the random process $X$. So in our model if we find the sufficient statistic $S$ and its properties, methods of statistical inference about $\phi$ can be easily determined. So we will concentrate on the study of the sufficient statistic.

**Theorem 2 (Factorization Theorem)** The statistic $S$ is sufficient for $\phi$ in the distribution $p_x(X)$ if and only if there exist functions $h(X)$ and $g(S(X), \phi)$ such that for all $\phi \in \mathbb{R}$

$$p_x(x; \phi) = h(x)g(s(x), \phi). \tag{2.20}$$

In our case, according to the Factorization Theorem, we have two sufficient statistics

$$Q_1(x) = \sum_{i=1}^{N} \sum_{j=1}^{n_i} x_{ij}, \tag{2.21}$$

$$Q_2(x) = \sum_{i=1}^{N-1} \sum_{k=i+1}^{N} x_{i,k}, \tag{2.22}$$

where $Q_1$ is a sufficient statistic for $\alpha$ and $Q_2$ sufficient for $\beta$.

**Theorem 3 (Central Limit Theorem)** Suppose $\{X_i : i = 1, 2, \ldots, n\}$ is a sequence of independent and identically distributed variables with $E(X_i) = \mu$ and $Var(X_i) = \sigma^2$, then as $n$ approaches infinity,

$$\sqrt{n} \left( \frac{1}{n} \sum_{i=1}^{n} X_i \right) \rightarrow N(0, \sigma^2). \tag{2.23}$$

As we see, the distributions of $Q_1$ and $Q_2$ seem too complicated for us to calculate the expectation and variance in the later simulation. According to a corresponding Central Limit Theorem for the Ising model (see [14]), when the sample is large enough we can approximately consider $Q_1$ as normally distributed which has been easily proved though $X_i$ and $X_{i,j}$ are dependent, and $Q_2$ as Chi-Square distributed.
2.5 Gibbs sampling

The Gibbs sampler is a spatial simulation tool of Markov Chain Monte Carlo (MCMC) techniques which provides a way to approximate the joint distribution function. It is an algorithm for generating sequences of samples from joint distributions. The main thought behind the Gibbs sampler is that we may only consider the conditional distribution with one variable rather than considering the much more complicated joint distribution (see [5]).

Given a state configuration of our big traders \( \{x_1, x_2, \ldots, x_N\} \) and the initial value \( x(0) \) at time 0, and a time sequence \( t = 0, 1, 2, \ldots \), the sampling starts with some initial values \( x_2(0), \ldots x_N(0) \) rendering the value \( x_1(1) \) according to the conditional distribution \( p(x_1|x_2, \ldots, x_N) \). In this way we can obtain the value \( x_2(1) \) from the value \( x_1(1), x_3(0) \ldots x_N(0) \). The sampler works as follows:

\[
\begin{align*}
  x_1(1) &\sim p(x_1|x_2(0), x_3(0), \ldots, x_N(0)), \\
  x_2(1) &\sim p(x_2|x_1(1), x_3(0), \ldots, x_N(0)), \\
  & \quad \ldots \ldots \ldots \\
  x_N(1) &\sim p(x_N|x_1(1), x_2(1), \ldots, x_{N-1}(1)).
\end{align*}
\]

As mentioned before, we only have two states: 1 and -1. The sampler updates the values of the companies according to the following condition:

\[
\begin{align*}
  x_i(t) &= 1 \quad \text{if} \quad p(x_i = 1|x^t) > u_i(t), \\
  x_i(t) &= -1 \quad \text{otherwise},
\end{align*}
\]

where \( \{u_i(t), i = 1, 2, \ldots N, t = 0, 1, 2 \ldots\} \) is a sequence of independent random variables generated from the uniform distribution on the interval \([0, 1]\).

Then we can get the samples of the big traders. In the same way we also can obtain the samples of the suppliers \( \{x_{ij}, i = 1, 2, \ldots N, j = 1, 2, \ldots n_i\} \).

The program code is given in Appendix B.
Chapter 3

Results

In this chapter the theoretical and simulation results are presented.

3.1 Theoretical results

**Theorem 4** $Q_1(x)$ is sufficient for $\alpha$, $Q_2(x)$ is sufficient for $\beta$, and the expectations of $Q_1(x)$ and $Q_2(x)$ are

\[
E(Q_1(x)) = \frac{d}{d\alpha} \ln Z, \quad (3.1)
\]

\[
E(Q_2(x)) = \frac{d}{d\beta} \ln Z. \quad (3.2)
\]

and the variances of $Q_1(x)$ and $Q_2(x)$ are

\[
Var(Q_1(x)) = \frac{d^2}{d^2\alpha} \ln Z - \left( \frac{d}{d\alpha} \ln Z \right)^2, \quad (3.3)
\]

\[
Var(Q_2(x)) = \frac{d^2}{d^2\beta} \ln Z - \left( \frac{d}{d\beta} \ln Z \right)^2. \quad (3.4)
\]

The expected value of $Q_1(x)$ is given by

\[
E(Q_1) = \frac{\sum_x Q_1 \exp(\alpha Q_1 + \beta Q_2)}{\sum_x \exp(\alpha Q_1 + \beta Q_2)}, \quad (3.5)
\]
Figure 3.1: The expected value of $S_1$ for $\alpha = \beta$ on the interval $[-10, 10]$, the black line corresponds to the case $n_1 = n_2 = 2$, the black dashed corresponds to $n_1 = n_2 = n_3 = 2$, the red line corresponds to $n_1 = n_2 = n_3 = n_4 = 2$ and the blue line corresponds to $n_1 = n_2 = n_3 = n_4 = n_5 = 2$.

Figure 3.2: The expected value of $S_2$ for $\alpha = \beta$ on the interval $[-10, 10]$, the black line corresponds to the case $n_1 = n_2 = 2$, the black dashed corresponds to $n_1 = n_2 = n_3 = 2$, the red line corresponds to $n_1 = n_2 = n_3 = n_4 = 2$ and the blue line corresponds to $n_1 = n_2 = n_3 = n_4 = n_5 = 2$.

The expected value of $Q_2(x)$ is given by

$$E(Q_2) = \frac{\sum_x Q_2 \exp(\alpha Q_1 + \beta Q_2)}{\sum_x \exp(\alpha Q_1 + \beta Q_2)}, \quad (3.6)$$
Figure 3.3: The expected value of $S_1$ for $\alpha = \beta$ on the interval $[-10, 10]$, the black line corresponds to the case $n_1 = n_2 = 2$, the black dashed corresponds to $n_1 = n_2 = n_3 = 2$, the red line corresponds to $n_1 = n_2 = n_3 = n_4 = 2$ and the blue line corresponds to $n_1 = n_2 = n_3 = n_4 = n_5 = 2$.

Figure 3.4: The variance of $S_1$ for $\alpha = \beta$ on the interval $[-10, 10]$, the black line corresponds to the case $n_1 = n_2 = 2$, the black dashed corresponds to $n_1 = n_2 = n_3 = 2$, the red line corresponds to $n_1 = n_2 = n_3 = n_4 = 2$ and the blue line corresponds to $n_1 = n_2 = n_3 = n_4 = n_5 = 2$.

the variance of $Q_1(x)$ is given by

$$Var(Q_1) = \frac{\sum_x Q_1^2 \exp(\alpha Q_1 + \beta Q_2)}{\sum_x \exp(\alpha Q_1 + \beta Q_2)} - \left( \frac{\sum_x Q_2 \exp(\alpha Q_1 + \beta Q_2)}{\sum_x \exp(\alpha Q_1 + \beta Q_2)} \right)^2,$$  

(3.7)
Figure 3.5: The variance of $S_2$ for $\alpha = \beta$ on the interval $[-10, 10]$, the black line corresponds to the case $n_1 = n_2 = 2$, the black dashed corresponds to $n_1 = n_2 = n_3 = 2$, the red line corresponds to $n_1 = n_2 = n_3 = n_4 = 2$ and the blue line corresponds to $n_1 = n_2 = n_3 = n_4 = n_5 = 2$.

The variance of $Q_2(x)$ is given by

$$\text{Var}(Q_2) = \frac{\sum_x Q_2^2 \exp(\alpha Q_1 + \beta Q_2)}{\sum_x \exp(\alpha Q_1 + \beta Q_2)} \left( \frac{\sum_x Q_2 \exp(\alpha Q_1 + \beta Q_2)}{\sum_x \exp(\alpha Q_1 + \beta Q_2)} \right)^2. \quad (3.8)$$

In order to make our graph more comparable, we normalize $Q_1$ and $Q_2$ as $S_1$ and $S_2$ respectively

$$S_1(x) = \frac{1}{\sum_{i=1}^N n_i} Q_1(x), \quad (3.9)$$

$$S_2(x) = \frac{2}{N(N-1)} Q_2(x). \quad (3.10)$$

The distributions of $S_1$ and $S_2$ have been simulated by using the software R. In the simulations of the distributions, we consider $\alpha = \beta = 0.05$, the number of big traders $N = 20$ and 1000 samples. Figures 3.6, 3.7 indicate that when the sample size is large, $S_1$ is approximately normally distributed and $S_2$ is approximate Chi-Square distribution. According to the Central Limit Theorem, one may deduce that $S_1$ is approximately normally distributed. For $S_2$, we can prove that it tends to the Chi-Square distribution by the
following argument

$$\left( \sum_{i=1}^{N} X_i \right)^2 = (X_1^2 + X_2^2 + \ldots + X_N^2) + 2(X_1X_2 + \ldots X_{N-1}X_N)$$

$$= \sum_{i=1}^{N} X_i^2 + 2 \sum_{i=1}^{N-1} \sum_{k=i+1}^{N} X_iX_k,$$
where $X_i^2$ is always 1 since the value of $X_i$ is 1 or −1. thus

$$\sum_{i=1}^{N-1} \sum_{k=i+1}^{N} X_i X_k = \frac{\left( \sum_{i=1}^{N} X_i \right)^2 - N}{2}.$$  

According to the Central Limit Theorem, $\sum_i X_i$ tends to the normal distribution, consequently $\left( \sum_{i=1}^{N} X_i \right)^2$ is approximately the Chi-square distribution. As the result $Q_2 = \sum_{i=1}^{N-1} \sum_{k=i+1}^{N} X_i X_k$ is approximately Chi-Square distributed with one degree of freedom. $S_2$ then obviously tends to the Chi-Square distribution.

We have also made the exact plots of the expectation and variance of $S_1$ and $S_2$ using the software R. Because the number of terms grows exponentially as the number of companies grows, we only can calculate when $N = 3, 4, 5$ and $\alpha = \beta$ on the interval $[-10, 10]$.

Figure 3.8: The expected value of $S_1$ for the cases: black line corresponds to the case $n_1 = n_2 = n_3 = 2$, red dashed corresponds to $n_1 = n_2 = 3$. In the left plot $\alpha = 2\beta$, in the middle $\alpha = \beta$, in the right $2\alpha = \beta$.

Figure 3.1 and Figure 3.2 show the expected values of the normalized sufficient statistics $S_1$ and $S_2$ and Figure 3.4 and Figure 3.5 show the variances of $S_1$ and $S_2$. We calculated the moments for the cases with $N = 3$ and $n_1 = n_3 = 2, n_2 = 3$, with $N = 4$ and $n_1 = n_2 = n_3 = 2, n_4 = 1$ and with $N = 5$ and $n_1 = n_5 = 2, n_2 = n_3 = n_4 = 1$. The figures indicate that when the number of companies grows, the expectation and variance curves tend to be closer.
Figure 3.9: The variance of $S_1$ for the cases: black line corresponds to the case $n_1 = n_2 = n_3 = 2$, red dashed corresponds to $n_1 = n_2 = 3$. In the left plot $\alpha = 2\beta$, in the middle $\alpha = \beta$, in the right $2\alpha = \beta$.

Figure 3.10: The expected value of $S_1$ for the cases: black line corresponds to the case $n_1 = n_2 = n_3 = 2$, red dashed corresponds to $n_1 = n_2 = 3$. In the left plot $\alpha = 2\beta$, in the middle $\alpha = \beta$, in the right $2\alpha = \beta$.

Figure 3.8 and Figure 3.9 show that when the number of suppliers are the same, the expected values and variances of $S_1$ are not a function of the number of the big traders when the total number of suppliers is constant. And Figure 3.10 and Figure 3.11 indicate when the number of big traders is constant, then the expected values and variances of $S_2$ are not a function of the number of the suppliers.
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Figure 3.11: The variance of $S_2$ for the cases: black line corresponds to the case $n_1 = n_2 = n_3 = 2$, red dashed corresponds to $n_1 = n_2 = 3$. In the left plot $\alpha = 2\beta$, in the middle $\alpha = \beta$, in the right $2\alpha = \beta$.

3.2 The maximum likelihood estimation

The maximum likelihood estimation is a method for estimating the parameters in statistical models. Our likelihood estimation of $\alpha$ and $\beta$ is to maximize the joint probability function $p_X(x; \alpha, \beta)$ with respect to $\alpha$ and $\beta$, where $x$ is a sample of the random process $X$. To simplify the calculation, we maximize the log of the joint probability. The log-likelihood function is

$$L(\alpha, \beta) = \ln \prod_{k=1}^{m} \frac{\exp(\alpha S_1(x_k) + \beta S_2(x_k))}{Z_{\alpha,\beta}}$$

$$= \alpha \sum_{k=1}^{m} S_1(x_k) + \beta \sum_{k=1}^{m} S_2(x_k) - n \ln Z_{\alpha,\beta},$$

therefore

$$\frac{\partial L(\alpha, \beta)}{\partial \alpha} = \sum_{k=1}^{m} S_1(x_k) - nE_{\alpha,\beta}(S_1(X)) = 0,$$

$$\frac{\partial L(\alpha, \beta)}{\partial \beta} = \sum_{k=1}^{m} S_2(x_k) - nE_{\alpha,\beta}(S_2(X)) = 0.$$
no difference in the figures because the expected value and variance of $S_1$ change very little when the value of $\beta$ changes, also the expected value and variance of $S_2$ change very little when the value of $\alpha$ changes.

**Example 1** Let us consider a sample with three big traders and two suppliers in each sector, the sample being

$$
\begin{pmatrix}
1 & 1 & 1 \\
-1 & 1 & 1 \\
1 & 1 & 1
\end{pmatrix},
$$

where the values in the first columns of the matrices are the values of the big traders and others are the values of suppliers. Then we obtain $S_1 = 0.3333$ and $S_2 = -0.3333$. And we can also calculate the exact values of $E(S_1), V(S_1), E(S_2)$ and $V(S_2)$. Some values are listed in Table 3.1.

| $(\alpha, \beta)$ | $E(S_1)$ | $V(S_1)$ | $E(S_2)$ | $V(S_2)$ | $|E(S_1) - s_1|$ | $|E(S_2) - s_2|$ |
|-------------------|----------|----------|----------|----------|----------------|----------------|
| (2, -10)          | 0.3215   | 0.2528   | -0.3333  | 0.1116   | 0.0085         | 0              |
| (-10, -10)        | -0.9311  | 0.8890   | -0.3333  | 0.1116   | 1.2644         | 0              |
| (-5, -5)          | -0.6822  | 0.5546   | -0.3327  | 0.1115   | 1.0155         | 0.0006         |
| (0, 0)            | 0        | 0        | 0        | 0        | 0.3333         | 0.3333         |
| (10, -2)          | 0.9311   | 0.8890   | -0.3030  | 0.1312   | 0.5978         | 0.0303         |

Table 3.1: The values of $(\alpha, \beta)$ which correspond to the expected values and the value of $S_1$ and $S_2$.

*From Table 3.1, we can see that when the values of $(\alpha, \beta)$ are $(2, -10)$,*

$$
S_1(x) = E_{\alpha,\beta}(S_1(X)),
$$

$$
S_2(x) = E_{\alpha,\beta}(S_2(X)),
$$

*the two equations are satisfied. So $\hat{\alpha} = 2, \hat{\beta} = -10$ are the maximum likelihood estimations for this sample.*

Figure 3.12: The expected value of $S_1$ when $\beta = 0.1, 0.5, 1, 5$ respectively.
Chapter 3. Results

3.3 Hypothesis test

Examples based on simulations

Hypothesis testing is a methodology in theoretical statistics to examine statistical material following a probability distribution, which is a function of some parameters, for evidence for a given statement about one of these parameters. The pre-determined probability under which the result is unlikely to occur is called the significance level. Usually there are two hypothesis, the null hypothesis $H_0$ and the alternative hypothesis $H_1$. The result from the samples are compared with the known result under the significance level and if they are not correspondent with each other, the null hypothesis is rejected.

Example 2 Our first test is to test whether big traders are dependent with the suppliers in their own sectors. Using hypothesis test, we set null hypothesis $H_0 : \alpha = 0$, alternative hypothesis $H_1 : \alpha \neq 0$. We randomly take three samples with three big traders and two suppliers in each sector, the samples are as following

$$\begin{pmatrix} -1 & -1 & 1 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ -1 & -1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & -1 \end{pmatrix},$$
where the values in the first columns of the matrices are the values of the big traders and others are the values of suppliers. Then we obtain the average value of the samples $S_1 = 0.3333$. Similar to the procedure in Section 3.1, we get the distribution by large samples and approximately consider $S_1$ as normally distributed $N(0.003333, 0.079179)$ when $\alpha = 0, \beta = 1$. Therefore we can calculate the Z-score

$$Z = \frac{S_1 - E_{\alpha,\beta}(S_1)}{\sqrt{\frac{\text{Var}_{\alpha,\beta}(S_1)}{n}}} = 2.843642,$$

which is bigger than

$$Z_{0.05} = 1.96,$$

where 0.05 is the significance level. So when $\beta = 1$ the null hypothesis $H_0$ is rejected in favor of $H_1$. For any other values of $\beta$ like 0.1 and 2, we get the same results.

Similarly we also illustrate a hypothesis test of whether the dependence between the big traders and the suppliers is positive and one test of whether the dependence is negative. For the former, $H_0 : \alpha = 0$ and $H_1 : \alpha > 0$

$$Z = 2.843642 > Z_{0.05} = 1.64,$$

which means that $H_0$ is rejected in favor of $H_1$. For the latter, $H_0 : \alpha = 0$ and $H_1 : \alpha < 0$

$$Z = 2.843642 > -Z_{0.05} = -1.64,$$

implying that $H_0$ is accepted in favor of $H_1$.

For testing whether the big traders are positive dependent with each other, we have the null hypothesis $H_0 : \beta = 0$ and the alternative hypothesis $H_1 : \beta > 0$. From Section 3.1, we know that the $S_2$ value of the $k$th sample $x_k$ is

$$S_2^k = \frac{2}{N(N-1)} Q_2(x_k) = \frac{2}{N(N-1)} \left( \sum_{i=1}^{N} x_i(k) \right)^2 - N,$$

so we can obtain

$$u_k = \frac{1}{\sigma^2} (N - 1) S_2^k + N) = \left( \frac{\sum_{i=1}^{N} x_i(k)}{\sigma} \right)^2,$$

and thus we may construct

$$U = \frac{1}{K} \sum_{k=1}^{K} u_k.$$
which tends to the chi square distribution $\chi^2(K)$ with $K$ degrees of freedom. 1000 samples are used and the variance of $\sum_{i=1}^{N} x_i$ is equal to 0.8782, $U = 8.991369$ is bigger than $\chi^2_{0.05}(1000) = 3.841$, consequently $H_0$ is rejected in favor of $H_1$ at the level 0.05 of significance.
Chapter 4

Discussion

In this chapter the theoretical and simulation results are discussed and the work which we can do in the future is presented.

4.1 Conclusions

We can see from the plots in Section 3.1 that the expectations and variances of $S_1$ and $S_2$ are increasing when the values of the interacting parameters $\alpha$ and $\beta$ increase. This can be explained by the fact that when the companies interact with each other strongly enough, the values of the companies tend to be the same with their neighbors. This means if there is strong interaction in the market, the default of one company can influence its neighbors to default and the non-default situation of one company also influences its neighbors to have good situations. It is very logical that if the big trader in one sector defaults and can not pay the money it owes, all the supply companies which are strongly dependent on the big trader will default too. And also among the big traders which have business relations with each other, the default of one company will influence others to default.

We can also see from Figure 3.4 and Figure 3.5 that the more companies in the market, the steeper is the peak of the variance. The reason for this can be that the less neighbors one company has, the easier it can decide which value it will be. For instance, if one company has two neighbors in which one defaults and the other does not, it will equally likely default or non-default. If it only has one neighbor, the default of the neighbor will most probably cause it to default.

Figure 3.8 and Figure 3.9 show that when the number of suppliers are
the same, the expected values and variances of $S_1$ are independent with the number of the big traders. And Figure 3.10 and Figure 3.11 indicate when the number of big traders are constant, the expected values and variances of $S_2$ are independent with the number of the suppliers. The estimation in Section 3.2 shows when given a sample, the maximum likelihood estimator of $\alpha$ and $\beta$ can be found. In Section 3.3, the results of the hypothesis tests indicate that there are some dependence between the big traders and the suppliers in the same sectors as well as among the big traders at the level 0.05 of significance.

4.2 Future work

As the coming work in the near future, the test of our models with real data is required. To do the test, we need the information of defaults among companies and also the pairwise interactions between all the companies. We are not able to get this information right now because these data of the default is not free and the interaction among these companies are usually private and not visible. One way to get this data is to access the Statistics Sweden (SCB), We have tried several times but without any results.

Instead of the Ising model, the Potts model with three states can also be interesting. And we can also take the external field into consideration.
Notation

\[ P(A) \quad \text{Probability of the event } A \]
\[ E(X) \quad \text{Expectation of the random variable } X. \]
\[ Var(X) \quad \text{Variance of the random variable } X. \]
\[ P(A|B) \quad \text{Conditional probability of the event } A \]
\[ \quad \text{given the event } B. \]
Bibliography


*Fifty Years of the Exact Solution of the Two-Dimensional Ising Model by Onsager*. Current Science, 69 (10), 821 – 861.


*Markov Random Fields and Their Application*. Contemporary Mathematical V.1, American Mathematical Society.


An Introduction to Credit Default Contagion. Financial forum, 4, 92 – 94.

Modelling Bonds and Credit Default Swaps using a Structural Model with Contagion. 60 (3), 252 – 262.


Chapter 5

Appendix A

Theorems with proof

Theorem 5  $S_1$ is a sufficient of $\alpha$, $S_2$ is a sufficient of $\beta$, here the expectation of $S_1$ and $S_2$ are as follow formula

\[ E(S_1) = \frac{d}{d\alpha} \ln Z \quad (5.1) \]
\[ E(S_2) = \frac{d}{d\beta} \ln Z \quad (5.2) \]

Proof:

\[
E(S_1) = \sum_s s P(S_1 = s) = \sum_s \sum_{x,c:S_1(x,c)=s} \frac{1}{Z} \exp(\alpha S_1 + \beta S_2)
\]
\[
= \frac{1}{Z} \sum_s \sum_{x,c:S_1=s} \exp(\alpha s + \beta S_2)
\]
\[
= \frac{1}{Z} \sum_s \sum_{x,c:S_1=s} s \exp(\alpha s + \beta S_2)
\]
\[
= \frac{1}{Z} \sum_s \sum_{x,c:S_1=s} \frac{d}{d\alpha} \exp(\alpha s + \beta S_2)
\]
\[
= \frac{1}{Z} \frac{d}{d\alpha} \sum_s \sum_{x,c:S_1=s} \exp(\alpha s + \beta S_2)
\]
\[
= \frac{d}{d\alpha} \sum_{x,c:S_1=s} \exp(\alpha s + \beta S_2)
\]
\[
= \frac{d}{d\alpha} \sum_{x,c:S_1=s} \frac{\exp(\alpha S_1 + \beta S_2)}{Z}
\]
\[
= \frac{d}{d\alpha} \sum_{x,c:S_1=s} \frac{\exp(\alpha S_1 + \beta S_2)}{Z}
\]
\[ \frac{d}{d\alpha} \frac{Z}{Z} = \frac{d}{d\alpha} \ln Z \]

**Theorem 6** \( S_1 \) is a sufficient of \( \alpha \), \( S_2 \) is a sufficient of \( \beta \), here the variance of \( S_1 \) and \( S_2 \) are as follow formula

\[
V(S_1) = \frac{d^2}{d^2\alpha} \ln Z - \left( \frac{d}{d\alpha} \ln Z \right)^2 \tag{5.3}
\]

**Proof:**

\[
V(S_1) = E(s_1^2) - E(s_1)^2
\]

\[
E(s_1^2) = \sum_s s P(S_1 = s)
\]

\[
= \frac{1}{Z} \sum_s s^2 \sum_{x,c:S_1 = s} \exp(\alpha s + \beta S_2)
\]

\[
= \frac{1}{Z} \sum_s s \sum_{x,c:S_1 = s} s^2 \exp(\alpha s + \beta S_2)
\]

\[
= \frac{1}{Z} \sum_s \sum_{x,c:S_1 = s} \frac{d^2}{d^2\alpha} \exp(\alpha s + \beta S_2)
\]

\[
= \frac{1}{Z} \frac{d^2}{d^2\alpha} \sum_s \sum_{x,c:S_1 = s} \exp(\alpha s + \beta S_2)
\]

\[
= \frac{\frac{d^2}{d^2\alpha} \sum_s \sum_{x,c:S_1 = s} \exp(\alpha s_1 + \beta S_2)}{Z}
\]

\[
= \frac{\frac{d^2}{d^2\alpha} \sum_{x,c:S_1 = s} \exp(\alpha s_1 + \beta S_2)}{Z}
\]

\[
= \frac{d^2}{d^2\alpha} \frac{Z}{Z}
\]

\[
= \frac{d^2}{d^2\alpha} \ln Z
\]

So, we can get the formula

\[
V(S_1) = \frac{d^2}{d^2\alpha} \ln Z - \left( \frac{d}{d\alpha} \ln Z \right)^2
\]

30
\[ p(X_i = x_i | X^i = x^i) = \frac{p(X)}{p(X^i = x^i)} \]

\[ = \frac{\frac{1}{Z_{\alpha,\beta}} \exp\{\alpha \sum_{i=1}^{N} \sum_{j=1}^{n_i} x_i x_{i,j} + \beta \sum_{i=1}^{N-1} \sum_{k=i+1}^{N} x_i x_k\}}{\sum_{x \subseteq \{-1,1\}} \exp\{\alpha \sum_{i=1}^{N} \sum_{j=1}^{n_i} x_i x_{i,j} + \beta \sum_{i=1}^{N-1} \sum_{k=i+1}^{N} x_i x_k\}} \]

and also

\[ p(X_{i,j} = x_{i,j} | X_i = x_i) = \frac{p(X)}{p(X_i = x_i)} \]
Chapter 6

Appendix B

-------The test when 2*alpha=beta-------

h <- 2000
alpha <- 0.01*(1:h-1000)
beta <- 0.01*(1:h-1000)
oes1 <- 0
oes2 <- 0
ovs1 <- 0
ovs2 <- 0
numofsites <- c(3,3)
y1 <- 0
for(i in 1:h) {
  y1 <- main(alpha[i],alpha[i]/2,numofsites) %the first sample of the test
  es1[i] <- y1[1]
  es2[i] <- y1[3]
  vs1[i] <- y1[2]-y1[1]^2
  vs2[i] <- y1[4]-y1[3]^2
  y1 <- 0
}

ses1 <- 0
ses2 <- 0
svs1 <- 0
svs2 <- 0
numofsites <- c(2,2,2) %the second sample of the test
y2 <- 0
for(i in 1:h) {
  y2 <- main(alpha[i],alpha[i]/2,numofsites)
ses1[i] <- y2[1]
ses2[i] <- y2[3]
y2 <- 0
}

------the exact value of the expectation and variance of s1 and s2------

---decimal to binary---
dec2bin <- function(m,N){
    answ <- 0*1:N
temp <- m
    for(i in 1:N){
        step <- N-i
        if(temp>=2^step){
            answ[i] <- 1
            temp <- temp-2^step
        }
    }
    return(answ)
}

---count s1 and s2---
s1count <- function(x,numofsites){
    bigc <- length(numofsites)
    sm <- sum(numofsites)
    accm <- 0
    m <- 0
    answ <- 0
    for(i in 1:bigc){
        accm <- accm+m
        m <- numofsites[i]
        for(j in 1:m)
            answ <- answ+x[i]*x[bigc+accm+j]
    }
    return(answ/sm)
}

s2count <- function(x,numofsites){
    bigc <- length(numofsites)
answ <- 0
for(i in 1:(bigc-1)){
    for(j in (i+1):bigc)
        answ <- answ+x[i]*x[j]
}
answ <- answ*2/(bigc*(bigc-1))
return(answ)
}

main <- function(alpha,beta,numofsites){
    numofconf <- sum(numofsites) + length(numofsites)  %number of configurations
    n <- 2^numofconf
    neib <- n*(n-1)/2  %number of neighbors
    Z <- 0
    A <- 0
    B <- 0
    A2 <- 0
    B2 <- 0
    temp <- 0
    for(K in 1:n){
        x <- 2*dec2bin((K-1),numofconf)-1
        s1 <- s1count(x,numofsites)
        s2 <- s2count(x,numofsites)
        temp <- exp(alpha*s1+beta*s2)
        Z <- Z+temp
        A <- A+s1*temp  %the expectation of s1
        B <- B+s2*temp  %the expectation of s2
        A2 <- A2+(s1^2)*temp  %the variance of s1
        B2 <- B2+(s2^2)*temp  %the variance of s2
    }
    return(answ)
}

h <- 2000
---set the scale of alpha and beta---
alpha <- 0.01*(1:h-1000)
beta <- 0.01*(1:h-1000)
oses1 <- 0
oses2 <- 0
ovs1 <- 0
ovs2 <- 0
---the first sample of the test---
numofsites <- c(2,2)
y1 <- 0
for(i in 1:h) {
y1 <- main(alpha[i],beta[i],numofsites)
oses1[i] <- y1[1]
oses2[i] <- y1[3]
ovs1[i] <- y1[2]-y1[1]^2
y1 <- 0
}

% the second sample of test
ses1 <- 0
ses2 <- 0
svs1 <- 0
svs2 <- 0
numofsites <- c(2,2,2)
y2 <- 0
for(i in 1:h) {
y2 <- main(alpha[i],beta[i],numofsites)
ses1[i] <- y2[1]
ses2[i] <- y2[3]
y2 <- 0
}

% the third sample of test
tes1 <- 0
tes2 <- 0
tvs1 <- 0
tvs2 <- 0
numofsites <- c(2,2,2,2)
y3 <- 0
for(i in 1:h) {
    y3 <- main(alpha[i],beta[i],numofsites)
    tes1[i] <- y3[1]
    tes2[i] <- y3[3]
    tvs1[i] <- y3[2]-y3[1]^2
    y3 <- 0
}

% the fourth sample of test
fes1 <- 0
fes2 <- 0
fvs1 <- 0
fvs2 <- 0
numofsites <- c(2,2,2,2)
y4 <- 0
for(i in 1:h) {
    y4 <- main(alpha[i],beta[i],numofsites)
    fes1[i] <- y4[1]
    fes2[i] <- y4[3]
    fvs1[i] <- y4[2]-y4[1]^2
    y4 <- 0
}

% the fifth sample of test
wes1 <- 0
wes2 <- 0
wvs1 <- 0
wvs2 <- 0
numofsites <- c(2,2,2,2,2,2)
y5 <- 0
for(i in 1:h) {
    y5 <- main(alpha[i],beta[i],numofsites)
    wes1[i] <- y5[1]
    wes2[i] <- y5[3]
    wvs1[i] <- y5[2]-y5[1]^2
    wvs2[i] <- y5[4]-y5[3]^2
    y5 <- 0
}
% the sixth sample of test
ses1 <- 0
ses2 <- 0
svs1 <- 0
svs2 <- 0
numofsites <- c(2,2,2,2)
y6 <- 0
for(i in 1:h) {
  y6 <- main(alpha[i],beta[i],numofsites)
  ses1[i] <- y6[1]
  ses2[i] <- y6[3]
  svs1[i] <- y6[2]-y6[1]^2
  svs2[i] <- y6[4]-y6[3]^2
  y6 <- 0
}
% show the result of different sample
plot(alpha,oes1,type="l",xlab="alpha",ylab="E[S1]")
lines(alpha,ses1,lty=2)
lines(alpha,tes1,lty=3)
lines(alpha,ses1,lty=1,col="red")
lines(alpha,tes1,lty=2,col="blue")
lines(alpha,tes1,lty=3,col="green")

plot(alpha,oes2,type="l",xlab="alpha",ylab="E[S2]")
lines(alpha,ses2,lty=2)
lines(alpha,ses2,lty=3)

plot(alpha,ovs1,type="l",xlab="alpha",ylab="V[S1]")
lines(alpha,svs1,lty=2)
lines(alpha,tvs1,lty=3)

plot(alpha,ovs2,type="l",xlab="alpha",ylab="V[S2]")
lines(alpha,tvs2,lty=2)
lines(alpha,svs2,lty=3)

------Get the density of s1 and s2------

funcs <- function(n,alpha,beta){
  m <- max(n)
N <- length(n)
x <- matrix(0,N,m)
p <- matrix(0,N,m)
u <- runif(m*N)
sm <- sum(n)
tick <- 1
---the initial random sample---
for(i in 1:N){
  for(j in 1:m){
    if(j<=n[i]){%
      x[i,j] <- 2*(u[tick]>0.5)-1
      tick <- tick+1
    }
  }
}
a <- 0
b <- 0
for(i in 1:N){
  a <- x[i,1]+a
}
for(i in 1:N){
  for(j in 2:m){%
    if(x[i,j] == 0)
      x[i,j] <- 0
    else{
      b <- b+x[i,j]
    }
  }
}
---the probability of suppliers---
p[i,j] <- exp(alpha*x[i,1])/(2*cosh(alpha*x[i,1]))
u <- runif(m*N)
if(p[i,j]>u)
  x[i,j] <- 1
else
  x[i,j] <- -1
}
---the probabilty of big traders---
p[i,1] <- exp(alpha*a+beta*b)/(2*cosh(alpha*a+beta*b))
b <- 0
a <- 0
u <- runif(m*N)
if(p[i,1]>u)
x[i,1] <- 1
else
  x[i,1] <- -1
for(k in 1:N){
a <- x[k,1]+a
}

s1 <- 0
s2 <- 0
% sum of s1 and s2
for(i in 1:N){
  for(j in 2:m){
    s1 <- s1+x[i,1]*x[i,j]
  }
}
for(i in 1:(N-1)){
  for(j in (i+1):N){
    s2 <- s2+x[i,1]*x[j,1]
  }
}
---normalize s1 and s2---
s1 <- s1/sm
s2 <- s2*2/(N*(N-1))
s12 <- c(s1,s2)
return(s12)

---simulating 1000 times---
M <- 1000
% set the initial value of alpha and beta
alpha <- 0.01
beta <- 0.01
% the sample of test
n <- c(4,5,7,4,2,5,3,4,4,6,5,4,3,2,3,2,3,6,7,5)
y <- 0
sums1 <- 0
s1pic <- 0
s2pic <- 0
for(i in 1:M) {
  y <- funcs(n,alpha,beta)
s1pic[i] <- y[1]
sums1 <- sums1+y[1]
s2pic[i] <- y[2]
y <- 0
}
sums1 <- sums1/M
vars1 <- 0
for(k in 1:M){
  vars1 <- vars1+(s1pic[i]-sums1)^2
}
vars1 <- vars1/(M-1)
%show the result of distribution
xs1 <- 0.01*(-50):50
ys1 <- dnorm(xs1, mean = sums1, sd = sqrt(vars1))
hist(s1pic)
lines(xs1,100*ys1)