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Stable Numerical Methods for PDE Models of Asian Options

Master's Thesis in Financial Mathematics

Adam Rehurek

MASTER'S PROGRAMME IN FINANCIAL MATHEMATICS

School of Information Science, Computer and Electrical Engineering
Halmstad University

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Supervisor: Prof. Dr. Matthias Ehrhardt
Examiner: Prof. Ljudmila A. Bordag
External referees: Prof. Mikhail Babich

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Department of Mathematics, Physics and Electrical Engineering
School of Information Science, Computer and Electrical Engineering
Halmstad University

Preface

The area of path-dependent financial derivatives is in last years widely discussed. Via solutions of the respective PDEs, we discuss in depth the numerical pricing of Asian option contracts which payoff depends on average value of the underlying asset, since after the recent economic crisis their property of low volatile payoff seems to be reasonably attractive.

My thanks go to Prof. Dr. Matthias Ehrhardt for his helpfulness, guidance and support I was provided while writing this thesis. With appreciation I acknowledge many inspiring conversations with students and professors on both Halmstad University, Halmstad, Sweden and Comenius University, Bratislava, Slovakia, especially with Michal Takac. Last but certainly not least, I am grateful to my family for endless encouragement and support, I was given throughout my studies.

Abstract

Asian options are exotic financial derivative products which price must be calculated by numerical evaluation. In this thesis, we study certain ways of solving partial differential equations, which are associated with these derivatives. Since standard numerical techniques for Asian options are often incorrect and impractical, we discuss their variations, which are efficiently applicable for handling frequent numerical instabilities reflected in form of oscillatory solutions. We will show that this crucial problem can be treated and eliminated by adopting flux limiting techniques, which are total variation dimishing.

Contents

Chapter 1

Introduction

1.1 Options

Although the financial sector has a relatively long history, since the early 1970's it grew much quicker than ever before. As a result, a lot of newly introduced financial derivatives are dealt on the markets nowadays, providing their owner with wide range of investment opportunities and strategies. One group of such financial products are option contracts, considered and analyzed in this thesis.

Roughly speaking, an option is a derivative security, which gives its holder the right (but not the obligation) to buy/sell a certain asset for the prespecified price at some prespecified date in the future. Simplest example is a plain vanilla European call option. Thanks to this security, the holder may at a prescribed time, known as maturity or expiration date T, purchase a predefined asset, known as the underlying, for a prespecified amount of money, known as the strike price K.

The value of the option differs over the time with respect to various factors, mainly with respect to the actual price of the underlying. At the time of expiry, the price of the option is depicted by the so-called terminal or final payoff. In case of vanilla European call option, it is given by

$$(S(T) - K)^+$$

where $(q)^+ = max(q, 0)$. By $S(t)$ we denote the price of the underlying at time $0 \leq t \leq T$. If one speaks about a plain vanilla European put option, the right to purchase the underlying is changed into the right to sell it. The terminal payoff takes then the form

$$(K - S(T))^+.$$

1.2 Asian Options

Asian options, sometimes also called average options, are path-dependent option contracts which payoff depends on the average value of the asset price over some prespecified period of time.

Via these exotic financial instruments, the holder is provided with an efficient and reasonable protection against possible harm caused by inconvenient fluctuations in price of the underlying. This is especially the case when price movements are speculatively attempted near the expiry date. For the issuer, Asian options simply represent the ability to attain a better forecasting of the terminal position and therefore be more relaxed in dealing with the maturity situation.

As for the occurrence of such products, Asian options are mainly traded on markets concerning commodities, energies, foreign exchange rates and interest rates, since end-users on these markets tend to be exposed to average prices over an extended period of time. Thanks to these contracts, the counterparts are allowed to hedge more effectively their running cash flows and lower their potential losses (or profits at the same time).

On the exchange rate markets, one possible example to illustrate their usage is an exporting company with ongoing currency exposures. This company is selling products in one currency but for the production process throughout the year must purchase raw materials in a different currency. For such a company, the possession of Asian option issued on the appropriate exchange rate is clearly more than convenient. On commodity markets, one can imagine the situation of a company, which is obliged to buy some commodity at the specified date each year and afterwards is selling it further throughout the following year. In fact, on American commodity markets the greatest amount of options traded on oil and gold are nowadays of Asian type. Attractiveness of Asian options is also caused by their relatively low prices. Premiums for these options tend to be lower than those of comparable vanilla call or put options. This is because the volatility in the average price of the underlying asset tends to be lower than the volatility of the asset price itself.

As we already mentioned, the main feature of all Asian options is that their terminal payoff is a function of averaged asset prices over some time period prior to expiry. The averaging of this price can be done in several ways, with respect to certain selection of characteristics taken into consideration.

First of all, one might consider either discrete or continuous sampling of price of the underlying. This decision noticeably affects both method of calculation as well as the obtained solution. Secondly, there is a difference between applying arithmetic or geometric averaging. One can consider either mean of the asset price or the exponential of the mean of the logarithm of the asset price. In addition, one can decide about the period over which the average price is calculated. The period does not have to coincide with the life time of the option. Finally, there is the question about weighted or unweighted average, since recent prices for instance might be for pricing the option in some sense much more important (see Appendix for more details).

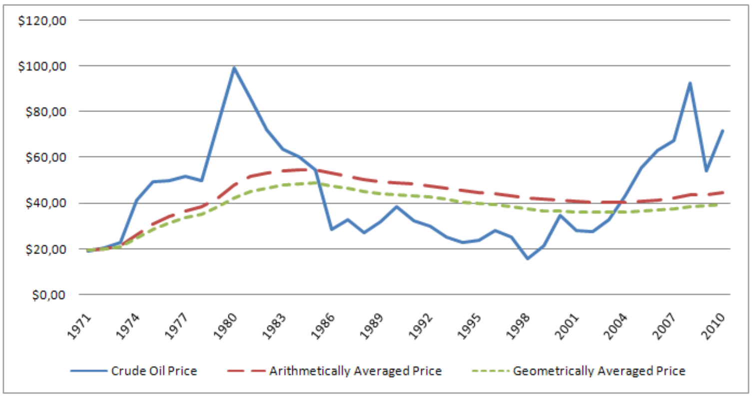

Figure 1.1: *Comparison of continuous arithmetic and geometric averaging illustrated on historical prices of crude oil (inflation adjusted). Source of data: www.inflationdata.com.*

For each different choice of these previously mentioned characteristics, the resulting solution might clearly differ. Because there is no way how to objectively decide, which selection is more and which is less accurate, all possible combinations have been already analyzed and have led to various techniques of pricing Asian options.

In fact, there are even more features, according to which Asian options can be distinguished. The basic grouping is naturally into call options and put options. Another classification is into European type options and American type options. This means, that like in case of vanilla options there exist contracts with and without early exercise opportunity. Other specific feature is the way in which the average is incorporated into the payoff function. If we denote the average value by $A(T)$, the following four basic payoff functions are defined

$$
\begin{array}{ll}
(A(T)-K)^+ & \textit{average rate call} \\
(K-A(T))^+ & \textit{average rate put} \\
(S(T)-A(T))^+ & \textit{average strike call} \\
(A(T)-S(T))^+ & \textit{average strike put}
\end{array}
$$

The rate options are sometimes called price options, or fixed strike options; the strike options are also called floating strike options. If we compare the depicted payoffs to those of plain vanilla options, for an average rate call option the price term $S(T)$ is replaced by the average price term $A(T)$, whereas for the average strike option, the term $A(T)$ replaces the strike term K.

1.3 Numerical Evaluation Techniques

Since exact analytic formulas for average rate and average strike options generally do not exist, they are in point of fact much more difficult to evaluate than common vanilla options. Furthermore, applying numerical methods classically used for pricing financial instruments often carries serious disadvantages.

For example, in financial mathematics commonly used partial differential equation (PDE) techniques are with regard to Asian options quite inaccurate, cf. [1]. Monte Carlo simulations with a specific variance reduction method proposed by Kemna and Vorst [7] work well only considering European type options and are still relatively slow. Conventional binomial lattice methods are effectively unusable due to their huge requirements of computer memory.

As for American type Asian options, Hull and White [5] proposed a modified binomial lattice method. Unfortunately, the convergence of this algorithm is questionable. Neave [9] used on a binomial lattice a frequency distribution approach. Here, calculations of order N^4, where N is the number of time steps in the lattice are required. Barraquand and Pudet [1] proposed an unconditionally convergent "forward shooting grid" technique. In this thesis, we shall investigate modifications of finite difference (FD) methods.

As a matter of fact, the price of Asian option contract can be obtained as a numerical solution of a PDE problem, that has one dimension in time and two dimensions in space. We will refer to this equation as a two-dimensional PDE. As we show later on, this PDE has convection-diffusion character, but without diffusion term in one of the spatial dimensions. This is very often the reason of many numerical difficulties, especially occurance of spurious oscillations. Moreover, for FD methods both the time and space complexities grow exponentially in the number of state variables. Therefore one should try to lower their number.

In most cases, the above mentioned two-dimensional PDE can be reduced into a PDE with only one spatial dimension. There are at least three possible ways how to make this procedure. The first possible reduction, formulated by Ingersoll [6] is applicable for case of average strike options. The same property has the transformation of Wilmott, Dewinne and Howison [13]. The last reduction method was proposed by Rogers and Shi [11] and can be applied only for Asian options of European type.

In general, the one-dimensional PDE obtained from all previously introduced transformations is difficult to solve numerically. The reason is a none or very small diffusion term in the derived PDE, which eventually causes solutions to oscillate. To eliminate these numerical instabilities, we shall formulate modifications to common methods to cope with this delicate problem.

One approach how to get rid of the oscillations caused by classically used centrally weighted schemes is to apply first-order upstream weighting for the convective term (used in computational fluid dynamics). By this procedure, artificial diffusion is locally implemented into calculation and handles the oscillations. Unfortunately, one must point out that as a side-effect this leads to results with excessive false diffusion. To additionally eliminate this phenomenon, high order non-linear flux limiter can be applied. As we shall demonstrate on the following pages, after such adjustment the resulting solutions are truly sufficiently accurate.

In Chapter 2 of this work, we are concerned with the derivation of PDE models to price Asian options. Chapter 3 is dedicated to depict problems, that might occur while numerically solving Asian option's evaluation. Without loss of relevance, this illustration is done on standard European vanilla options with a feature of small value of diffusion term, typical for average options. In Chapter 4 we present numerical simulations and finally conclude our work in Chapter 5.

Chapter 2

PDE Models

2.1 Derivation of Two-dimensional Models

2.1.1 The Black-Scholes equation for path-dependent options

First we derive the general model for the valuation of all path-dependent options.

Consider the price function $V(S(t), I(t), t)$. This function depends on the time t, price process $S(t)$ and the newly introduced path-dependent variable

$$I(t) = \int_0^t f(S(\tau), \tau)d\tau, \tag{2.1}$$

where the function $f(.)$ is specific for each possible path-dependent option considered. Since the variable I does not depend on the current asset price S, the option price V is a function of three independent variables. The stock price $S(t)$ follows a classical geometrical Brownian motion (GBM)

$$dS = rSdt + \sigma SdB, \tag{2.2}$$

where r represents the risk free interest rate, the variable σ is the volatility and by dB we denote the standard Brownian motion. To apply Itô's lemma on V, we additionally need to know the stochastic differential equation (SDE) for variable I. This can be found very easily, since

$$I + dI = I(t + dt) = \int_0^{t+dt} f(S(\tau), \tau)d\tau = \int_0^t f(S(\tau), \tau)d\tau + f(S(t), t)dt.$$

From this we clearly see, that

$$dI = f(S(t), t)dt. \tag{2.3}$$

Note that there is no stochastic component dB, only a drift component dt.

Now we are allowed to apply the multidimensional Itô's lemma on the value function $V(S(t), I(t), t)$ (see Appendix for the general form of this formula). This procedure provides us with the expression

$$dV = \sigma S \frac{\partial V}{\partial S} dB + \left(\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} + \frac{\partial V}{\partial t} + f(S,t)\frac{\partial V}{\partial I} \right) dt. \tag{2.4}$$

Let us now construct a risk-free portfolio Π by

$$\begin{aligned} \Pi &= V - \Delta S, \\ d\Pi &= dV - \Delta dS, \end{aligned} \tag{2.5}$$

i.e., from a long position in option and short position in Δ-times underlying asset. Substituting (2.2) and (2.4) into (2.5) yields

$$\begin{aligned} d\Pi =& \left(\sigma S \frac{\partial V}{\partial S} - \Delta \sigma S \right) dB \\ &+ \left(\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} + \frac{\partial V}{\partial t} + f(S,t)\frac{\partial V}{\partial I} - \Delta r S \right) dt \\ =& \left(\sigma S \left(\frac{\partial V}{\partial S} - \Delta \right) \right) dB \\ &+ \left(\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\left(\frac{\partial V}{\partial S} - \Delta \right) + \frac{\partial V}{\partial t} + f(S,t)\frac{\partial V}{\partial I} \right) dt. \end{aligned}$$

To eliminate stochastic fluctuations from our portfolio, we should set $\Delta = \partial V / \partial S$. One is left with

$$d\Pi = \left(\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} + f(S,t)\frac{\partial V}{\partial I} \right) dt. \tag{2.6}$$

If we adopt now the assumption of no arbitrage opportunities on the market, i.e., the change of our portfolio should coincide with the change of corresponding value of money deposited on the bank account earning risk-free interest rate, we obtain

$$d\Pi = r\Pi dt = r(V - \Delta S)dt = r(V - \frac{\partial V}{\partial S} S)dt. \tag{2.7}$$

Putting together (2.6) and (2.7) leads to the general form of the Black-Scholes PDE for pricing path-dependent option contracts

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} + f(S,t)\frac{\partial V}{\partial I} - rV = 0. \tag{2.8}$$

2.1.2 The Black-Scholes equation for Asian options

As was already stated, various forms of the function $f(.)$ specifies (2.8) for valuing different path-dependent options. Considering the case of average rate and average strike options, two basic possibilities arise. In the first one, we let the average value be defined as a running sum via the formula

$$I(t) = \int_0^t S(\tau) d\tau.$$

For variable $I(t)$, first derivative with respect to time gives us the function $f(.)$ as

$$\frac{dI}{dt} = S(t) = f(S(t), t).$$

The Black-Scholes equation for Asian options in terms of variable $I(t)$ takes then the form

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} + S\frac{\partial V}{\partial I} - rV = 0. \tag{2.9}$$

Second common and equivalent concept for formulating average value is usually denoted $A(t)$ and defined by

$$A(t) = \frac{I(t)}{t} = \frac{1}{t}\int_0^t S(\tau) d\tau.$$

Differentiation with respect to variable t gives function $f(.)$ as expression

$$\frac{dA}{dt} = \frac{1}{t}S(t) - \frac{1}{t^2}\int_0^t S(\tau)d\tau = \frac{1}{t}\left(S(t) - \frac{1}{t}\int_0^t S(\tau)d\tau\right) =$$
$$= \frac{1}{t}\Big((S(t) - A(t)\Big) = f(S(t), t).$$

Therefore, in terms of $A(t)$ the Black-Scholes PDE for Asian options takes the form

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} + \frac{1}{t}(S - A)\frac{\partial V}{\partial A} - rV = 0. \tag{2.10}$$

As one can see, both depicted forms of the Black-Scholes PDE have a two-dimensional character. In other words, both PDEs include partial derivatives of the option price with respect to time and the two other state variables. However, neither equation (2.9) nor equation (2.10) has a diffusion term (second spatial partial derivative over respective variable) in the second spatial dimension. In fact, this feature causes many numerical instabilities while numerically solving with standard finite difference methods. We shall investigate this in-depth in the following chapter.

2.2 Reduction to One-dimensional Models

The more spatial dimensions the Black-Scholes equation has, the more complexity of FD methods is naturally incorporated into the process of pricing particular options. In fact, both time and space complexities grow exponentially in the number of state variables. The two-dimensionality of the derived Black-Scholes PDEs for valuing Asian options is therefore a rather unpleasant property.

Nevertheless, this undesirable feature can vanish, since in most cases certain transformations are available which lead to reduction in spatial dimensions. This is not true only for American type average rate options where each time full two-dimensional PDE must be solved. We proceed here with illustrations of possible ways, how the reduction into one-dimensional PDEs can be done.

2.2.1 The Ingersoll method

As shown by Ingersoll [6], one possible reduction for average strike options might be achieved by introducing the new variable $R = S/I$. Considering the original pricing equation for function $V(S(t), I(t), t)$ in form (2.9), appropriate boundary conditions are

$$\begin{aligned} V(0, I, t) &= 0, \\ \lim_{S\to\infty} \frac{\partial V}{\partial S}(S, I, t) &= 1, \\ \lim_{I\to\infty} V(S, I, t) &= 0, \\ V(S_T, I_T, T) &= \left(S_T - \frac{I_T}{T} \right)^+ . \end{aligned}$$

Since in equation (2.9) appears only first partial derivative with respect to the variable I, only one boundary condition in I direction is sufficient. Note that all previously depicted boundary conditions together with equation (2.9) are lineary homogeneous in variables S and I. Therefore homogeneity holds for the option price V as well.

After the transformation $R = S/I$, the option price transforms into $V(S, I, t) = IG(R, t)$. Furthermore,

$$\begin{aligned} \partial V/\partial S &= \partial G/\partial R, \\ \partial^2 V/\partial S^2 &= (\partial^2 G/\partial R^2)/I, \\ \partial V/\partial I &= G - R(\partial G/\partial R), \\ \partial V/\partial t &= I\partial G/\partial t. \end{aligned}$$

In terms of the new value function G and new variable R, the pricing equation and boundary conditions become

$$\frac{1}{2}\sigma^2 R^2 \frac{\partial^2 G}{\partial R^2} + (rR - R^2)\frac{\partial G}{\partial R} + (R - r)G + \frac{\partial G}{\partial t} = 0, \tag{2.11}$$

$$\begin{aligned} G(0,t) &= 0, \\ \lim_{R\to\infty} \frac{\partial G}{\partial R}(R,t) &= 1, \\ G(R_T, T) &= (R_T - 1/T)^+, \end{aligned}$$

This approach can not be considered with average rate options, where the assumption of homogeneity in variables S and I is not fulfilled for the terminal condition.

2.2.2 The Method of Wilmott, Dewinne and Howison

In contrast to the previously described method, Wilmott, Dewinne and Howison [13] proposed a change of variables of inverse type:

$$R_t = \frac{I_t}{S_t} = \frac{1}{S_t}\int_0^t S(\tau)d\tau.$$

This causes the price function V to be transformed into a new function H with the relation $V(S, I, t) = SH(R, t)$. The terminal payoff becomes

$$\begin{aligned} V(S,I,T) &= \left(S_T - \frac{1}{T}I_T\right)^+ = S_T\left(1 - \frac{1}{TS_T}I_T\right)^+ \\ &= S_T\left(1 - \frac{1}{T}R_T\right)^+ = S_T H(R_T, T). \end{aligned}$$

From

$$\begin{aligned} R_{t+dt} &= R_t + dR_t, \\ dS_t &= rS_t dt + \sigma S_t dB_t, \end{aligned}$$

follows the SDE

$$dR_t = (1 + (\sigma^2 - r)R_t)dt - \sigma R_t dB_t.$$

In terms of H and R, equation (2.9) takes now the one-dimensional form

$$\frac{\partial H}{\partial t} + \frac{1}{2}\sigma^2 R^2 \frac{\partial^2 H}{\partial R^2} + (1 - rR)\frac{\partial H}{\partial R} = 0. \tag{2.12}$$

As for appropriate boundary conditions,

$$\lim_{R\to\infty} H(R,t) = 0,$$
$$\frac{\partial H}{\partial t}(0,t) + \frac{\partial H}{\partial R}(0,t) = 0,$$
$$H(R_T,T) = (1 - R_T/T)^+.$$

Similarly to the previous algorithm, this reduction works only in case of average strike options.

2.2.3 The Method of Rogers and Shi

The approach formulated by Rogers and Shi [11] reduces the space dimensions by introducing the new state variable

$$x = \frac{K - \int_0^t S(\tau)\mu(d\tau)}{S_t},$$

where μ is a probability measure with density $\rho(t)$ on the interval $(0,T)$. More precisely, $\rho(t) = 1/T$ for average rate options and $\rho(t) = 1/T - \delta(T-t)$ for average strike options, where δ is a delta function.

The two-dimensional PDE for Asian options transforms in this case into a one-dimensional equation of the form

$$\frac{\partial W}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 W}{\partial x^2} - (\rho(t) + rx)\frac{\partial W}{\partial x} = 0. \tag{2.13}$$

The terminal conditions are

$$w(x,T) = x^- \tag{2.14}$$

for average rate call option and

$$w(x,T) = (x+1)^- \tag{2.15}$$

for average strike put option, where $(q)^- = max(-q,0)$.

In the case where μ is uniform on $[0,T]$, the price of Asian option with initial price S_0, maturity T and fixed strike K is $S_0 W(K/S_0, 0)$. For an option with floating strike but all other parameters the same is the price equal to $S_0 W(0,0)$.

Let us remark, that on one hand the Rogers and Shi framework holds for both average rate and average strike options. But on the other hand, it is not applicable for Asian options of American type.

Chapter 3

Numerical Discretization

In this chapter we will focus on some basic ways in which discretization of considered option models can be done. We will try to illustrate, what are the possible problems and complications which might occur while applying classical discretization techniques and what could one eventually do to get rid of them. Our goal is therefore to analyze the classical way of treatment and subsequently try to do effective adjustments so that one ends up with practical method, which definitely provides fitting and accurate solutions.

Depiction of this matter is first done on classical European vanilla options. Afterwards, to make our point clear and understandable, transformation into terms of Asian options is done.

3.1 Vanilla Options Discretization

As already stated, in case of Asian option contracts standard numerical discretization leads often to oscillatory solutions. This feature also stands for classical plain vanilla options which are convection-dominated (this property is discussed and explained later on). Instable spurious solutions are naturally untrustworthy (see Figure (3.1)). The numerical procedure must be therefore in all such cases clearly treated in quite a different way. The following text explains, how to proceed.

European plain vanilla options can be priced by solving the well known linear parabolic Black-Scholes PDE

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0, \tag{3.1}$$

subject to appropriate boundary conditions and proper terminal condition.

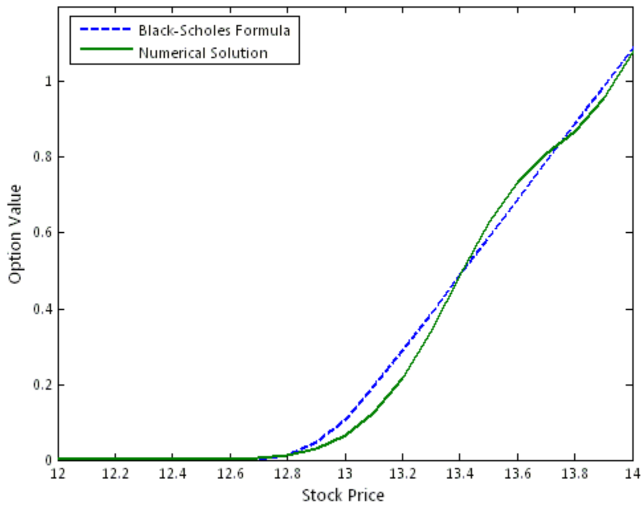

Figure 3.1: *Plain vanilla European Call price. Comparison of the Black-Scholes analytical solution and explicit numerical solution. ($K = 15$, $T = 1$, $r = 0.15$, $\sigma = 0.01$, $\Delta S = 0.1$, $\Delta\tau = 0.01$)*

For European call option, terminal and boundary conditions take the form

$$V(S(T), T) = (S(T) - K)^+,$$

$$V(0, t) = 0,$$

$$V(S(t), t) \sim S(t) - Ke^{-r(T-t)} \qquad for\ S(t) \to \infty,$$

respectively, whereas for European put option, the terminal condition is

$$V(S(T), T) = (K - S(T))^+$$

and the boundary conditions read

$$V(0, t) = Ke^{-r(T-t)},$$

$$V(S(t), t) \sim 0 \qquad for\ S(t) \to \infty.$$

Since equation (3.1) is backward in time, one first needs to perform transformation leading to a forward equation. This might be done by simply introducing the new reversed time variable, $\tau = T - t$. Variable τ obviously runs from 0 to T and equation (3.1) after such substitution becomes

$$\frac{\partial V}{\partial \tau} = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - (-rS)\frac{\partial V}{\partial S} - rV. \tag{3.2}$$

This equation is understood to be a convection-diffusion equation, where the term

$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}$$

is reffered to as a diffusion term and

$$(-rS)\frac{\partial V}{\partial S}$$

as a convective term. If in particular the ratio between these two terms, known as the Péclet number (convective coeff./diffusion coeff.), is very large or in other words, diffusion term is very small comparing to convective term ($\sigma \ll r$), the equation is said to be convection-dominated. This is the case of all PDEs for pricing Asian options. The numerical solution then behaves as if it was not parabolic equation but a hyperbolic one. Finding solutions to such equation is therefore much difficult and demanding.

After applying on equation (3.2) one of possible finite volume discretization approaches, derived by Roache [10], we get the value at cell i at time step $n+1$ represented as

$$\begin{aligned} \frac{V_i^{n+1} - V_i^n}{\Delta\tau} &= \theta F_{i-\frac{1}{2}}^{n+1} - \theta F_{i+\frac{1}{2}}^{n+1} + \theta f_i^{n+1} \\ &\quad + (1-\theta)F_{i-\frac{1}{2}}^{n} - (1-\theta)F_{i+\frac{1}{2}}^{n} + (1-\theta)f_i^{n}. \end{aligned} \tag{3.3}$$

The lower index characterizes for variable its spatial position and upper index respective time layer. Variable θ stands for temporal weighting and holds, that $0 \le \theta \le 1$. For a fully-explicit method $\theta = 0$, for a fully-implicit method $\theta = 1$ and to get the Crank-Nicolson method, one should let $\theta = \frac{1}{2}$. Terms denoted by F are referred to as flux terms, those denoted by f as source/sink terms. As for their detailed form,

$$F_{i-\frac{1}{2}}^{n+1} = \frac{1}{\Delta S_i}\left[\Big(-\frac{1}{2}\sigma^2 S_i^2\Big)\frac{V_i^{n+1} - V_{i-1}^{n+1}}{\Delta S_{i-\frac{1}{2}}} + (-rS_i)V_{i-\frac{1}{2}}^{n+1}\right], \tag{3.4}$$

$$F_{i+\frac{1}{2}}^{n+1} = \frac{1}{\Delta S_i}\left[\Big(-\frac{1}{2}\sigma^2 S_i^2\Big)\frac{V_{i+1}^{n+1} - V_{i}^{n+1}}{\Delta S_{i+\frac{1}{2}}} + (-rS_i)V_{i+\frac{1}{2}}^{n+1}\right], \tag{3.5}$$

$$f_i^{n+1} = (-r)V_i^{n+1}, \tag{3.6}$$

where

$$\Delta S_i = \frac{S_{i+1} - S_{i-1}}{2}\ , \ \Delta S_{i+\frac{1}{2}} = S_{i+1} - S_i.$$

Sometimes, the log transformation $y = ln(S)$ applied on equation (3.1) is suggested to prevent some minor computational problems. After such a transformation, equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 V}{\partial y^2} + (r - \frac{1}{2}\sigma^2)\frac{\partial V}{\partial y} - rV = 0 \tag{3.7}$$

is obtained. Converting this into a forward equation in time and applying finite volume discretization depicted in equation (3.3) provides flux term in the form

$$F^{n+1}_{i+\frac{1}{2}} = \frac{1}{\Delta y_i}\left[\Big(-\frac{1}{2}\sigma^2\Big)\frac{V^{n+1}_{i+1} - V^{n+1}_i}{\Delta y_{i+\frac{1}{2}}} + \Big(-r + \frac{\sigma^2}{2}\Big)V^{n+1}_{i+\frac{1}{2}}\right]. \tag{3.8}$$

As one may clearly see, in contrast to previous case, diffusion and convective coefficients became constant, what makes the computation somehow easier to perform.

3.2 Schemes for Handling Oscillatory Solutions

The way in which we decided to deal with the appearance of spurious solutions is improving the treatment of discretizing convective terms $V^{n+1}_{i-\frac{1}{2}}$ and $V^{n+1}_{i+\frac{1}{2}}$ arising as components of previously illustrated flux terms.

3.2.1 The central weighting scheme

One basic and common scheme for dealing with these convective terms is central weighting discretization in the form

$$V^{n+1}_{i+\frac{1}{2}} = \frac{V^{n+1}_{i+1} + V^{n+1}_i}{2}, \tag{3.9}$$

which is second-order accurate when using uniform grids. As long as we apply this approach, to guarantee accurate and oscillatory-free solutions, one needs to satisfy two crucial conditions (for their concise derivation see [15]). The first one is the so-called Péclet condition. Considering the original equation (3.1), the Péclet condition is in the form

$$\frac{1}{\Delta S_{i-\frac{1}{2}}} > \frac{r}{\sigma^2 S_i}, \tag{3.10}$$

for the log-transformed equation (3.7) in form

$$\frac{1}{\Delta y_{i-\frac{1}{2}}} > \frac{|r - \frac{\sigma^2}{2}|}{\sigma^2}. \tag{3.11}$$

Let us point out one problem, handled by the log transformation, which is satisfying the Péclet condition if $S \to 0$. Considering (3.10), for $S_0 = 0$ the implied condition $\frac{\sigma^2}{r} > 1$ must be fulfilled which may not be true. In the latter case, none such a strict relation must be guaranteed.

The second additional condition that must be taken into consideration during the discretizing procedure takes the form

$$\frac{1}{(1-\theta)\Delta\tau} > \frac{\sigma^2 S_i^2}{2}\left(\frac{1}{\Delta S_{i-\frac{1}{2}}\Delta S_i} + \frac{1}{\Delta S_{i+\frac{1}{2}}\Delta S_i}\right) + r \tag{3.12}$$

and

$$\frac{1}{(1-\theta)\Delta\tau} > \frac{\sigma^2}{2}\left(\frac{1}{\Delta y_{i-\frac{1}{2}}\Delta y_i} + \frac{1}{\Delta y_{i+\frac{1}{2}}\Delta y_i}\right) + r \tag{3.13}$$

for cases (3.1) and (3.7) respectively.

Notice that if r is significantly large compared to σ (considered equation is convection-dominated), to meet both additional and Péclet conditions one must use a very fine grid spacing. To avoid this limiting and sometimes extremely demanding requirement, one may try to synthetically affect the convection-diffusion ratio by numerically enlarging true diffusion of the model by artificial diffusion. One possible implementation of this approach is by introducing a first-order upstream weighting scheme, that is often used in computational fluid dynamics.

3.2.2 The first-order upstream weighting scheme

This weighting method is first-order accurate on a uniform lattice. Its general form is

$$V_{i+\frac{1}{2}}^{n+1} = \begin{cases} V_i^{n+1} & \text{if } -rS_i \geq 0, \\ V_{i+1}^{n+1} & \text{otherwise.} \end{cases}$$

Since we consider the Black-Scholes equation (3.1), the relation $rS_i \geq 0$ holds. Therefore $V_{i+\frac{1}{2}}^{n+1} = V_{up}^{n+1} = V_{i+1}^{n+1}$.

Because evident one-sided differencing is applied through this scheme, requested synthetic diffusion is truly incorporated into the computational process and the resulting solutions are after fulfilling only one condition

$$\frac{1}{(1-\theta)\Delta\tau} > \frac{\sigma^2 S_i^2}{2}\left(\frac{1}{\Delta S_{i-\frac{1}{2}}\Delta S_i} + \frac{1}{\Delta S_{i+\frac{1}{2}}\Delta S_i}\right) + \frac{rS_i}{\Delta S_i} + r, \tag{3.14}$$

no longer unstable in terms of producing spurious oscillations. However, the method still has some disadvantages. Among others, the main inconvenience is certainly the fact, that artificial diffusion supplemented into numerical calculation causes too diffused and hence inaccurate solutions. For the sake of reliable outcomes, presence of this excessive diffusion must be further adjusted. We shall focus on this issue in the following subsection.

3.2.3 The van Leer flux limiter

In order to obtain reasonably accurate solutions without the threat of dealing with both spurious oscillations and excessive diffusion, one has the possibility to introduce the van Leer flux limiter. The aim of this approach is to augment true diffusion of the model by additional numerical diffusion only if necessary to prevent instabilities. The general formula for this method is

$$V_{i+\frac{1}{2}}^{n+1} = V_{up}^{n+1} + \frac{\phi(q_{i+\frac{1}{2}}^{n+1})}{2}(V_{down}^{n+1} - V_{up}^{n+1}), \tag{3.15}$$

where the limiter function $\phi(.)$ is defined as

$$\phi(q) = \frac{|q| + q}{1 + |q|} \tag{3.16}$$

and the variable q is defined as

$$q_{i+\frac{1}{2}}^{n+1} = \frac{V_{up}^{n+1} - V_{2up}^{n+1}}{S_{2up} - S_{up}} \bigg/ \frac{V_{down}^{n+1} - V_{up}^{n+1}}{S_{up} - S_{down}}. \tag{3.17}$$

The crucial feature of this scheme is the fact, that artificial diffusion is added into calculation only at specific grid points. In particular, at points where the gradient is steep, which as a result handles the former problem of diffused solutions. Consequently, the method has second-order accuracy on regions which are not affected by synthetic diffusion.

The guarantee of providing oscillation-free solutions is assured by scheme's total variation dimishing (TVD) property. TVD means that for the total variation of the solution, defined as

$$TV(V^{n+1}) = \sum_i |V_{i+1}^{n+1} - V_i^{n+1}|,$$

the following relation holds

$$TV(V^{n+1}) \leq TV(V^n).$$

This outlined inequality clearly doesn't allow any instabilities contained in solution, since any undesirable oscillations would naturally cause total variation to increase and so to violate this relation.

3.3 Asian Options Discretization

Although in the previous text we were concerned with plain European vanilla options and their numerical treatment, now our goal is to put illustrated discretization ideas into terms of Asian options. The already depicted gradual improvement of handling problems, which may occur while numerically pricing special case of vanilla options, can be relatively easily transformed and applied on Asian option contracts. This section is dedicated exactly to this matter.

To recall, Asian options can be priced by solving two-dimensional PDE, either in form (2.9) or in form (2.10). As we already illustrated, there are certain ways how the two-dimensionality can be eliminated and so one just needs to deal with one-dimensional PDE which may not be that demanding. Discretization can be afterwards applied on an only one-dimensional PDE which is very favourable. Let us depict discretization treatment of two presented PDEs, which is done analogically to the case of plain vanilla options.

If we consider the reduction proposed by Ingersoll, one deals with the PDE

$$\frac{1}{2}\sigma^2 R^2 \frac{\partial^2 G}{\partial R^2} + (rR - R^2)\frac{\partial G}{\partial R} + (R - r)G + \frac{\partial G}{\partial t} = 0.$$

After the transformation into a forward equation, done by introducing the reversed time variable $\tau = T - t$, one obtains the PDE

$$\frac{\partial G}{\partial \tau} = \frac{1}{2}\sigma^2 R^2 \frac{\partial^2 G}{\partial R^2} - (-rR + R^2)\frac{\partial G}{\partial R} + (R - r)G. \tag{3.18}$$

Now, if we apply the discretization scheme (3.3), the flux terms take the form

$$F_{i-\frac{1}{2}}^{n+1} = \frac{1}{\Delta R_i}\left[\left(-\frac{1}{2}\sigma^2 R_i^2\right)\frac{G_i^{n+1} - G_{i-1}^{n+1}}{\Delta R_{i-\frac{1}{2}}} + (-rR_i + R_i^2)G_{i-\frac{1}{2}}^{n+1}\right], \tag{3.19}$$

$$F_{i+\frac{1}{2}}^{n+1} = \frac{1}{\Delta R_i}\left[\left(-\frac{1}{2}\sigma^2 R_i^2\right)\frac{G_{i+1}^{n+1} - G_{i}^{n+1}}{\Delta R_{i+\frac{1}{2}}} + (-rR_i + R_i^2)G_{i+\frac{1}{2}}^{n+1}\right] \tag{3.20}$$

and the sink term takes the form

$$f_i^{n+1} = (R_i - r)G_i^{n+1}. \tag{3.21}$$

In case of considering the one-dimensional PDE

$$\frac{\partial H}{\partial t} + \frac{1}{2}\sigma^2 R^2 \frac{\partial^2 H}{\partial R^2} + (1 - rR)\frac{\partial H}{\partial R} = 0$$

proposed by Wilmott, Dewinne and Howison, transformation to forward equation leads to PDE

$$\frac{\partial H}{\partial \tau} = \frac{1}{2}\sigma^2 R^2 \frac{\partial^2 H}{\partial R^2} - (rR - 1)\frac{\partial H}{\partial R}.$$

Discretization scheme (3.3) may be then applied which provides flux terms in the form

$$F_{i-\frac{1}{2}}^{n+1} = \frac{1}{\Delta R_i}\left[\left(-\frac{1}{2}\sigma^2 R_i^2\right)\frac{H_i^{n+1} - H_{i-1}^{n+1}}{\Delta R_{i-\frac{1}{2}}} + (rR_i - 1)H_{i-\frac{1}{2}}^{n+1}\right], \tag{3.22}$$

$$F_{i+\frac{1}{2}}^{n+1} = \frac{1}{\Delta R_i}\left[\left(-\frac{1}{2}\sigma^2 R_i^2\right)\frac{H_{i+1}^{n+1} - H_{i}^{n+1}}{\Delta R_{i+\frac{1}{2}}} + (rR_i - 1)H_{i+\frac{1}{2}}^{n+1}\right] \tag{3.23}$$

The sink term vanishes since considered PDE includes only partial derivatives of H.

For both methods, convective terms $G_{i-\frac{1}{2}}^{n+1}$ and $G_{i+\frac{1}{2}}^{n+1}$ (alternatively $H_{i-\frac{1}{2}}^{n+1}$ and $H_{i+\frac{1}{2}}^{n+1}$) shall be treated just like we illustrated in section 3.2.

Chapter 4

Stable Numerical Valuation of Options

This chapter of the thesis is dedicated to practical results, which we obtained by gradually applying all previously mentioned theoretical methods and knowledge. To perform numerical calculations, we decided to use the mathematical software *MATLAB*. Two examples of the respective algorithms created in order to price the options can be found in Appendix.

4.1 Valuation of Vanilla Options

As already stated, problems which occur while pricing Asian options might also arise while pricing plain vanilla options. This is the case especially if the pricing PDE is convection-dominated. On the following pages, we concern with exactly this situation. We shall try to numerically evaluate the plain vanilla call option, while intentionally considering small volatility and large interest rate.

First of all, we try to calculate the price of this option using usual concept of central weighting scheme. After substitution done into flux terms (3.4) and (3.5) with respect to formula (3.9), substituting flux terms and sink terms into scheme (3.3) and putting together all coefficients standing next to the same grid points, one obtains an expression of the type

$$\alpha_i^{n+1} V_{i-1}^{n+1} + \beta_i^{n+1} V_i^{n+1} + \gamma_i^{n+1} V_{i+1}^{n+1} = \alpha_i^n V_{i-1}^n + \beta_i^n V_i^n + \gamma_i^n V_{i+1}^n, \qquad (4.1)$$

which is well known from the finite difference methods. In particular, for central weighting scheme, the coefficients read

$$\alpha_i^{n+1} = \frac{\theta\Delta\tau}{2}(-\sigma^2 i^2 + ri), \qquad \alpha_i^{n} = -\frac{(1-\theta)\Delta\tau}{2}(-\sigma^2 i^2 + ri),$$
$$\beta_i^{n+1} = 1 - \theta\Delta\tau(-\sigma^2 i^2 - r), \qquad \beta_i^{n} = 1 + (1-\theta)\Delta\tau(-\sigma^2 i^2 - r),$$
$$\gamma_i^{n+1} = \frac{\theta\Delta\tau}{2}(-\sigma^2 i^2 - ri), \qquad \gamma_i^{n} = -\frac{(1-\theta)\Delta\tau}{2}(-\sigma^2 i^2 - ri).$$

After setting appropriate boundary conditions and terminal condition and after decision about temporal weighting ($0 \leq \theta \leq 1$), finally the equation (4.1) can be written in the *MATLAB* code and the premium of the considered call option can be calculated.

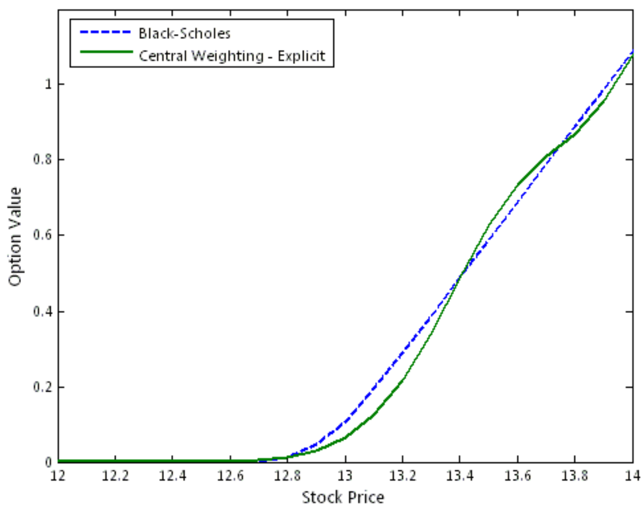

Figure 4.1: *European Call price. Comparison of the Black-Scholes analytical solution and numerical solution, considering the central weighting scheme. ($K =$ 15, $T = 1$, $r = 0.15$, $\sigma = 0.01$, $\Delta S = 0.1$, $\Delta\tau = 0.01$, $\theta = 0$)*

Let us note once again that the 'convection-dominated' feature typical for PDE models of Asian options is guaranteed by considering unrealistic values of the risk free interest rate and volatility, in particular $r = 0.15$ and $\sigma = 0.01$. As for other parameters, the strike price $K = 15$ and the time to maturity $T = 1$. We perform the computation while using uniform grid spacing $\Delta S = 0.1$, $\Delta\tau = 0.01$.

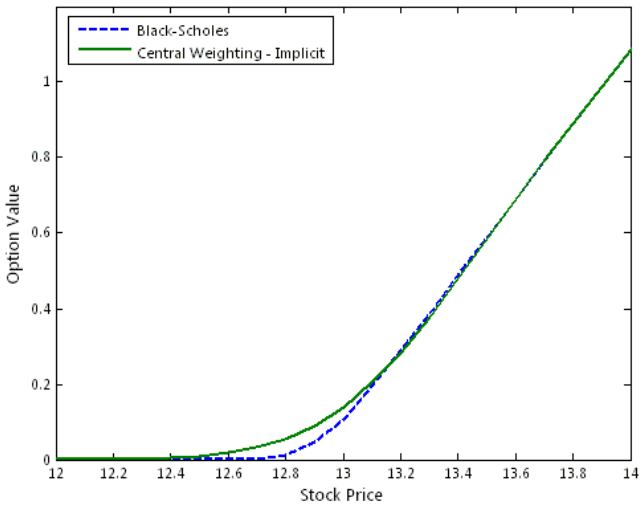

Figure 4.2: *European Call price. Comparison of the Black-Scholes analytical solution and numerical solution, considering the central weighting scheme.* $(K = 15,\ T = 1,\ r = 0.15,\ \sigma = 0.01,\ \Delta S = 0.1,\ \Delta\tau = 0.01,\ \theta = 1)$

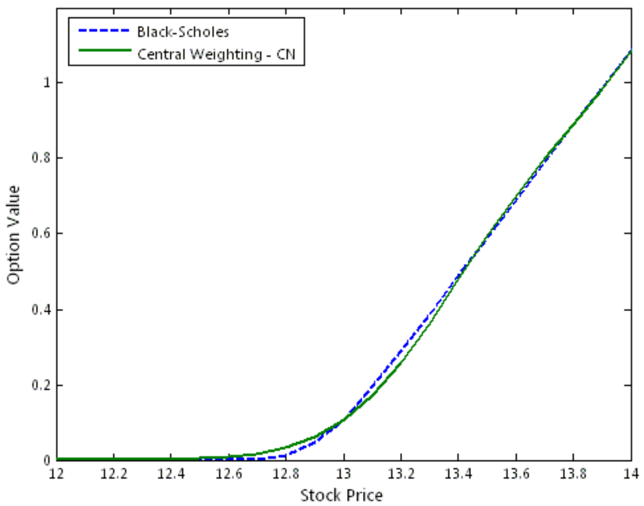

Figure 4.3: *European Call price. Comparison of the Black-Scholes analytical solution and numerical solution, considering the central weighting scheme.* $(K = 15,\ T = 1,\ r = 0.15,\ \sigma = 0.01,\ \Delta S = 0.1,\ \Delta\tau = 0.01,\ \theta = \frac{1}{2})$

Illustrated results prove that the Crank-Nicolson temporal weighting provides us with the most precise results in terms of general accuracy. However, as one may clearly see, while using the central weighting scheme and applying explicit discretization method (Figure (4.1)), the algorithm leads to apparent oscillatory results as predicted. As a matter of fact, in this particular case the oscillations are caused by violation of the Péclet condition, which may very easily happen if solving Asian option PDEs. This is definitely not quite desirable and neither considering implicit approach depicted on Figure (4.2) nor Crank-Nicolson approach shown on Figure (4.3) helps to improve handling the oscillations. Therefore, in order to obtain non-oscillatory outcomes, instead of the central weighting approach one needs to apply some other more suitable method.

To move on with this matter and so to get rid of the unpleasent occurance of spurious oscillations, consider now the first-order upstream weighting scheme. If one derives the equation of the form (4.1) for this particular case, the coefficients read

$$\begin{aligned}
\alpha_i^{n+1} &= \frac{\theta\Delta\tau}{2}(-\sigma^2 i^2), & \alpha_i^n &= -\frac{(1-\theta)\Delta\tau}{2}(-\sigma^2 i^2),\\
\beta_i^{n+1} &= 1-\theta\Delta\tau(-\sigma^2 i^2 - ri - r), & \beta_i^n &= 1+(1-\theta)\Delta\tau(-\sigma^2 i^2 - ri - r),\\
\gamma_i^{n+1} &= \theta\Delta\tau(-\frac{1}{2}\sigma^2 i^2 - ri), & \gamma_i^n &= -(1-\theta)\Delta\tau(-\frac{1}{2}\sigma^2 i^2 - ri).
\end{aligned}$$

To recall, this approach is based on the fact, that true diffusion of the model is enlarged with some intended artificial diffusion. As a result, the discretized PDE locally behaves as non-convection-dominated and so the feature of providing oscillatory solution positively disappears. Values depicted on the Figure (4.4) proves our point. On this figure, plotted against the analytical Black-Scholes solution, you may see results we obtained by calculating price of the plain vanilla European call option considering upstream weighting. Outcomes from all three basic temporal weighting approaches are illustrated.

On the one hand, oscillations truly disappeared from the obtained solution. On the other hand, as suspected, the artificial diffusion incorporated into computational process seriously harmed the solution and caused it to become too diffuse. Unfortunately, such results are therefore absolutely not reliable and the upstream weighting scheme can not be accepted to be the proper method for solving such PDEs.

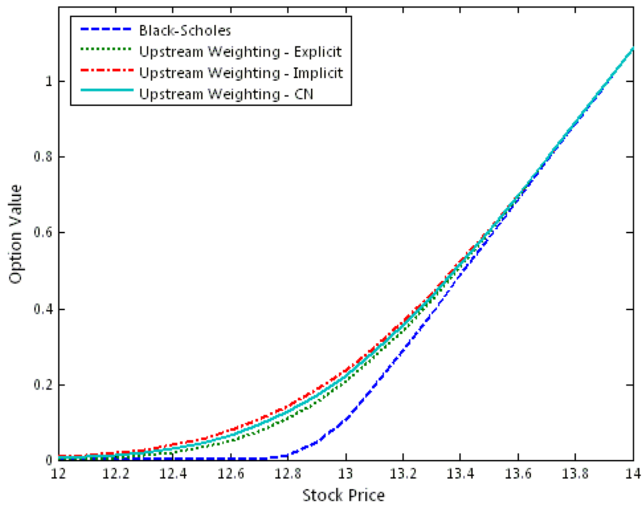

Figure 4.4: *European Call price. Comparison of the Black-Scholes analytical solution and numerical solutions, considering the first-order upstream weighting scheme. ($K =$ 15, $T = 1$, $r = 0.15$, $\sigma = 0.01$, $\Delta S = 0.1$, $\Delta\tau = 0.01$)*

Although supplementing the artificial diffusion into the model seems to be right for handling problems with the instabilities, it must be treated differently and more cautiously than by the first-order upstream weighting scheme so that the solution persists to be accurate. Let us therefore introduce the van Leer flux limiter techniques into our calculations.

As derived and explained previously in theoretical part of this text, considering flux limiter is expected to add the numerical diffusion only if necessary to prevent oscillations. The coefficients of equation (4.1) takes now the form

$$
\begin{aligned}
\alpha_i^{n+1} &= \theta\Delta\tau(-\frac{1}{2}\sigma^2 i^2 + ri\phi_+^n),\\
\beta_i^{n+1} &= 1 - \theta\Delta\tau(-\sigma^2 i^2 - ri(1 - \phi_+^n - \phi_-^n) - r),\\
\gamma_i^{n+1} &= \theta\Delta\tau(-\frac{1}{2}\sigma^2 i^2 - ri(1 - \phi_-^n)),\\
\alpha_i^n &= -(1-\theta)\Delta\tau(-\frac{1}{2}\sigma^2 i^2 + ri\phi_+^n),\\
\beta_i^n &= 1 + (1-\theta)\Delta\tau(-\sigma^2 i^2 - ri(1 - \phi_+^n - \phi_-^n) - r),\\
\gamma_i^n &= -(1-\theta)\Delta\tau(-\frac{1}{2}\sigma^2 i^2 - ri(1 - \phi_-^n)).
\end{aligned}
$$

The variables ϕ_+^n and ϕ_-^n represent the values of the limiter functions

$$\phi_+^n = \frac{\phi(q_{i+\frac{1}{2}}^n)}{2}, \qquad \phi_-^n = \frac{\phi(q_{i-\frac{1}{2}}^n)}{2}$$

explained in detail in section 3.2.3. To simplify the computation, we decided to substitute values ϕ_+^{n+1} and ϕ_-^{n+1} (originally appearing in the flux terms $F_{i-\frac{1}{2}}^{n+1}$ and $F_{i+\frac{1}{2}}^{n+1}$) with values calculated at the previous time layer. Following results show, that this simplification does not significantly affect accuracy of this method.

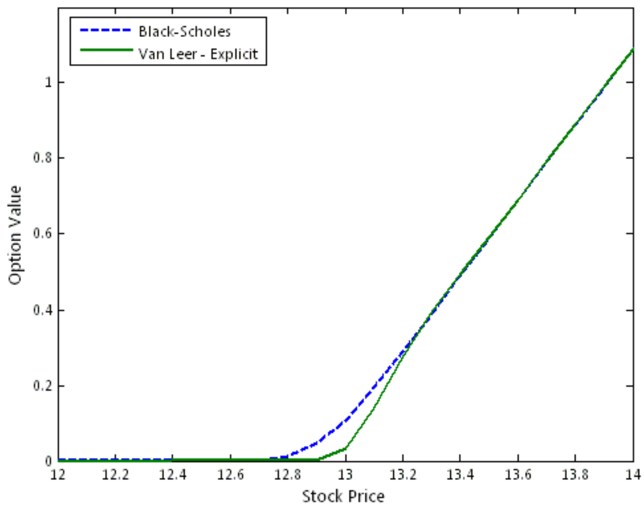

Figure 4.5: *European Call price. Comparison of the Black-Scholes analytical solution and numerical solution, considering the van Leer flux limiter scheme. ($K =$ 15, $T = 1$, $r = 0.15$, $\sigma = 0.01$, $\Delta S = 0.1$, $\Delta\tau = 0.01$, $\theta = 0$)*

Figure (4.5) and Figure (4.6) illustrate, that although we incorporated the van Leer flux limiter, while using uniform grid spacing $\Delta S = 0.1$ and $\Delta\tau = 0.01$, the explicit and implicit method provide results that are still diffused and do not fit properly the call premiums given by the Black-Scholes analytical solution (our benchmark). However, reasonably accurate results might be calculated by these approaches while using finer grid spacing, e.g. considering time spacing $\Delta\tau = 0.001$, but after such modification the computation becomes naturally much more time-demanding. This is true mainly in case of the implicit method. Therefore, one shall focus on the Crank-Nicolson form of the van Leer flux limiter scheme.

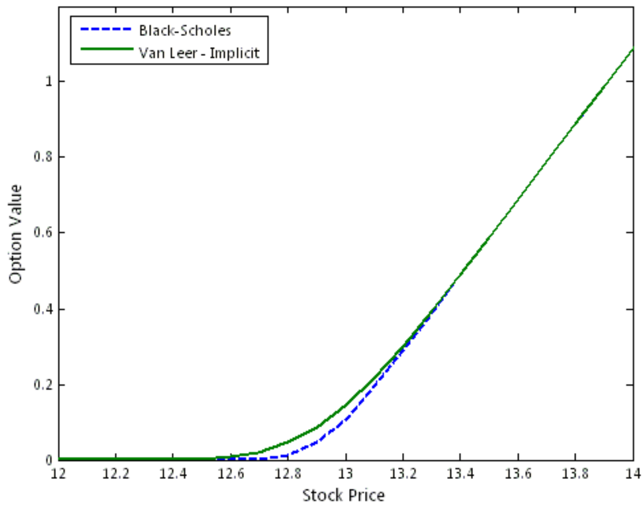

Figure 4.6: *European Call price. Comparison of the Black-Scholes analytical solution and numerical solution, considering the van Leer flux limiter scheme. ($K =$ 15, $T = 1$, $r = 0.15$, $\sigma = 0.01$, $\Delta S = 0.1$, $\Delta\tau = 0.01$, $\theta = 1$)*

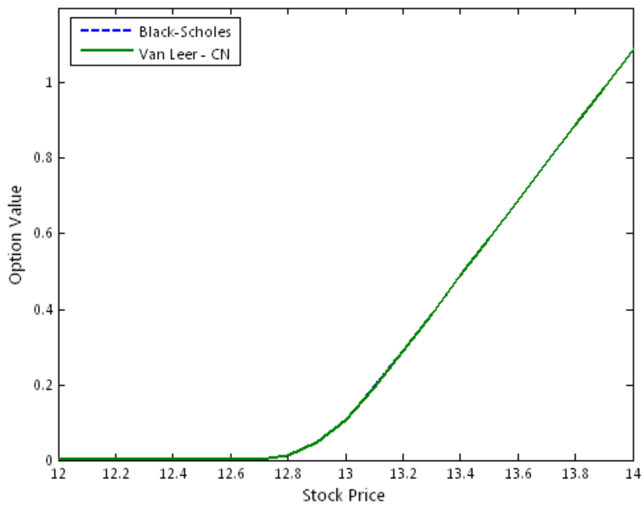

Figure 4.7: *European Call price. Comparison of the Black-Scholes analytical solution and numerical solution, considering the van Leer flux limiter scheme. ($K =$ 15, $T = 1$, $r = 0.15$, $\sigma = 0.01$, $\Delta S = 0.1$, $\Delta\tau = 0.01$, $\theta = \frac{1}{2}$)*

Judging from the Figure (4.7), the van Leer flux limiter technique jointly with the Crank-Nicolson temporal weighting approach is truly the answer to the established problems. As we declared in the theoretical part, this method really effectively handles instabilities of the resulting solution while adding just appropriate measure of the artificial diffusion so that the outcome is accurate and not over-diffused. If considering grid spacing $\Delta S = 0.1$ and $\Delta\tau = 0.01$, as depicted on Figure (4.7), prices of the call option given by the Black-Scholes analytical formula and those provided by the algorithm based on van Leer flux limiting are hardly distinguishable. As illustrated on Figure (4.8), the difference between these two premium curves is clearly minimal and the obtained solution can be undoubtedly accepted as highly accurate.

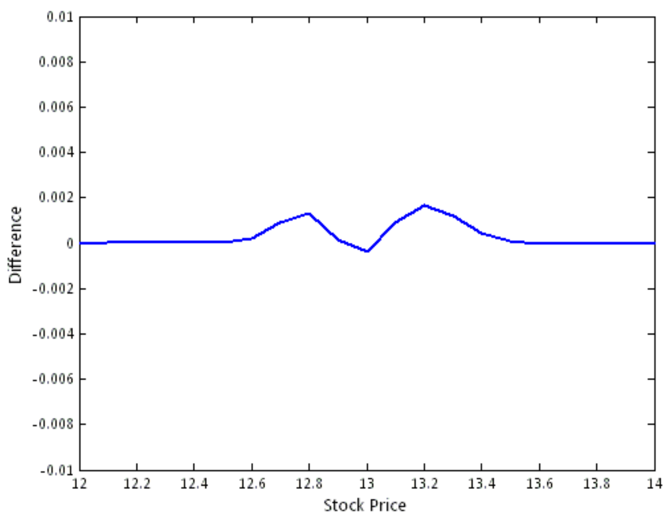

Figure 4.8: *European Call price. Difference between Black-Scholes analytical solution and numerical solution, considering the CN van Leer flux limiter scheme.* *(*$K = 15$*,* $T = 1$*,* $r = 0.15$*,* $\sigma = 0.01$*,* $\Delta S = 0.1$*,* $\Delta\tau = 0.01$*,* $\theta = \frac{1}{2}$*)*

4.2 Valuation of Asian Options

To properly verify the statement, that the van Leer flux limiting truly provides us with highly accurate premiums while numerically pricing Asian options, it is certainly reasonable to present some practical results.

European Floating Strike Call premiums, S_0=100

r	T	σ	Hansen & Jørgensen	van Leer flux limiting
0.03	1/12	0.2	1.390	1.3852
		0.3	2.060	2.0512
		0.4	2.720	2.7161
	4/12	0.2	2.910	2.9146
		0.3	4.230	4.2331
		0.4	5.550	5.5522
	7/12	0.2	3.950	3.9547
		0.3	5.690	5.6920
		0.4	7.420	7.4285
0.05	1/12	0.2	1.430	1.4277
		0.3	2.100	2.0928
		0.4	2.760	2.7571
	4/12	0.2	3.080	3.0858
		0.3	4.390	4.3981
		0.4	5.710	5.7124
	7/12	0.2	4.250	4.2576
		0.3	5.980	5.9792
		0.4	7.700	7.7044
0.05	1/12	0.2	1.490	1.4709
		0.3	2.140	2.1348
		0.4	2.800	2.7984
	4/12	0.2	3.260	3.2628
		0.3	4.560	4.5667
		0.4	5.870	5.8751
	7/12	0.2	4.570	4.5732
		0.3	6.270	6.2743
		0.4	7.980	7.9857

Table 4.1: Comparison of the European Floating Strike Call premiums calculated while using the van Leer flux limiting (C-N temporal weighting) and premiums calculated by Hansen & Jørgensen.

In order to do so, we decided to consider the transformed PDE (2.12), proposed by Wilmott, Dewinne and Howison, which as far as we are concerned has not been studied yet in terms of the van Leer flux limiting. Based on this PDE, we have created the *MATLAB* algorithm which numerically solves this equation (the code can be found in Appendix). We employed the Crank-Nicolson temporal weighting method. The representative European Floating Strike Call premiums calculated with this algorithm are illustrated in Table 4.1 and are compared to premiums calculated by Hansen & Jørgensen [4] (our benchmark).

To briefly comment on the obtained values, as declared, the van Leer flux limiting truly appears to be efficient and very convenient approach while numerically pricing the Asian options. All the premiums in Table 4.1 provided by our algorithm differ from the benchmark premiums in less than 0.02. In order to reach such a high accuracy of results for various values of parameter T and still persist low execution time (we point out, that each illustrated price in Table 4.1 was calculated within 20 seconds.), it was crucial to determine the appropriate values for grid spacing. Hence, we present here to the reader three basic commands, that led to premiums in Table 4.1.

$price = vLCN\_AsianCall(100, 0.03, 1/12, 0.2, 0.2, 0.0005, 0.0005)$
$price = vLCN\_AsianCall(100, 0.03, 4/12, 0.2, 0.5, 0.002, 0.0005)$
$price = vLCN\_AsianCall(100, 0.03, 7/12, 0.2, 0.8, 0.004, 0.001)$

Let us also shortly comment on the issue of various flux limiters. Originally, after calculating the Asian option premiums with the van Leer flux limiter, we wanted to compare these to premiums calculated with some other known limiters. However, after proving such a high accuracy of the van Leer approach, this task seemed to us quite unreasonable.

In conclusion, at this point we are certainly allowed to state, that numerically solving convection-dominated PDEs such as the one for pricing vanilla options considered in previous section or those discribing prices of the Asian options by introducing the van Leer flux limiter can be understood to be the right choice, since the computation is not significantly time-demanding and provided results are proven to be highly accurate.

Chapter 5

Conclusion

Dealing wide range of financial derivatives is nowadays part of everyday routine on the markets all around the developed world. The valuation of all such liquid products is therefore expected to be clear, transparent, fast and most of all, accurate. This master's thesis was dedicated to fair and precise pricing of Asian option contracts, especially to the matter of handling the instabilities which frequently occur while numerically valuing these options via solution of the respective PDEs.

In Chapter 1, we introduced the reader into the concept of the Asian type options, mentioning huge variety of their specific properties and unique features. We pointed out, that although original PDEs for their evaluation are two-dimensional in space, various reductions into one-dimensional PDEs are available. Since this procedure allows potential pricing algorithm to become much less time-demanding, such a simplification is considered to be greatly convenient. At the end of this part, we went very briefly through a couple of ways and methods, how numerical pricing of option contracts might be performed.

Next chapter was dedicated to detailed derivation of PDE models associated to these derivatives. Starting from the classical geometrical Brownian motion, we introduced new path-dependent variable and applied the multidimensional Itô's lemma. After constructing the risk-free portfolio and adopting the no arbitrage assumption, we ended up with the general PDE for pricing path-dependent options. Its specific form for the case of Asian options was derived as a following step. The second part of Chapter 2 was then concerned with concise depiction of three possible methods, in which reduction of the PDE from two into one spatial dimension can be done.

The crucial property of being 'convection-dominated', typical for PDE models of Asian options, was explained in Chapter 3. Without loss of relevance, depiction of this matter was done on the well known Black-Scholes

PDE for pricing plain vanilla options. Problems, arising from this unpleasant feature were subsequently introduced. The finite volume discretization approach was illustrated together with the special form of its flux terms and sink terms and on its basis appropriate methods for dealing with instabilities of provided numerical solution were shown. In particular, the central weighting scheme was presented, which leads to accurate solution only if two relatively strict conditions are fulfilled. Therefore, the first-order upstream weighting scheme was introduced and both advantages and disadvantages of this appproach were discussed. Afterwards the van Leer flux limiter technique, which might be seen as the efficient adjustment to the previous method, was depicted. Its TVD property which implies providing reasonably accurate numerical solutions was theoreticaly demonstrated.

Finally, Chapter 4 consisted of practical results we obtained by creating several *MATLAB* algorithms. In its first part, we analysed valuation of specially designed plain vanilla options. As expected, under certain assumptions and while using fixed uniform grid spacing in both time and space, the central weighting scheme classicaly used in discretization analysis did not provide sufficient results. Visible oscillations occuring while using all three basic temporal weighting schemes were illustrated. So the upstream weighting scheme was analysed. It was proved by the outcomes, that this method truly eliminates any oscillations from the obtained solution. However, while introducing the artificial diffusion into the computation, this scheme affected the solution to become over-diffused. This newly arising problem was effectively managed later on by incorporationg the van Leer flux limiter. Through this method the artificial diffusion was added into the computation just appropriately. The van Leer flux limiter scheme provided us with highly accurate results, while in particular the calculated vanilla premiums differed from the Black-Scholes analytical benchmark in less than 0.002. The second part of this chapter was then eventually dedicated to illustration and verification of efficiency of the analysed approach in terms of the Asian options. Based on the obtained results we were allowed to state, that the van Leer flux limiting can be without dispute accepted as convenient method for pricing these exotic derivatives.

To sum up, the thesis illustrated the concept of the spurious oscillations occuring while pricing convection-dominated PDEs such as those associated with Asian option models and practically proven efficiency and applicability of the van Leer flux limiter approach, through which these problems might by handled.

Notation

PDE	Partial differential equation.
SDE	Stochastic differential equation.
t	Time.
T	Expiration time.
K	Strike price of an option.
$S, S(t)$	Price of the underlying asset at time t.
$I, I(t), A, A(t)$	Average price of the underlying asset at time t.
$V, V(S, I, t)$	Price of the option at time t, asset's price S and asset's average price I.
Π	Risk-free portfolio.
$(q)^+$	$max(q, 0)$.
$(q)^-$	$max(-q, 0)$.
r	Risk-free interest rate.
σ	Constant volatility.
τ	Reversed time variable ($\tau = T - t$).
θ	Temporal weighting parameter ($0 \leq \theta \leq 1$).
$\phi(.)$	Limiter function.
ΔS	Spacial step.
$\Delta\tau$	Time step.
F_i^n	Flux term at time level n and spatial level i.
f_i^n	Source/sink term at time level n and spatial level i.

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Appendix

1. Different Methods of Averaging

In case of continuous sampling, one deals with a price process S_t. If this process S_t is observed at discrete time moments t_i, say equidistantly with the time distance $h := T/n$, one is dealing with a times series S_{t_1}, S_{t_2}, . . . , S_{t_n}. If we denote the average value of asset price $A(T)$ and indices for arithmetical and geometrical average a and g respectively, in the discrete case

$$A_a(T) = \frac{1}{n}\sum_{i=1}^{n} S_{t_i} = \frac{1}{T}h\sum_{i=1}^{n} S_{t_i},$$

$$A_g(T) = \left(\prod_{i=1}^{n} S_{t_i}\right)^{1/n} = exp\left(\frac{1}{n}log\prod_{i=1}^{n} S_{t_i}\right) = exp\left(\frac{1}{n}\sum_{i=1}^{n} logS_{t_i}\right).$$

For continuously sampled price process, average value can be expressed as follows:

$$A_a(T) = \frac{1}{T}\int_0^T S_t dt,$$

$$A_g(T) = exp\left(\frac{1}{T}\int_0^T logS_t dt\right).$$

As for weighted averaging, one possible weightening method in case of arithmetic average is by the formula

$$A_a^w(T) = \frac{1}{\int_0^T a(t)dt}\int_0^T a(T-t)S_t dt,$$

where the kernel function $a(.) \geq 0$, $\int_0^\infty a(t)dt < \infty$ is usually defined as $a(t) = e^{-\lambda t}$ for some constant $\lambda > 0$.

2. Multidimensional Itô's lemma

Assume, that the stochastic processes $X_t^{(i)}$ for $i = 1, ..., n$ can be expressed through the following stochastic differential equations:

$$\frac{dX_t^{(1)}}{X_t^{(1)}} = \mu^{(1)} dt + \tilde{\sigma}^{(1)} dB_t^{(1)},$$

$$\frac{dX_t^{(2)}}{X_t^{(2)}} = \mu^{(2)} dt + \tilde{\sigma}^{(2)} dB_t^{(2)},$$

$$\vdots$$

$$\frac{dX_t^{(n)}}{X_t^{(n)}} = \mu^{(n)} dt + \tilde{\sigma}^{(n)} dB_t^{(n)},$$

where $d[B_t^{(i)}, B_t^{(j)}] = \rho_{ij} dt$.

Then, the general form of Itô's lemma for any function $f(X_t^{(1)}, ..., X_t^{(n)}, t)$ reads

$$\begin{aligned} df(X_t^{(1)}, ..., X_t^{(n)}, t) = \\ & \left[\left(\frac{\partial}{\partial t} + \sum_{i=1}^{n} \mu^{(i)} X_t^{(i)} \frac{\partial}{\partial X_t^{(i)}} \right.\right. \\ & \left.\left. + \frac{1}{2} \sum_{i,j=1}^{n} X_t^{(i)} X_t^{(j)} \tilde{\sigma}^{(i)} \tilde{\sigma}^{(j)} \rho_{ij} \frac{\partial^2}{\partial X_t^{(i)} \partial X_t^{(j)}} \right) f \right] dt \\ & + \sum_{i=1}^{n} \frac{\partial f}{\partial X_t^{(i)}} \tilde{\sigma}^{(i)} dB_t^{(i)}. \end{aligned}$$

3. MATLAB code for European Vanilla Call options - the C-N van Leer flux limiter method

```
function price = vLCN_VanillaCall(S0 ,K, r, T, sigma, Smax, dS, dt)

% Copyright (c) 2011 by Adam Rehurek
%*************************************************************%
% vLCN_VanillaCall.m - supplementary program to
% Adam Rehurek (2011) Stable Numerical Methods for PDE Models of
% Asian Options (Master's thesis), Department of Mathematics,
% Physics and Electrical Engineering, Halmstad University, Sweden
%*************************************************************%

% set up grid and adjust increments if necessary
M = round(Smax/dS);
dS = Smax/M;
N = round(T/dt);
dt = T/N;
matval = zeros(M+1,N+1);
vetS = linspace(0,Smax,M+1)';
veti = 0:M;
vetj = 0:N;

% set up boundary conditions
matval(:,1) = max(vetS-K,0);
matval(1,:)= 0;
matval(M+1,:)  = Smax - K*exp(-r*dt*(vetj));

for j=2:1:N+1
  % calculate values of the limiter function
  acomp = zeros(M+1,1);
  bcomp = zeros(M+1,1);
  ccomp = zeros(M+1,1);
  for i=2:M
    a=matval(i-1,j-1);
    b=matval(i,j-1);
    c=matval(i+1,j-1);
    if (a-b)==0
      ae1=0;
    else
```

```
      ae1=(((abs((b-c)/(a-b))+(b-c)/(a-b))/(1+abs((b-c)/...
        ...(a-b))))/2);
    end
    if i<M
      d=matval(i+2,j-1);
      if (b-c)==0
        ae2=0;
      else
        ae2=(((abs((c-d)/(b-c))+(c-d)/(b-c))/...
          ...(1+abs((c-d)/(b-c))))/2);
      end
    else
      if (b-c)==0
        ae2=0;
      else
        ae2=ae1;
      end
    end
    acomp(i,1)=ae1;
    bcomp(i,1)=ae1+ae2;
    ccomp(i,1)=ae2;
  end

  % set up coefficients
  alpha = 0.5*dt*( -0.5*(sigma^2)*(veti.^2)+r*veti.*acomp');
  beta = -dt*0.5*( -(sigma^2)*(veti.^2) - r*veti - r +...
      ...r*veti.*bcomp');
  gamma = 0.5*dt*( -0.5*(sigma^2)*(veti.^2) - r*veti +...
      ...r*ccomp'.*veti);
  M1 = diag(alpha(3:M),-1) + diag(1+beta(2:M)) +...
      ...diag(gamma(2:M-1),1);
  [L,U] = lu(M1);
  M2 = -diag(alpha(3:M),-1) + diag(1-beta(2:M)) -...
      ...diag(gamma(2:M-1),1);

  % calculate next time layer values
  matval(2:M,j) = U \ (L \ (M2*matval(2:M,j-1)));
end

price = interp1(vetS, matval(:,N+1), S0);
```

4. MATLAB code for European Floating Strike Call options - the C-N van Leer flux limiter method

```
function price = vLCN_AsianCall(S0, r, T, sigma, Smax, dS, dt)

% Copyright (c) 2011 by Adam Rehurek
%**************************************************************%
% vLCN_AsianCall.m - supplementary program to
% Adam Rehurek (2011) Stable Numerical Methods for PDE Models of
% Asian Options (Master's thesis), Department of Mathematics,
% Physics and Electrical Engineering, Halmstad University, Sweden
%**************************************************************%

% set up grid and adjust increments if necessary
M = round(Smax/dS);
dS = Smax/M;
N = round(T/dt);
dt = T/N;
matval = zeros(M+1,N+1);
vetS = linspace(0,Smax,M+1)';
veti = 0:M;
vetj = 0:N;

% set up terminal and upper boundary condition
matval(:,1) = max(1-vetS/T,0);
matval(M+1,:)  = 0;

for j=2:1:N+1
  %set up lower boundary condition
  matval(1,j) = -((dt/dS)*(matval(1,j-1)-matval(2,j-1)))+...
     ...matval(1,j-1);

  %calculate values of the limiter function
  lfacomP = zeros(M+1,1); %limiter function a-component(positive)
  lfbcomP = zeros(M+1,1);
  lfccomP = zeros(M+1,1);
  lfacomN = zeros(M+1,1); %limiter function a-component(negative)
  lfbcomN = zeros(M+1,1);
  lfccomN = zeros(M+1,1);
```

```
for i=2:M
  if (r*i*dS-1)<=0 %if velocity is positive, V_(up)=V_(i+1)
    a = matval(i-1,j-1);
    b = matval(i,j-1);
    c = matval(i+1,j-1);
    if ( a-b )==0
      lime1P = 0; %limiter expression 1(positive)
    else
      lime1P = (((abs((b-c)/(a-b))+(b-c)/(a-b))/...
        ...(1+abs((b-c)/(a-b))))/2);
    end
    if i<M
      d = matval(i+2,j-1);
      if ( b-c )==0
        lime2P = 0; %limiter expression 2(positive)
      else
        lime2P = (((abs((c-d)/(b-c))+(c-d)/(b-c))/...
          ...(1+abs((c-d)/(b-c))))/2);
      end
    else
      if ( b-c )==0
        lime2P = 0;
      else
        lime2P = lime1P;
      end
    end
  else
    lime1P = 0;
    lime2P = 0;
  end

  if (r*i*dS-1)>0 %if velocity is negative, V_(up)=V_(i)
    b = matval(i-1,j-1);
    c = matval(i,j-1);
    if i>2
      a = matval(i-2,j-1);
      if ( a-b )==0
        lime1N = 0; %limiter expression 1(negative)
      else
        lime1N = (((abs((b-c)/(a-b))+(b-c)/(a-b))/...
          ...(1+abs((b-c)/(a-b))))/2);
```

```
      end
    else
      lime1N = 0;
    end
    d = matval(i+1,j-1);
    if ( b-c )==0
      lime2N = 0; %limiter expression 2 (negative)
    else
      lime2N = (((abs((c-d)/(b-c))+(c-d)/(b-c))/...
        ...(1+abs((c-d)/(b-c))))/2);
    end
  else
    lime1N = 0;
    lime2N = 0;
  end
  lfacomP(i,1) = lime1P;
  lfbcomP(i,1) = lime1P+lime2P;
  lfccomP(i,1) = lime2P;
  lfacomN(i,1) = lime1N;
  lfbcomN(i,1) = lime1N+lime2N;
  lfccomN(i,1) = lime2N;
end

% set up coefficients, if velocity is positive
deltaP = zeros(M+1,1);
alphaP = -0.5*dt*(0.5*sigma^2*veti.*veti+...
  ...(r*veti-1/dS).*lfacomP');
betaP = 0.5*dt*(sigma^2*veti.^2-r*veti+1/dS+...
  ...(r*veti-1/dS).*lfbcomP');
gammaP = 0.5*dt*(-(1/2)*sigma^2*veti.^2+(r*veti-1/dS)-...
  ...(r*veti-1/dS).*lfccomP');
% set up coefficients, if velocity is negative
deltaN = -0.5*dt*(r*veti-1/dS).*lfacomN';
alphaN = dt*0.5*(-(1/2)*sigma^2*veti.^2-(r*veti-1/dS)+...
  ...(r*veti-1/dS).*lfbcomN');
betaN = 0.5*dt*(sigma^2*veti.^2+r*veti-1/dS-...
  ...(r*veti-1/dS).*lfccomN' );
gammaN = -0.5*dt*((1/2)*sigma^2*veti.^2);

delta = zeros(M+1,1);
alpha = zeros(M+1,1);
```

```
  beta = zeros(M+1,1);
  gamma = zeros(M+1,1);
  for i=2:M
    if (r*i*dS-1)<=0
      delta(i) = deltaP(i);
      alpha(i) = alphaP(i);
      beta(i) = betaP(i);
      gamma(i) = gammaP(i);
    else
      delta(i) = deltaN(i);
      alpha(i) = alphaN(i);
      beta(i) = betaN(i);
      gamma(i) = gammaN(i);
    end
  end
  M1 = diag(delta(4:M),-2)+diag(alpha(3:M),-1)+...
    ...diag(1+beta(2:M))+diag(gamma(2:M-1),1);
  [L,U] = lu(M1);
  M2 = -diag(delta(4:M),-2)-diag(alpha(3:M),-1)+...
    ...diag(1-beta(2:M))-diag(gamma(2:M-1),1);
  aux = zeros(M-1,1);
  auxx = zeros(M-1,1);
  aux(1) = alpha(2)*matval(1,j);
  aux(M-1) = gamma(M)*matval(M+1,j);
  auxx(1) = -alpha(2)*matval(1,j-1);
  auxx(M-1) = -gamma(M)*matval(M+1,j-1);
  % calculate next time layer values
  matval(2:M,j) = U \ (L \ (M2*matval(2:M,j-1)+auxx-aux));
end

price = S0*interp1(vetS,matval(:,N+1),0);
```