Technical report, IDE1131 , September 7, 2011

Energy Derivatives Pricing

Master's Thesis in Financial Mathematics

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MASTER'S PROGRAMME IN FINANCIAL MATHEMATICS

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Technical Report IDE1131

Master's thesis in Financial Mathematics, 15 ECTS credits

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September 7, 2011

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Preface

This paper would not have been possible without our supervisor Mikhail L. Nechaev. We would like to thank him for the support and the guidance through our thesis. We are also deeply grateful to our Program coordinator Ljudmila A. Bordag for the inspiration and wise advices.

Abstract

In this paper we examine energy derivatives pricing. The previous studies considered the same source of uncertainty for the spot and the futures prices. We investigate the problem of futures pricing with two independent sources of risk. In general the structure of the oil and gas futures markets is closely related to some stock indices. Therefore, we develop a model for the futures market and compound derivatives with pricing in accordance with the correspondent index. We derive a framework for energy derivatives pricing, compute the price of the European call option on futures and corresponding hedging strategy. We calculate the price of the European call option adjusted for an index level, study the American put option on futures and corresponding hedging strategies.

Contents

Chapter 1

Introduction

The development of the financial market is followed by the rapid growth of the derivatives market. New types of financial instruments are being constructed. Nevertheless, there exist different frameworks for pricing futures and option contracts, each of them has its own advantages and disadvantages.
The classical approach to the futures pricing implies that, in the case of the constant risk-free interest rate, the contract's price is presented by the price of underlying assets corrected to the time to maturity

$$F_t = S_t e^{r(T-t)},$$

while in more general settings

$$F_t = \mathsf{E}_{\mathsf{Q}}[S_T|\mathscr{F}_t],$$

where Q is a risk-neutral measure.
Black-Scholes (1973) introduced the model, where the spot price dynamics is given by the standard geometrical Brownian motion with the constant, time independent parameters μ and σ

$$dS_t = S_t(\mu dt + \sigma dW_t).$$

The two factor mean-reverting model was described by Gibson-Schwarz (1990) by introduction of the second risk factor — convenience yield δ — into the model

$$dS_t = S_t(r - \delta_t)dt + S_t\sigma_S dW_{S,t},$$

where the convenience yield is a mean reverting stochastic process of the Ornstein-Uhlenbeck type

$$d\delta_t = \kappa_\delta(\alpha_\delta - \delta_t)dt + \sigma_\delta dW_{\delta,t}.$$

The framework has higher explanatory power in contrast to the Black-Scholes approach.
The spot price and the convenience yield were the object of investigation of many models and futures contracts were not considered as a separate assets. Consequently, futures prices were defined just endogenously. Another disadvantage of the framework is that the convenience yield turns out to be unobservable.
Clewlow-Strickland (1999) [1] proposed a multi-factor model for valuating futures prices

$$dF(T,t) = F(t,T)\sum_{i=1}^{n}\sigma_i(t,T)dW_t^i,$$

where $W_t^i,\ i = 1,\dots,n$ are independent Brownian motions. The framework is consistent with market futures prices and volatilities.
All the previous approaches propose the same sources of risk for the spot and the futures prices. It can be unjustified for energy derivatives, which can have additional sources of risk. In this sector of financial market the volume of deals with derivatives exceed significantly spot market trading, while for some commodities spot market is not representative at all.
For this reason we consider a framework for the energy derivatives pricing, where individual source of uncertainty — an independent Brownian motion — is added to a futures price process. In our case the spot price follows the standard Brownian motion. Based on this model we price the European type contingent claim.
Energy futures contracts are widespread financial tools, because economy has a straight dependence on the oil and gas prices. In oil-exporting countries the structure of oil and gas futures markets is closely related to the term structure of GDP index. Taking into consideration this correlation, we construct a financial instrument with lower volatility. We price the futures contract with worth, expressed through the index value. Relying on the model, we provide an explicit pricing formula for European call option and consider the optimal stopping problem for the put option of American type.
The paper is structured as follows. In Chapter 2 we present essentials of probability theory and derive a framework for energy derivatives pricing. In Chapter 3 we compute European option on futures and corresponding hedging strategy. Chapter 4 shows calculations of price of European option adjusted for an index level. In Chapter 5 we study perpetual American put option on futures. Chapter 6 investigates the investment problem. Chapter 7 contains the main conclusions.

Chapter 2

The General Theory

In Chapter 2 we derive a model of futures market. The classical framework refers to a bank interest rate as a parameter of the term structure. We calibrate it by a stochastic forward rate which, firstly, does not coincide with a bank interest rate, secondly, is a ratio of a forward price to a spot one. We consider a model with two sources of uncertainty. One describes the behavior of the underlying assets and the other reflects the movement of the forward rate. We provide an explicit pricing formula for the futures contract.

2.1 Some essentials of the probability theory

This Section introduces some definitions and propositions, which we use in this paper.
For our future considerations we present a filtered probability space.

Definition 1 (Probability Space) $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}, \mathsf{P})$ *filtered* ***probability space*** *with set of outcomes* Ω*, sigma algebra* $\mathscr{F}$*, filtration* $\{\mathscr{F}_t\}$ *and probability measure* P.

On this probability space we consider (B, S, F)-market, where

- $B = (B_t)_{t\geq 0}$ is a risk-free assets, and
- $S = (S_t)_{t\geq 0}$ is a spot price of the risk assets, and
- $F^{T_i} = (F_t^{T_i})_{t\geq 0},\ i = 1, \ldots, n$ are prices of futures on the risk assets with expiration dates $T_i \geq t$,

where $(B_t)_{t\geq 0},\ (S_t)_{t\geq 0},\ (F_t^{T_i})_{t\geq 0}$ are adapted to the filtration $\{\mathscr{F}_t\}$, which satisfies standard conditions.

Let predictable stochastic process

$$\pi = (\theta, \delta, \gamma) \tag{2.1}$$

present a portfolio, where $\theta = (\theta_t)_{t\geq 0}$ is a number of bonds, $\delta = (\delta_t)_{t\geq 0}$ is a number of stocks, $\gamma = (\gamma_t^1, \dots, \gamma_t^n)_{t\geq 0}$ is a number of futures and θ, δ, γ are $\mathscr{F}_{s,s<t}$ –measurable. The value of the portfolio π at the moment t amounts to

$$X_t^\pi = \theta_t B_t + \delta_t S_t, \ t \geq 0, \tag{2.2}$$

where futures are not included, because no payments between a writer and a holder occur at the moment of a new contract origination.
The portfolio π is self-financing, if

$$dX_t^\pi = \theta_t dB_t + \delta_t dS_t + \sum_{i=1}^{n} \gamma_t^i dF_t^{T_i}, \tag{2.3}$$

in other words, the changes in the portfolio value can be caused by the changes in the prices of the portfolio assets only.
Moreover, we require the market to be non-arbitrage and complete. Following Shiryaev [4] we define the martingale criterion of the absence of arbitrage.

Definition 2 (Martingale measure) *The **martingale measure** is a probability measure* Q *equivalent to the physical one, such that the processes*

$$\frac{S}{B} \textit{and } F_t^{T_i} \tag{2.4}$$

are Q*-martingales, i.e.*

$$\mathsf{E}_\mathsf{Q}\left|\frac{S_t}{B_t}\right| < \infty \textit{ and } \mathsf{E}_\mathsf{Q}|F_t^{T_i}| < \infty \tag{2.5}$$

for all $i = 1, \dots, n$ *and* $t \geq 0$ *and*

$$\mathsf{E}_\mathsf{Q}\left(\frac{S_t}{B_t}\middle|\mathscr{F}_{u,u<t}\right) = \frac{S_u}{B_u} \textit{ and } \mathsf{E}_\mathsf{Q}(F_t^{T_i}|\mathscr{F}_{u,u<t}) = F_u^{T_i}(\mathsf{Q}-\textit{almost surely}), \tag{2.6}$$

E_Q *is an expectation with respect to the martingale measure* Q*.*

Proposition 1 *There are no arbitrage opportunities in the (B,S,F)-market if and only if a set of martingale measures is not empty.*

Proof: We can follow the standard proof, but we have to show that the discounted capital is a martingale with respect to the martingale measure.

$$
\begin{aligned}
d\left(\frac{X_t^\pi}{B_t}\right) &= \frac{dX_t^\pi}{B_t} + X_t^\pi d\left(\frac{1}{B_t}\right) \\
&= \frac{\theta_t dB_t + \delta_t dS_t + \sum_{i=1}^n \gamma_t^i dF_t^{T_i}}{B_t} + \Big(\theta_t B_t + \delta_t S_t\Big) d\left(\frac{1}{B_t}\right) \\
&= \theta_t\Big(\frac{dB_t}{B_t} + B_t d\left(\frac{1}{B_t}\right)\Big) + \delta_t\Big(\frac{dS_t}{B_t} + S_t d\left(\frac{1}{B_t}\right)\Big) + \sum_{i=1}^n \gamma_t^i \frac{dF_t^{T_i}}{B_t} \\
&= \theta_t d\left(\frac{B_t}{B_t}\right) + \delta_t d\left(\frac{S_t}{B_t}\right) + \sum_{i=1}^n \gamma_t^i \frac{dF_t^{T_i}}{B_t} \\
&= \delta_t d\left(\frac{S_t}{B_t}\right) + \sum_{i=1}^n \gamma_t^i \frac{dF_t^{T_i}}{B_t}. \qquad (2.7)
\end{aligned}
$$

To show that there is no arbitrage in the market we have to change the initial physical measure P for the risk-neutral probability measure Q. Following Shreve [7], we define a change of measure.

Definition 3 (Radon-Nikodým derivative) *Let $(\Omega, \mathscr{F}, \mathsf{P})$ be a probability space, let Q be another probability measure on $(\Omega, \mathscr{F})$ that is equivalent to P, and let Z be an almost surely positive random variable that relates P and Q via*

$$\mathsf{Q}(A) = \int_A Z(\omega) d\mathsf{P}(A). \qquad (2.8)$$

*Then Z is called the **Radon-Nikodým derivative** of Q with respect to P, and we write*

$$Z = \frac{d\mathsf{Q}}{d\mathsf{P}}. \qquad (2.9)$$

To present a process as a martingale, following Bulinski and Shiryaev [5] we use Itô-Clark theorem.

Theorem 1 (Itô-Clark) *Let $W = (W_t)_{t\geq 0}$ be a Brownian motion, $\mathbb{F}^W = \{\mathscr{F}_t^W\}_{t\geq 0}$, $\mathscr{F}_t^W$ be a natural filtration of the Brownian motion.*
X is a random variable $X \in \mathscr{F}_T^W$ then the following statements are correct

1. *If $\mathsf{E}[X^2] < \infty$, then there exists a stochastic process φ_s such that*

$$\mathsf{E}\left[\int_0^T \varphi_s^2 ds\right] < \infty, \qquad (2.10)$$

and

$$X = \mathsf{E}[X] + \int_0^T \varphi_s dW_s. \tag{2.11}$$

2. *If* $\mathsf{E}|X| < \infty$, *then there exists a stochastic process* φ_s *such that*

$$\mathsf{P}\left(\int_0^T \varphi_s^2 ds < \infty \right) = 1, \tag{2.12}$$

and

$$X = \mathsf{E}[X] + \int_0^T \varphi_s dW_s. \tag{2.13}$$

3. *If* X *is a positive random variable* (*i.e.* $\mathsf{P}(X > 0) = 1$) *and* $\mathsf{E}X < \infty$, *then there exists a stochastic process* φ_s *such that*

$$\mathsf{P}\left(\int_0^T \varphi_s^2 ds < \infty \right) = 1, \tag{2.14}$$

and X *could be presented in the form*

$$X = \mathsf{E}[X] \exp\left\{ - \int_0^t \varphi_s dW_s - \frac{1}{2} \int_0^t \varphi_s^2 ds \right\}, t \geq 0. \tag{2.15}$$

From (2.11) and (2.15) it follows that

1) every square integrable martingale $M = \{M_t\}, M_t \in \mathscr{F}_T^W$ has a unique representation

$$M_t = M_0 + \int_0^t \varphi_s dW_s, \ \mathsf{E}\left[\int_0^T \varphi_s^2 ds \right] < \infty; \tag{2.16}$$

and

2) every positive martingale $M = \{M_t\}_{t \geq 0}, \mathsf{E}[|M_t|] < \infty, M_t \in \mathscr{F}_T^W$ has a unique representation

$$M_t = M_0 \exp\left\{ - \int_0^t \varphi_s dW_s - \frac{1}{2} \int_0^t \varphi_s^2 ds \right\}, t \geq 0. \tag{2.17}$$

For a construction of martingale measures we use Girsanov theorem.

Theorem 2 (Girsanov) *We consider a process* $X = (X_t, \mathscr{F}_t)_{t\geq 0}$, *defined on a filtered probability space* $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathsf{P})$, *which is Itô process with differential*

$$dX_t = a_t(\omega)dt + dW_t, \quad x_0 = 0, \tag{2.18}$$

where $a = (a_t(\omega), \mathscr{F}_t)_{t\geq 0}$ *is a process satisfying the condition*

$$\mathsf{P}(\int_0^t |a_s(\omega)|ds < \infty) = 1, \quad t \geq 0, \tag{2.19}$$

and $W = (W_t, \mathscr{F}_t)_{t\geq 0}$ *is a standard Brownian motion.*

The process $\widetilde{W} = \left(\widetilde{W}_t, \mathscr{F}_t\right)_{t\geq 0}$ *with*

$$\widetilde{W}_t = W_t - \int_0^t (\widetilde{\varphi}_s - \varphi_s)ds \tag{2.20}$$

is standard Brownian motion with respect to the measure Q *and*

$$dX_t = \widetilde{\varphi}_t(\omega)dt + d\widetilde{W}_t. \tag{2.21}$$

Corollary 1 *Let*

$$X_t = W_t - \phi \int_0^t \varphi_s ds, \tag{2.22}$$

where $\phi \in \mathbb{R}$, *and let*

$$Z_t^\phi = \exp\left\{ -\phi \int_0^t \varphi_s dW_s - \frac{\phi^2}{2} \int_0^t \varphi_s^2 ds \right\}. \tag{2.23}$$

Assume that $\mathsf{E}Z_\infty^\phi = 1$ *and set* $d\mathsf{Q}^\phi = Z_\infty^\phi d\mathsf{P}$. *Then the process* $X = \left(X_t\right)_{t\geq 0}$ *is a standard Brownian motion with respect to measure* Q^ϕ.
If $\mathsf{E}Z_T^\phi = 1$ *for some finite* T, *then* $X = \left(X_t\right)_{t\geq 0}$ *is a standard Brownian motion on the interval* $[0, T]$ *with respect to measure* Q_T^ϕ *such that* $d\mathsf{Q}_T^\phi = Z_T^\phi d\mathsf{P}_T$, *where* $\mathsf{P}_T = \mathsf{P}|\mathscr{F}_T$.

2.2 The futures market model

In this Section we provide the general information about the structure of the price processes in our model, and construct a price process of the futures contracts. As a result, we obtain the explicit formulæ.
Futures is a type of contract which obligates the writer and the buyer to execute the contract at the agreed-upon price on the maturity date. The fundamental characteristics of the futures prices are

- the volatility of the futures contract is higher than the volatility of the underlying assets and decreases with time,
- the price of the futures contract at the expiration equals the price of the underlying assets

$$F_T^T = S_T. \tag{2.24}$$

Within the classical framework the value of the futures F_t can be calculated by compounding the present value of a non-dividend paying assets S_t at the time t to the expiry date T by the rate of risk-free return r

$$F_t = S_t e^{r(T-t)}. \tag{2.25}$$

The disadvantage of the model is a constant discounting parameter. Therefore it fits only the simplest term structure of interest rates. In reality, there exists no single risk-free return. Different types of bonds are traded in the market. The set of them represents a complicated term structure. Besides that, the futures as a traded contract has its own source of uncertainty. For our model to be more consistent with the market we consider the stochastic forward rate

$$F_t^T = S_t \exp\left\{\int_t^T f(t,v)dv\right\}, 0 \leq t \leq T, \tag{2.26}$$

where the price process of the underlying assets S_t is given by the geometrical Brownian motion

$$dS_t = S_t\left(\mu dt + \sigma dW_t^1\right), \tag{2.27}$$

and $f(t,T)$ is the forward rate given by the stochastic integral equation

$$f(t,T) = f(0,T) + \int_0^t \xi(u,T)du + \int_0^t \sigma(u,T)dW_u^2. \tag{2.28}$$

Note, that $f(t,T)$ is not a bank interest rate, but the rate of difference in prices of the underlying assets and the futures, which reflects the structure of the derivatives market.

To derive the differential of futures process, firstly, we rewrite the forward rate process in the differential form

$$df(t,T) = \xi(t,T)dt + \sigma(t,T)dW_t^2, \tag{2.29}$$

then define the differential of $\int_t^T f(t,v)dv$ in the following way

$$d\left(\int_t^T f(t,v)dv\right) = -f(t,t) + \int_t^T df(t,v)dv. \tag{2.30}$$

It is easy to see, that the left-hand side includes t both in the integration limit and in the integrand. Consequently, as a result of the differentiation we get the sum of two terms. The first one is the result of taking differential with respect to the lower limit of the integral. The second is the integral of the differential of $f(t,v)$. Using (2.29) and (2.30) we immediately obtain

$$d\left(\int_t^T f(t,v)dv\right) = -f(t,t) + \int_t^T \left[\xi(t,T)dt + \sigma(t,T)dW_t^2\right]dv. \tag{2.31}$$

Then we represent (2.31) as a martingale. For this purpose we assume, that $\xi(t,T)$ and $\sigma(t,T)$ are integrable, and reverse the order of integration

$$\int_t^T \xi(t,v)dtdv = \int_t^T \xi(t,v)dvdt = \xi^*(t,T)dt, \tag{2.32}$$

$$\int_t^T \sigma(t,v)dW_t^2 dv = \int_t^T \sigma(t,v)dvdW_t^2 = \sigma^*(t,T)dW_t^2, \tag{2.33}$$

and finally get

$$d\left(\int_t^T f(t,v)dv\right) = -f(t,t) + \xi^*(t,T)dt + \sigma^*(t,T)dW_t^2. \tag{2.34}$$

Applying the Itô formula in the form

$$df(t,X(t)) = \frac{\partial f(t,X(t))}{\partial t}dt + \frac{\partial f(t,X(t))}{\partial x}dX(t) + \frac{1}{2}\frac{\partial^2 f(t,X(t))}{\partial x^2}d\langle X;X\rangle_t$$

to $e^{\int_t^T f(t,v)dv} = g\left(\int_t^T f(t,v)dv\right)$, where $g(x) = e^x$, we can find the differential

$$\begin{aligned} d\left(e^{\int_t^T f(t,v)dv}\right) &= g'\left(\int_t^T f(t,v)dv\right)d\left(\int_t^T f(t,v)dv\right) \\ &\quad + \frac{1}{2}g'\left(\int_t^T f(t,v)dv\right)\left[dg''\left(\int_t^T f(t,v)dv\right)\right]^2 \\ &= \exp\left\{\int_t^T f(t,v)dv\right\}\Big[-f(t,t) + \xi^*(t,T) \\ &\quad + \frac{1}{2}\big(\sigma^*(t,T)\big)^2\Big]dt + \exp\left\{\int_t^T f(t,v)dv\right\}\sigma^*(t,T)dW_t^2. \end{aligned} \tag{2.35}$$

Further, we compute the differential of the futures process itself. We use the Itô product rule to define the futures contract price process

$$\begin{aligned} dF_t &= d(S_t)e^{\int_t^T f(t,v)dv} + S_t d(e^{\int_t^T f(t,v)dv}) + d\langle S_t; e^{\int_t^T f(t,v)dv}\rangle \\ &= F_t\Bigg[\Big(\mu - f(t,t) + \xi^*(t,T) + \frac{1}{2}\big(\sigma^*(t,T)\big)^2\Big)dt \\ &\quad + \sigma dW_t^1 + \sigma^*(t,T)dW_t^2\Bigg]. \end{aligned} \tag{2.36}$$

For the sake of simplicity, we suppose that $\alpha = \mu - f(t,t) + \xi^*(t,T) + \frac{1}{2}\big(\sigma^*(t,T)\big)^2$ is a constant and $\sigma^*(t,T) = \beta(T-t)$. Then the futures price process is defined by the following stochastic differential equation

$$dF_t^T = F_t^T\Big(\alpha dt + \sigma dW_t^1 + \beta(T-t)dW_t^2\Big), \tag{2.37}$$

which has an exact solution

$$\begin{aligned} F_t^T &= F_0^T \exp\Big\{\alpha t - \frac{\sigma^2}{2}t + \sigma W_t^1 \\ &\quad + \int_0^t \beta(T-s)dW_s^2 - \frac{1}{2}\int_0^t \beta^2(T-s)^2 ds\Big\}. \end{aligned} \tag{2.38}$$

Chapter 3

The European Call on Futures

This Chapter is devoted to the European type options. Here we provide the analytical solution of the valuation problem of the European contingent claim on the futures contract. Also we derive the hedging strategy for this financial instrument.

The European option is a contingent claim, that gives the right to a holder to buy or sell the certain amount of the underlying assets at the certain date for the agreed-upon price.

We consider the market model with the futures contracts $F(t,T_1),\ F(t,T_2)$ and the risk-free bank account B_t

$$\begin{cases} dF(t,T_i) = F(t,T_i)\Big(\alpha dt + \sigma dW_t^1 + \beta(T_i - t)dW_t^2\Big), i = 1,2, \\ dB_t = rB_t dt,\ B_0 = 1, \end{cases}$$

where $\alpha, \sigma, \beta, r \in \mathbb{R}_+^1$, W_t^1 and W_t^2 are independent Brownian motions.

As we consider the European call option, the pay-off or gain function is

$$f_T = \left[\begin{array}{l} (F(T,T) - K), \text{for } F(T,T) > K, \\ 0, \text{otherwise,} \end{array} \right. \tag{3.1}$$

where $F(T,T)$ is the futures price at the maturity and K is the exercise price.

Proposition 2 (European Call Option) *If the market is complete and arbitrage-free, then the option price is an expectation of the discounted gain function with respect to a martingale measure*

$$\mathbb{C} = B_0 \mathsf{E}_Q \left[\frac{f_T}{B_T} \right]. \tag{3.2}$$

3.1 Arbitrage and completeness

First of all, we have to prove, that there exist no arbitrage opportunities and that we can hedge any security in the market. Towards this end, we show, that there is a unique martingale measure in the market and according to (Proposition 1) the market is complete and non-arbitrage.
Let us consider two probability measures P and Q on a filtered space $(\Omega, \mathscr{F}, F)$ and assume, that Q is absolutely continuous with respect to P. Then we look for a density process of Q with respect to P, i.e. a martingale Z on $(\Omega, \mathscr{F}, F, \mathsf{P})$ such that $\forall t \in \mathbb{R}^+$, Z_t is the Radon-Nikodým derivative $\frac{d\mathsf{Q}_{|\mathscr{F}_t}}{d\mathsf{P}_{|\mathscr{F}_t}}$ of the restrictions of Q and P to $(\Omega, \mathscr{F}_t)$.
Let Z_t be a density process for two Brownian motions in the form

$$Z_t = \exp\left\{ - \int_0^t \varphi_s dW_s^1 - \int_0^t \frac{\varphi_s^2}{2} ds - \int_0^t \psi_s dW_s^2 - \int_0^t \frac{\psi_s^2}{2} ds \right\}. \tag{3.3}$$

The process $F(t, T_i)Z_t$ should be a martingale with respect to the physical measure; i.e. the drift coefficient in the differential $d\Big(F(t, T_i)Z_t\Big)$ should be equal to 0

$$\begin{aligned} d\Big(F(t,T_i)Z_t\Big) &= Z_t dF(t,T_i) + F(t,T_i)dZ_t + d\langle F(T_i), Z\rangle_t \\ &= F(t,T_i)Z_t\Bigg(\Big(\alpha dt + \sigma dW_t^1 + \beta(T_i - t)dW_t^2\Big) \\ &\quad -\Big(\varphi_t dW_t^1 + \psi_t dW_t^2\Big) - \Big(\varphi_t\sigma + \psi_t\beta(T_i - t)\Big)dt\Bigg) \\ &= F(t,T_i)Z_t\Bigg(\Big(\alpha - \varphi_t\sigma - \psi_t\beta(T_i - t)\Big)dt \\ &\quad + \Big(\sigma - \varphi_t\Big)dW_t^1 + \Big(\beta(T_i - t) - \psi_t\Big)dW_t^2\Bigg). \end{aligned} \tag{3.4}$$

Since $F(t, T_i)Z_t \neq 0$, we obtain

$$\alpha - \varphi_t\sigma - \psi_t\beta(T_i - t) = 0, \text{ for } i = 1, 2. \tag{3.5}$$

For two futures contracts we have the set of equations with two unknown variables

$$\begin{cases} \alpha - \varphi_t\sigma - \psi_t\beta(T_1 - t) = 0, \\ \alpha - \varphi_t\sigma - \varphi_t\beta(T_2 - t) = 0. \end{cases} \tag{3.6}$$

We represent ψ and φ through our model parameters

$$\begin{cases} \varphi_t = \frac{\alpha\left(\beta(T_2-t)-\beta(T_1-t)\right)+\left(\alpha-\alpha\right)+\varphi_t(\sigma-\sigma)}{\sigma\left(\beta(T_2-t)-\beta(T_1-t)\right)}, \\ \psi_t = \frac{(\alpha-\alpha)+\varphi_t(\sigma-\sigma)}{\beta(T_2-t)-\beta(T_1-t)}. \end{cases} \tag{3.7}$$

As a result of the calculations, we get

$$\begin{cases} \varphi_t = \frac{\alpha}{\sigma}, \\ \psi_t = 0. \end{cases} \tag{3.8}$$

We see, that φ_t and ψ_t does not depend on t, consequently, the density process has a unique representation. Thus we conclude, that the market is non-arbitrage and complete.
Obviously, if the term corresponding to the second Brownian motion is zero, then the part of the process $F(t, T_i)$ based on W_t^2 is a martingale with respect to the initial measure P.
We place the values into the density process formula (3.3)

$$\begin{aligned} Z_t &= \exp\Big\{-\int_0^t \frac{\alpha}{\sigma} dW_s^1 - \int_0^t \frac{\left(\frac{\alpha}{\sigma}\right)^2}{2} ds\Big\} \\ &= \exp\Big\{-\frac{\alpha}{\sigma} W_t^1 - \frac{\alpha^2}{2\sigma^2} t\Big\}. \end{aligned} \tag{3.9}$$

3.2 Pricing of the European call option on futures

To price the European call option, we make the process $F(t, T_1)$ a martingale with respect to the martingale measure Q. Using Girsanov theorem (Theorem 2), we derive

$$\begin{aligned} F(t, T_1) &= F(0, T_1) \exp\Big\{(\alpha t - \frac{\sigma^2}{2} t - \frac{1}{2}\int_0^t \beta^2(t-s)^2 ds + \sigma W_t^1 \\ &\quad + \int_0^t \beta(t-s) dW_s^2\Big\} \\ &= F(0, T_1) \exp\Big\{-\frac{\sigma^2}{2} t + \sigma\big(W_t^1 + \frac{\alpha}{\sigma} t\big) - \frac{1}{2}\int_0^t \beta^2(t-s)^2 ds \\ &\quad + \int_0^t \beta(t-s) dW_s^2\Big\}. \end{aligned} \tag{3.10}$$

It is easy to see, that the term in the parentheses multiplied by σ is a Brownian motion. Thus we can make the substitution

$$W_t^* = W_t^1 + \frac{\alpha}{\sigma} t. \tag{3.11}$$

In conclusion, we have

$$\begin{aligned} F(T_1, T_1) &= F(0, T_1) \exp\Big\{ -\frac{\sigma^2}{2} T_1 + \sigma W_{T_1}^* - \frac{1}{2} \int_0^{T_1} \beta^2 (T_1 - s)^2 ds \\ &\quad + \int_0^{T_1} \beta(T_1 - s) dW_s^2 \Big\}, \end{aligned} \tag{3.12}$$

and in the differential form

$$dF(t, T_1) = F(t, T_1) \Big(\sigma dW_t^* + \beta(T_1 - t) dW_t^2 \Big). \tag{3.13}$$

When discounted, the process (3.12) becomes a martingale with respect to the physical measure P.
Using (3.2) and (3.12), we compute the option price

$$\begin{aligned} \mathbb{C} &= \mathsf{E}_{\mathsf{Q}} \Big[e^{-rT_1} \big(F(T, T_1) - K \big)^+ \Big] = \mathsf{E}_{\mathsf{Q}} \Big[\Big(\frac{F(T, T_1)}{B_{T_1}} - \frac{K}{B_{T_1}} \Big)^+ \Big] \\ &= \mathsf{E}_{\mathsf{Q}} \Big[\Big(\frac{F(0, T_1)}{B_0} \exp\Big\{ -rT_1 - \frac{\sigma^2}{2} T_1 + \sigma W_{T_1}^* - \frac{1}{2} \int_0^{T_1} \beta^2 (T_1 - s)^2 ds \\ &\quad + \int_0^{T_1} \beta(T_1 - s) dW_s^2 \Big\} - \frac{K}{B_{T_1}} \Big)^+ \Big]. \end{aligned} \tag{3.14}$$

To simplify our calculations we substitute

$$x = \int_0^{T_1} \beta(T_1 - s) dW_s^2, \tag{3.15}$$

with the variance

$$\mathsf{Var}(x) = \int_0^{T_1} \beta^2 (T_1 - s)^2 ds. \tag{3.16}$$

Further, because of the independence of the Brownian motions, we obtain

$$\begin{aligned} \mathbb{C} &= \frac{1}{\sqrt{2\pi \mathsf{Var}(x)}} \int \Big[e^{-\frac{x^2}{2\mathsf{Var}(x)}} \mathsf{E}_{\mathsf{Q}} \Big(\frac{F(0,T_1)}{B_0} e^{-rT_1 - \frac{\sigma^2}{2} T + \sigma W_{T_1}^* - \frac{\mathsf{Var}(x)}{2} + x} - \frac{K}{B_{T_1}} \Big)^+ \Big] dx \\ &= \frac{1}{\sqrt{2\pi \mathsf{Var}(x)}} \int \Big[e^{-\frac{x^2}{2\mathsf{Var}(x)}} \mathsf{E}_{\mathsf{Q}} \Big(\frac{F(0,T_1)}{B_0} e^{\sigma W_{T_1}^* - \frac{\sigma^2}{2} T_1} e^{-rT_1 + x - \frac{\mathsf{Var}(x)}{2}} - \frac{K}{B_{T_1}} \Big)^+ \Big] dx \\ &= \frac{1}{\sqrt{2\pi \mathsf{Var}(x)}} \int \Big[e^{-\frac{x^2}{2\mathsf{Var}(x)}} \mathbb{C}_{\mathbb{BS}} \Big(\frac{F(0,T_1)}{B_0} e^{-rT_1 + x - \frac{\mathsf{Var}(x)}{2}}, T_1, \sigma, K, r \Big) \Big] dx. \end{aligned} \tag{3.17}$$

3.3 The hedging strategy against the European call option on futures

To find the hedging strategy we apply the Itô product rule to the discounted capital

$$d\frac{V(t,F(t,T))}{B_t} = \frac{1}{B_t}dV(t,F(t,T)) + V(t,F(t,T))d\frac{1}{B_t} + d\langle V(F(T)),\frac{1}{B}\rangle_t. \tag{3.18}$$

We are looking for the differential $dV(t,F(t,T))$, using the Itô formula and (3.13),

$$dV(t,F(t,T)) = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial F}dF_t + \frac{1}{2}\frac{\partial^2 V}{\partial F^2}dF_t dF_t, \tag{3.19}$$

where

$$dF_t dF_t = \Big(F(t,T)\Big)^2\Bigg(\sigma^2 + \beta^2(T-t)^2\Bigg)dt. \tag{3.20}$$

We insert this representation into (3.18)

$$\begin{aligned} d\frac{V(t,F(t,T))}{B_t} &= \frac{1}{B_t}\Bigg(\frac{\partial V(t,F(t,T))}{\partial t}dt + \frac{\partial V(t,F(t,T))}{\partial F}dF_t \\ &\quad + \frac{1}{2}\frac{\partial^2 V(t,F(t,T))}{\partial X^2}dF_t dF_t\Bigg) - V\big(t,F(t,T)\big)\frac{rdt}{B_t} \\ &= \frac{1}{B_t}\Bigg(\frac{\partial V(t,F(t,T))}{\partial t} - rV(t,F(t,T)) \\ &\quad + \frac{1}{2}\frac{\partial^2 V(t,F(t,T))}{\partial F_t^2}(F(t,T))^2\Big(\sigma^2 + \beta^2(T-t)^2\Big)\Bigg)dt \\ &\quad + \frac{\partial V(t,F(t,T))}{\partial F_t}\frac{dF_t}{B_t}. \end{aligned} \tag{3.21}$$

We equate the first term on the right-hand side to zero, as the discounted capital should be a martingale. Therefore we have

$$d\frac{V(t,F(t,T))}{B_t} = \frac{\partial V(t,F(t,T))}{\partial F_t}\frac{dF_t}{B_t}. \tag{3.22}$$

We can conclude, that

$$\begin{cases} \gamma_t = \frac{\partial V(t,F(t,T))}{\partial F_t}, \\ \theta_t = \frac{X_t^\pi - \gamma_t F(t,T)}{B_t}, \end{cases}$$

where the portfolio is

$$X_t^\pi = \gamma_t F(t,T) + \theta_t B_t. \tag{3.23}$$

We showed, how to hedge the option with the underlying futures contract. Our model allows to construct the hedging strategy with any other two futures. For this purpose we can express the underlying futures through them. Let's represent any futures contract through two others

$$dF_{T_3} = \gamma_t dF_{T_1} + \delta_t dF_{T_2}.$$

Then we get

$$\left\{ \begin{array}{l} F_{T_3} = \gamma_t F_{T_1} + \delta_t F_{T_2}, \\ T_3 F_{T_3} = \gamma_t T_1 F_{T_1} + \delta_t T_2 F_{T_2}. \end{array} \right. \tag{3.24}$$

Solve it and get

$$\left\{ \begin{array}{l} \gamma_t = \frac{F(0,T_3)}{F(0,T_1)} \exp\left\{ \int\limits_0^t \beta(T_3 - T_1) dW_s^2 - \frac{1}{2} \int\limits_0^t \beta^2 (T_3 - T_1)(T_3 + T_1 - 2s) ds \right\}^{\frac{T_2 - T_3}{T_2 - T_1}}, \\ \delta_t = \frac{F(0,T_3)}{F(0,T_2)} \exp\left\{ \int\limits_0^t \beta(T_3 - T_2) dW_s^2 - \frac{1}{2} \int\limits_0^t \beta^2 (T_3 - T_2)(T_3 + T_2 - 2s) ds \right\}^{\frac{T_3 - T_1}{T_2 - T_1}}. \end{array} \right. \tag{3.25}$$

Chapter 4

The European Call on Futures Adjusted for the Index Level

The Chapter 4 is devoted to the computations of the European type contingent claims on the futures adjusted for the index level. Below we introduce the index, which reflects the dynamics of the stock prices of the oil and gas companies. Then we adjust the model and provide the closed-form solution

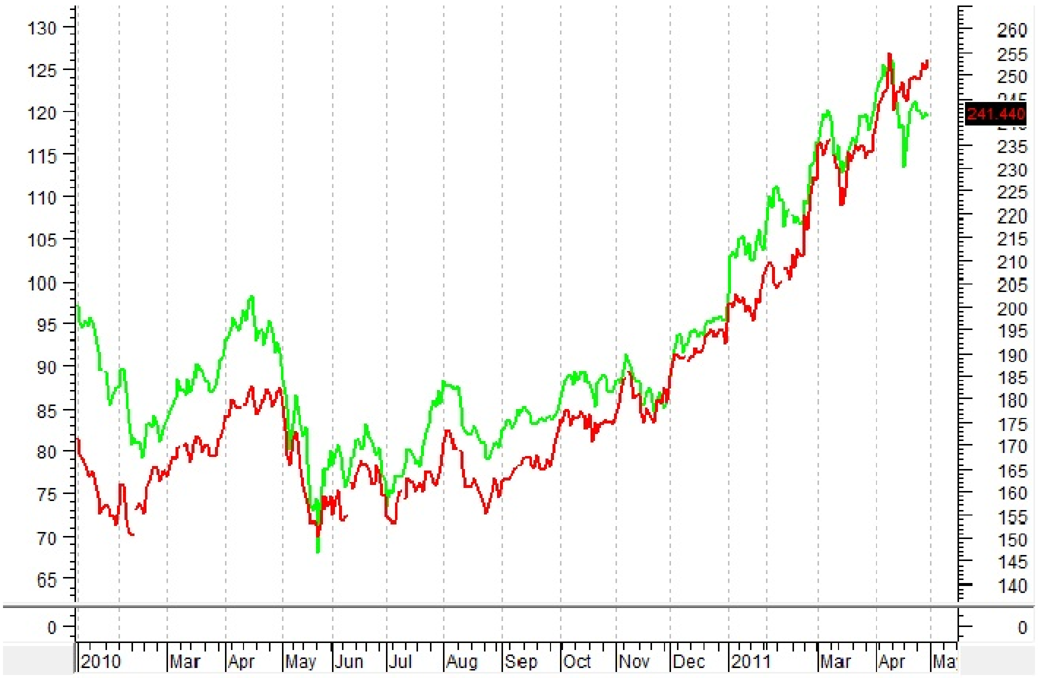

Figure 4.1: The comparison of the dynamics of RTS oial&gas index and futures contract on Brent

for the European call option and the hedging strategy. The figure (Figure 4) presents the real data from "Russian Trading System" Stock Exchange. It shows the dynamics of the oil futures contract and the index RTS oil&gas, which represents the dynamics of stocks of the hugest Russian oil companies. It is easy to notice, that the futures and index are correlated. We want to reduce the volatility by the construction of new assets which are nominated in units of index. In our model the index is defined by

$$I_t = I_0 \exp\{\eta t + \theta W_t^3\}, \tag{4.1}$$

and in the differential form we have

$$dI_t = I_t((\eta + \frac{\theta^2}{2})dt + \theta dW_t^3). \tag{4.2}$$

To hedge the obligation, which pays out $I(T)$ dollars at the time T, we construct the discounted futures price process

$$\begin{aligned} F_t^I &= \frac{F_t}{I_t} = \frac{F_0}{I_0} \exp\Bigg\{(\alpha - \frac{\sigma^2}{2} - \eta)t \\ &- \frac{1}{2}\int_0^t \beta^2(T-s)^2 ds + \sigma W_t^1 + \int_0^t \beta(T-s)dW_s^2 - \theta W_t^3\Bigg\}. \end{aligned} \tag{4.3}$$

The correlation coefficient between W_t^1 and W_t^3 is ρ. (As far as W_t^1 and W_t^3 are dependent, $\rho \neq 0$.)

4.1 The reduction of the number of Brownian motions

To price the European option on this futures contract, we switch the model to three independent Brownian motions

$$(W_1, W_2, W_3) \to (\widetilde{W}_1, \widetilde{W}_2, \widetilde{W}_3). \tag{4.4}$$

The differentials of the old Brownian motions are linear combinations of the differentials of the new Brownian motions

$$\begin{cases} dW_1 = \frac{\sigma_{11} d\widetilde{W}_1 + \sigma_{13} d\widetilde{W}_3}{\sigma_1}, & \sigma_1 = \sqrt{\sigma_{11}^2 + \sigma_{13}^2}, \\ dW_3 = \frac{\sigma_{31} d\widetilde{W}_1 + \sigma_{33} d\widetilde{W}_3}{\sigma_3}, & \sigma_3 = \sqrt{\sigma_{31}^2 + \sigma_{33}^2}. \end{cases} \tag{4.5}$$

We keep W_1 unchanged, so the first coefficient is equal to zero.

$$dW_1 = d\widetilde{W}_1 \Rightarrow \sigma_{13} = 0. \tag{4.6}$$

Then we use the information about the correlation between two Brownian motions

$$dW_1 dW_3 = \rho dt. \tag{4.7}$$

Simultaneously, we use the set (4.5) to find the product of two differentials.

$$dW_1 dW_3 = d\widetilde{W}_1 \frac{\sigma_{31} d\widetilde{W}_1 + \sigma_{33} d\widetilde{W}_3}{\sigma_3} = \frac{\sigma_{31}}{\sigma_3} dt. \tag{4.8}$$

It is easy to notice, that

$$\rho = \frac{\sigma_{31}}{\sigma_3}, \quad \sigma_3 = \sqrt{\sigma_{31}^2 + \sigma_{33}^2}. \tag{4.9}$$

Note, that the absolute values of the coefficients are not required, thus using their ratio we obtain the connection between the coefficients and ρ

$$\rho = \frac{\sigma_{31}}{\sqrt{\sigma_{31}^2 + \sigma_{33}^2}} = \frac{1}{\sqrt{1 + \left(\frac{\sigma_{33}}{\sigma_{31}}\right)^2}}, \tag{4.10}$$

and express the ratio of the unknown coefficients through ρ

$$\frac{\sigma_{33}}{\sigma_{31}} = \sqrt{\frac{1}{\rho^2} - 1} = \sqrt{\frac{1-\rho^2}{\rho^2}}. \tag{4.11}$$

We transform the formula for dW_3 from (4.5) in the same way as in (4.10), then place the value of the coefficients ratio (4.11) and get the expression through the known variables

$$\begin{aligned} dW_3 &= \frac{\sigma_{31} d\widetilde{W}_1 + \sigma_{33} d\widetilde{W}_3}{\sigma_3} = \frac{d\widetilde{W}_1 + \frac{\sigma_{33}}{\sigma_{31}} d\widetilde{W}_3}{\sqrt{1 + \left(\frac{\sigma_{33}}{\sigma_{31}}\right)^2}} = \frac{d\widetilde{W}_1 + \sqrt{\frac{1-\rho^2}{\rho^2}} d\widetilde{W}_3}{\sqrt{1 + \frac{1-\rho^2}{\rho^2}}} \\ &= \frac{d\widetilde{W}_1 + \sqrt{\frac{1-\rho^2}{\rho^2}} d\widetilde{W}_3}{\sqrt{\frac{\rho^2 + 1 - \rho^2}{\rho^2}}} = |\rho| d\widetilde{W}_1 + \sqrt{\frac{(1-\rho^2)\rho^2}{\rho^2}} d\widetilde{W}_3 \\ &= |\rho| d\widetilde{W}_1 + \sqrt{1-\rho^2} d\widetilde{W}_3. \end{aligned}$$

Finally, we obtain an exact form for the Brownian motions transformation

$$\begin{cases} dW_1 = d\widetilde{W}_1, \\ dW_3 = |\rho| d\widetilde{W}_1 + \sqrt{1-\rho^2} d\widetilde{W}_3. \end{cases}$$

As a result of the substitution, the indexed futures becomes

$$F_t^I = F_t^I\Big((\alpha - \frac{\sigma^2}{2} - \eta)t - \frac{1}{2}\int_0^t \beta^2(T-s)^2 ds + (\sigma - \theta|\rho|)\widetilde{W}_t^1 + \int_0^t \beta(T-s)dW_s^2 - \theta\sqrt{1-\rho^2}\widetilde{W}_t^3\Big). \quad (4.12)$$

Now three Brownian motions are independent. Notice, that $(\sigma - \theta|\rho|)$ and $\theta\sqrt{1-\rho^2}$ are constants. So we decide to join $\widetilde{W}_t^1$ and $\widetilde{W}_t^3$ into a new Brownian motion

$$\widetilde{W}_t = \frac{(\sigma - \theta|\rho|)\widetilde{W}_t^1 - \theta\sqrt{1-\rho^2}\widetilde{W}_t^3}{\sqrt{(\sigma-\theta|\rho|)^2 + \theta^2(1-\rho^2)}}, \quad (4.13)$$

with the expectation

$$\mathsf{E}\widetilde{W}_t = 0, \quad (4.14)$$

and the variance

$$\mathsf{Var}\widetilde{W}_t = \frac{(\sigma-\theta|\rho|)^2 t + \theta^2(1-\rho^2)t}{\Big(\sqrt{(\sigma-\theta|\rho|)^2 + \theta^2(1-\rho^2)}\Big)^2} = t. \quad (4.15)$$

Thus we substitute

$$(\sigma - \theta|\rho|)\widetilde{W}_t^1 - \theta\sqrt{1-\rho^2}\widetilde{W}_t^3 = \sqrt{(\sigma-\theta|\rho|)^2 + \theta^2(1-\rho^2)}\widetilde{W}_t. \quad (4.16)$$

For simplicity, we put $\lambda = \sqrt{(\sigma-\theta|\rho|)^2 + \theta^2(1-\rho^2)}$.
The modified futures price process is

$$F_t^I = F_t^I\Big((\alpha - \frac{\sigma^2}{2} - \eta)t - \frac{1}{2}\int_0^t \beta^2(T-s)^2 ds + \lambda\widetilde{W}_t + \int_0^t \beta(T-s)dW_s^2\Big). \quad (4.17)$$

By means of the Itô formula we look for the differential of the discounted futures price process, where we consider $F^I(t,T) = g(X_t),\ g(x) = e^x$. Therefore,

$$X_t = (\alpha - \frac{\sigma^2}{2} - \eta)t - \frac{1}{2}\int_0^t \beta^2(T-s)^2 ds + \lambda\widetilde{W}_t + \int_0^t \beta(T-s)dW_s^2, \quad (4.18)$$

and in the differential form

$$dX_t = (\alpha - \frac{\sigma^2}{2} - \eta - \frac{1}{2}\beta^2(T-t)^2)dt + \lambda d\widetilde{W}_t + \beta(T-t)dW_t^2. \quad (4.19)$$

As far as the futures price does not depend on the time explicitly, $\frac{\partial F^I(t,T)}{\partial t} = \frac{\partial g(x)}{\partial t} = 0$, we get

$$\begin{aligned}
dF_t^I &= F_t^I dX_t + \frac{1}{2} F_t^I dX_t dX_t \\
&= F_t^I \Big((\alpha - \frac{\sigma^2}{2} - \eta - \frac{1}{2}\beta^2 (T-t)^2) dt + \lambda d\widetilde{W}_t + \beta(T-t) dW_d^2 \Big) \\
&\quad + \frac{1}{2} F_t^I \Big(\lambda^2 + \beta^2 (T-t)^2 \Big) dt \\
&= F_t^I \Big((\alpha - \frac{\sigma^2}{2} - \eta + \frac{1}{2}\lambda^2) dt + \lambda d\widetilde{W}_t + \beta(T-t) dW_t^2 \Big). \qquad (4.20)
\end{aligned}$$

We obtained the formula for the differential of the discounted futures process with two independent Brownian motions.
We can proceed with the computation of the option price with the pay-off

$$f_T = \left[\begin{array}{l} (F^I(T,T) - K), \text{for } F^I(T,T) > K, \\ 0, \text{otherwise}, \end{array} \right. \qquad (4.21)$$

where $F^I(T,T)$ is the futures price at the maturity and K is the strike. We consider the market

$$\begin{cases} dF^I(t,T_i) = F^I(t,T_i) \Big((\alpha - \frac{\sigma^2}{2} - \eta + \frac{1}{2}\lambda^2) dt + \lambda d\widetilde{W}_t + \beta(T_i - t) dW_t^2 \Big), \\ dB_t = rB_t dt, \ B_0 = 1, \end{cases}$$

where $\alpha, \sigma, \eta, \lambda, \beta, r \in \mathbb{R}_+^1, \ i = 1,2$ and $\widetilde{W}_t$ and W_t^2 are independent Brownian motions.
As far as the underlying assets and the index process are correlated and the index reflects partly the movement of underlying assets, we expect the volatility of the transformed futures contracts to be lower than the volatility of the initial ones. In our notations, that means that $\lambda < \sigma$. This condition is satisfied, if

$$\sqrt{\big(\sigma - \theta|\rho|\big)^2 + \theta^2 \big(1 - |\rho|^2\big)} < \sigma. \qquad (4.22)$$

We raise both parts to the second power and consider the difference between them

$$\sigma^2 - 2\sigma\theta|\rho| + \big(\theta\rho\big)^2 + \theta^2 - \theta^2|\rho|^2 - \sigma^2 = -2\sigma\theta|\rho| + \theta^2 = \theta - 2\sigma|\rho|. \quad (4.23)$$

It should be less than zero. So we gather all parameters on the left-hand side and put the constant on the right-hand side

$$\frac{\theta}{\sigma|\rho|} \ < \ 2. \qquad (4.24)$$

We emphasize, that when (4.24) is satisfied, the volatility of the model reduces. Note, that the volatility of the index I_t can be larger than the adjusted to the correlation ratio volatility of the futures processes F_t^I, but maximum twice as much.

4.2 Arbitrage and completeness

The first step is to prove, that the market is arbitrage-free and complete, according to (Proposition 2).
As in the previous case, we consider a filtered space $(\Omega, \mathscr{F}, F)$ and two probability measures P and Q on it. We assume, that Q is absolutely continuous with respect to P. Then we look for a density process of Q relative to P, i.e. a martingale Z on $(\Omega, \mathscr{F}, F, \mathsf{P})$ such that $\forall t \in \mathbb{R}^+$, Z_t is the Radon-Nikodým derivative $\frac{d\mathsf{Q}_{|\mathscr{F}_t}}{d\mathsf{P}_{|\mathscr{F}_t}}$ of the restrictions of Q and P to $(\Omega, \mathscr{F}_t)$.
Under Girsanov theorem (Theorem 2), the density process for this market has the following form

$$Z_t = \exp\left\{ -\int_0^t \varphi_s d\widetilde{W}_s - \int_0^t \frac{\varphi_s^2}{2} ds - \int_0^t \psi_s dW_s^2 - \int_0^t \frac{\psi_s^2}{2} ds \right\}. \tag{4.25}$$

Using the same procedure as in the previous chapter, we show, that the process $F_t^I Z_t$ is a martingale with respect to the physical measure and its differential $d\Big(F_t^I Z_t\Big)$ is expressed through the differentials of the Brownian motions only

$$\begin{aligned} d\Big(F^I(t,T_i)Z_t\Big) &= F^I(t,T_i)dZ_t + Z_t dF^I(t,T_i) + d\langle F^I, Z_t\rangle_t \qquad (4.26)\\ &= F^I(t,T_i)Z_t\Bigg(-\Big(\varphi_t d\widetilde{W}_t + \psi_t dW_t^2\Big) \\ &\quad -\Big(\varphi_t\sigma + \psi_t\beta(T_i - t)\Big)dt + \Big(\alpha - \frac{\sigma^2}{2} - \eta + \frac{1}{2}\lambda^2\Big)dt \\ &\quad +\lambda d\widetilde{W}_t + \beta(T_i - t)dW_t^2\Bigg) \\ &= F^I(t,T_i)Z_t\Bigg(\Big(-\varphi_t + \lambda\Big)d\widetilde{W}_t + \Big(-\varphi_t + \beta(T_i - t)\Big)dW_t^2 \\ &\quad +\Big(-\varphi_t\sigma - \psi_t\beta(T_i - t) + \alpha - \frac{\sigma^2}{2} - \eta + \frac{1}{2}\lambda^2\Big)dt\Bigg). \qquad (4.27)\end{aligned}$$

The coefficient before dt equals 0

$$\alpha - \frac{\sigma^2}{2} - \eta + \frac{\lambda^2}{2} - \varphi_t\lambda - \psi_t\beta(T_i - t) = 0, \ i = 1, 2. \tag{4.28}$$

There are two unknown variables, so we need two equations. For this purpose we will use two different futures contracts

$$\begin{cases} \alpha - \frac{\sigma^2}{2} - \eta + \frac{\lambda^2}{2} - \varphi_t\lambda - \psi_t\beta(T_1 - t) = 0, \\ \alpha - \frac{\sigma^2}{2} - \eta + \frac{\lambda^2}{2} - \varphi_t\lambda - \psi_t\beta(T_2 - t) = 0. \end{cases} \tag{4.29}$$

Solving the set of equations, we get

$$\begin{cases} \varphi_t = \frac{(\alpha - \frac{\sigma^2}{2} - \eta + \frac{\lambda^2}{2})\left(\beta(T_1 - t) - \beta(T_2 - t)\right) + \left((\alpha - \frac{\sigma^2}{2} - \eta + \frac{\lambda^2}{2} - (\alpha - \frac{\sigma^2}{2} - \eta + \frac{\lambda^2}{2})\right) + \varphi(\lambda - \lambda)}{\lambda\left(\beta(T_1 - t) - \beta(T_2 - t)\right)}, \\ \psi_t = \frac{(\alpha - \frac{\sigma^2}{2} - \eta + \frac{\lambda^2}{2}) - (\alpha - \frac{\sigma^2}{2} - \eta + \frac{\lambda^2}{2}) + \varphi(\lambda - \lambda)}{\beta(T_1 - t) - \beta(T_2 - t)}. \end{cases} \tag{4.30}$$

As a result, we obtain

$$\begin{cases} \varphi_t = \frac{\alpha - \frac{\sigma^2}{2} - \eta + \frac{\lambda^2}{2}}{\lambda}, \\ \psi_t = 0. \end{cases} \tag{4.31}$$

As in the previous chapter, φ_t and ψ_t are constants, that leads to the uniqueness of the martingale measure. Therefore the market is non-arbitrage and complete. The density process is given by

$$\begin{aligned} Z_t &= \exp\left\{ - \int_0^t \frac{\alpha - \frac{\sigma^2}{2} - \eta + \frac{\lambda^2}{2}}{\lambda} d\widetilde{W}_s - \int_0^t \frac{\left(\frac{\alpha - \frac{\sigma^2}{2} - \eta + \frac{\lambda^2}{2}}{\lambda} \right)^2}{2} ds \right\} \\ &= \exp\left\{ - \frac{\alpha - \frac{\sigma^2}{2} - \eta + \frac{\lambda^2}{2}}{\lambda} \widetilde{W}_t - \frac{\left(\alpha - \frac{\sigma^2}{2} - \eta + \frac{\lambda^2}{2}\right)^2}{2\lambda^2} t \right\}. \end{aligned} \tag{4.32}$$

To price the European option, we apply Girsanov theorem (Theorem 2) to the futures process F_t^I

$$\begin{aligned} F_T^I &= F_0^I \exp\left\{ (\alpha - \frac{\sigma^2}{2} - \eta)T - \frac{1}{2}\int_0^T \beta^2(T - s)^2 ds \right. \\ &\quad \left. + \lambda\widetilde{W}_T + \int_0^T \beta(T - s) dW_s^2 \right\} \\ &= F_0^I \exp\left\{ - \frac{\lambda^2}{2}T + \lambda\left(\widetilde{W}_T - \frac{\eta - \alpha + \frac{\sigma^2}{2} - \frac{\lambda^2}{2}}{\lambda} T\right) \right. \\ &\quad \left. - \frac{1}{2}\int_0^T \beta^2(T - s)^2 ds + \int_0^T \beta(T - s) dW_s^2 \right\}, \end{aligned} \tag{4.33}$$

make the substitution

$$W_t^* = \widetilde{W}_t - \frac{\eta - \alpha + \frac{\sigma^2}{2} - \frac{\lambda^2}{2}}{\lambda} t, \tag{4.34}$$

and obtain

$$F_T^I = F_0^I \exp\left\{ -\frac{\lambda^2}{2} T + \lambda W_T^* - \frac{1}{2} \int_0^T \beta^2 (T-s)^2 ds + \int_0^T \beta(T-s) dW_s^2 \right\}, \tag{4.35}$$

in the differential form

$$dF_T^I = F_t^I \Big(\lambda dW_t^* + \beta(T-t) dW_t^2 \Big). \tag{4.36}$$

4.3 The price of the European call option on futures adjusted for the index level

We use the representation (4.35) to compute the price of the European call option on futures adjusted for the index level

$$\begin{aligned} \mathbb{C} &= \mathsf{E}^* \Big[e^{-rT} \big(F_T^I - K \big)^+ \Big] = \mathsf{E}^* \Big[\Big(\frac{F_T^I}{B_T} - \frac{K}{B_T} \big)^+ \Big] \\ &= \mathsf{E}^* \Big[\Big(\frac{F_0^I}{B_0} \exp\Big\{ -rT - \frac{\lambda^2}{2} T + \lambda W_T^* - \frac{1}{2} \int_0^T \beta^2 (T-s)^2 ds \\ &\quad + \int_0^T \beta(T-s) dW_s^2 \Big\} - \frac{K}{B_T} \Big)^+ \Big], \end{aligned} \tag{4.37}$$

as in the previous chapter

$$x = \int_0^T \beta(T-s) dW_s^2, \tag{4.38}$$

with the variance

$$\mathsf{Var}(x) = \int_0^T \beta^2 (T-s)^2 ds. \tag{4.39}$$

We use the fact of the independence of the Brownian motions to argue

$$\begin{aligned} \mathbb{C} &= \frac{1}{\sqrt{2\pi \mathsf{Var}(x)}} \int \Big[e^{-\frac{x^2}{2\mathsf{Var}(x)}} \mathsf{E}_\mathsf{Q} \Big(\frac{F_0^I}{B_0} e^{-rT - \frac{\lambda^2}{2} T + \lambda W_T^* - \frac{\mathsf{Var}(x)}{2} + x} - \frac{K}{B_T} \Big)^+ \Big] dx \\ &= \frac{1}{\sqrt{2\pi \mathsf{Var}(x)}} \int \Big[e^{-\frac{x^2}{2\mathsf{Var}(x)}} \mathsf{E}_\mathsf{Q} \Big(\frac{F_0^I}{B_0} e^{\lambda W_T^* - \frac{\lambda^2}{2} T} e^{-rT + x - \frac{\mathsf{Var}(x)}{2}} - \frac{K}{B_T} \Big)^+ \Big] dx \\ &= \frac{1}{\sqrt{2\pi \mathsf{Var}(x)}} \int \Big[e^{-\frac{x^2}{2\mathsf{Var}(x)}} \mathbb{C}_{\mathbb{BS}} \Big(\frac{F_0^I}{B_0} e^{-rT + x - \frac{\mathsf{Var}(x)}{2}}, T, \lambda, K, r \Big) \Big] dx. \end{aligned} \tag{4.40}$$

It is easy to see, that the formulæ for option pricing (3.17) and (4.40) are similar and both are expressed through the Black-Scholes formula. All parameters are the same, except the volatility. Therefore we can conclude, that the option with the higher volatility parameter is more expensive. If the condition (4.24) is satisfied, then the second option is cheaper. It implies, that we need to rebalance the hedging portfolio rarer.

4.4 The hedging strategy against the European call option on futures adjusted for the index level

This contingent claim can be hedged, as the market is complete. We use the same technique as in the previous chapter. We require that the discounted capital is a martingale. For this purpose we derive $dV(t, F^I(t,T))$ for (3.18)

$$dV(t, F^I(t,T)) = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial F^I}dF_t^I + \frac{1}{2}\frac{\partial^2 V}{\partial (F^I)^2}dF_t^I dF_t^I, \tag{4.41}$$

where dF_t^I is defined by (4.36) and

$$dF_t^I dF_t^I = \Big(F^I(t,T)\Big)^2 \Bigg(\lambda^2 + \beta^2(T-t)^2\Bigg)dt. \tag{4.42}$$

Thus we have

$$\begin{aligned} d\frac{V(t,F^I(t,T))}{B_t} &= \frac{1}{B_t}\Bigg(\frac{\partial V(t,F^I(t,T))}{\partial t}dt + \frac{\partial V(t,F^I(t,T))}{\partial F^I}dF_t^I \\ &\quad + \frac{1}{2}\frac{\partial^2 V(t,F^I(t,T))}{\partial X^2}dF_t^I dF_t^I\Bigg) - V\big(t,F^I(t,T)\big)\frac{rdt}{B_t} \\ &= \frac{1}{B_t}\Bigg(\frac{\partial V(t,F^I(t,T))}{\partial t} - rV(t,F^I(t,T)) \\ &\quad + \frac{1}{2}\frac{\partial^2 V(t,F^I(t,T))}{\partial (F_t^I)^2}(F^I(t,T))^2\Bigg(\lambda^2 + \beta^2(T-t)^2\Bigg)\Bigg)dt \\ &\quad + \frac{\partial V(t,F^I(t,T))}{\partial F_t^I}\frac{dF_t^I}{B_t}. \end{aligned} \tag{4.43}$$

As the first term on the right-hand side is equal to zero, we can state

$$d\frac{V(t,F_t^I)}{B_t} = \frac{\partial V(t,F^I(t,T))}{\partial F_t^I}\frac{dF_t^I}{B_t}. \tag{4.44}$$

From (4.44) we can see, that

$$\begin{cases} \gamma_t = \frac{\partial V(t,F_t^I)}{\partial F_t^I}, \\ \theta_t = \frac{X_t^\pi - \gamma_t F^I(t,T)}{B_t}, \end{cases}$$

where the portfolio is

$$X_t^\pi = \gamma_t F^I(t,T) + \theta_t B_t. \tag{4.45}$$

As in the previous chapter the option can be hedged with two different futures.

Chapter 5

The American Put on Futures Adjusted for the Index Level

In this Chapter we look for the prices of and the hedging strategies against the American type contingent claims. According to the general theory, the prices of the American and European call options coincide. Thus the most interesting is the problem of the American put option pricing. We study the cases of the infinite and finite horizons.
The Chapter is organized as follows. In the first Section we introduce the futures index, which allows us to remove the time dependence. In the second Section of the Chapter we discuss the perpetual American put option, which implies no expiration date. The last Section is devoted to the contingent claim with the restricted life time.
We consider the market

$$\begin{cases} dF^I(t,T_i) = F^I(t,T_i)\Big((\alpha - \frac{\sigma^2}{2} - \eta + \frac{1}{2}\lambda^2)dt + \lambda d\widetilde{W}_t + \beta(T_i - t)dW_t^2\Big), \\ dB_t = rB_t dt,\ B_0 = 1, \end{cases}$$

where $\alpha, \sigma, \eta, \lambda, \beta, r \in \mathbb{R}^1_+,\ i = 1, 2$ and $\widetilde{W}_t$ and W_t^2 are independent Brownian motions.

5.1 The futures index without maturity

Firstly, we construct the perpetual underlying assets. It should be based on the futures contracts in such a way to eliminate the changes in the the time to the expiration. It can be used both in the cases with and without the maturity. Hence, we can issue the option contracts with the longer life time than the underlying futures have. Following Tkachev (2010) [9] we use the index

$$dE_t = E_t\Bigg(\sum_{j=1}^{m}\nu_j\frac{dF^I(t,T_j)}{F^I(t,T_j)}\Bigg), \tag{5.1}$$

where ν_j is the weight of the contract with time T_j to the expiration date, $\sum_{j=1}^{m}\nu_j = 1$. We replace $dF^I(t,T_j)$ by the formula (4.20) and obtain

$$\begin{aligned} dE_t &= E_t\Bigg(\sum_{j=1}^{m}\nu_j\Big((\alpha - \frac{\sigma^2}{2} - \eta + \frac{1}{2}\lambda^2)dt + \lambda d\widetilde{W}_t + \beta\big(T_j - t\big)dW_t^2\Big)\Bigg) \\ &= E_t\Bigg((\alpha - \frac{\sigma^2}{2} - \eta + \frac{1}{2}\lambda^2)dt\sum_{j=1}^{m}\nu_j + \lambda d\widetilde{W}_t\sum_{j=1}^{m}\nu_j \\ &\quad + \sum_{j=1}^{m}\nu_j\beta\big(T_j - t\big)dW_t^2\Bigg) \\ &= E_t\Bigg((\alpha - \frac{\sigma^2}{2} - \eta + \frac{1}{2}\lambda^2)dt + \lambda d\widetilde{W}_t + \sum_{j=1}^{m}\nu_j\beta\big(T_j - t\big)dW_t^2\Bigg). \end{aligned} \tag{5.2}$$

For the sake of simplicity we denote $h = \alpha - \frac{\sigma^2}{2} - \eta + \frac{1}{2}\lambda^2$. We also designate $\sum_{j=1}^{m}\nu_j\big(T_j - t\big)$ as A. By this substitution we remove the dependence on the time to the expiration from the index and obtain

$$dE_t = E_t\Big(hdt + \lambda d\widetilde{W}_t + \beta A dW_t^2\Big), \tag{5.3}$$

and in the integral form

$$E_t = E_0\exp\Bigg\{\Big(h - \frac{\lambda^2}{2} - \frac{\beta^2A^2}{2}\Big)t + \lambda\widetilde{W}_t + \beta AW_t^2\Bigg\}. \tag{5.4}$$

We see, that the coefficients before the Brownian motions are constant, and we can replace two Brownian motions with the new one

$$\lambda W_t^* + \beta AW_t^2 = \sqrt{\lambda^2 + \beta^2A^2}\widehat{W}. \tag{5.5}$$

It is easy to see, that the expectation of $\widehat{W}$ is

$$\mathsf{E}\widehat{W} = 0, \tag{5.6}$$

and the variance is

$$\mathsf{Var}\widehat{W} = t. \tag{5.7}$$

We denote $\sqrt{\lambda^2+\beta^2A^2}$ as ζ.
Finally, we obtain the following form for the index in the differential

$$dE_t = E_t\left(hdt + \zeta d\widehat{W}_t\right), \tag{5.8}$$

and in the integral form

$$E_t = E_0 \exp\left\{\left(h - \frac{\zeta^2}{2}\right)t + \zeta\widehat{W}_t\right\}. \tag{5.9}$$

As the index is the underlying assets for the option contracts, the hedging strategy, presented in the standard way, requires the investments into the index and the bank account. But as far as the index is not traded, we have to express the index through the existing futures.
For the reason, that we can model the market by means of two futures contracts (see Section 3.3), we can claim, that

$$dE_t = \xi_1 dF^I(t,T_1) + \xi_2 dF^I(t,T_2), \tag{5.10}$$

where $\xi_i, i = 1,2$, are amounts of the correspondent futures contracts in the perpetual index. Since there are two Brownian motions in the futures, we will use (5.3)-representation for the index and (4.36) for the futures. We equate the coefficients before the differentials $d\widehat{W}_t$ and dW_t^2 and get the set of equations

$$\begin{cases} E_t = \xi_1 F_1 + \xi_2 F_2, \\ AE_t = \xi_1(T_1-t)F_1 + \xi_2(T_2-t)F_2. \end{cases} \tag{5.11}$$

Solving it, we get

$$\begin{cases} \xi_1 = \frac{E_t}{F^I(t,T_1)}\frac{(T_2-t)-A}{T_2-T_1}, \\ \xi_2 = \frac{E_t}{F^I(t,T_2)}\frac{A-(T_1-t)}{T_2-T_1}, \end{cases} \tag{5.12}$$

or, if we use (5.4) and (4.35),

$$\begin{cases} \xi_1 = \frac{E_0}{F^I(0,T_1)}\exp\left\{\frac{\beta^2A^2}{2} + \frac{1}{2}\int_0^t \beta^2(T_1-s)^2ds + \beta AW_t^2 - \int_0^t \beta(T_1-s)dW_s^2\right\}\frac{(T_2-t)-A}{T_2-T_1}, \\ \xi_2 = \frac{E_0}{F^I(0,T_2)}\exp\left\{\frac{\beta^2A^2}{2} + \frac{1}{2}\int_0^t \beta^2(T_2-s)^2ds + \beta AW_t^2 - \int_0^t \beta(T_2-s)dW_s^2\right\}\frac{A-(T_1-t)}{T_2-T_1}. \end{cases} \tag{5.13}$$

5.2 The perpetual American put option on the futures index

This Section investigates the perpetual American put option. According to Peskir and Shiryaev [6], the American put option price in the arbitrage-free market is

$$V(x) = \sup_{\tau} \mathsf{E}_x \left[e^{-r\tau} f_\tau \right], \tag{5.14}$$

where supremum is taken over all is the stopping times τ and f_τ is the pay-off function

$$f_t = \left[\begin{array}{l} (K - E_t), \text{for } E_t < K, \\ 0, \text{otherwise}, \end{array} \right. \tag{5.15}$$

where the futures index price E_t is the solution of the stochastic differential equation (5.8) under the martingale measure Q_x and K is the strike.
For the problem (5.14) the optimal solution is

$$\tau^* = \inf_{t \geq 0} \Big(E_t \leq b \Big), \tag{5.16}$$

where b is the exercise point.

5.2.1 The change of measure in the underlying assets

To solve the the stochastic differential equation (5.8) under the martingale measure Q_x, we consider a filtered space $(\Omega, \mathscr{F}, F)$ with two probability measures P and Q on it. Assume, that Q is absolutely continuous with respect to P. We look for a density process of Q relative to P, i.e. a martingale Z on $(\Omega, \mathscr{F}, F, \mathsf{P})$ such that $\forall t \in \mathbb{R}^+$, Z_t is the Radon-Nikodým derivative $\frac{d\mathsf{Q}_{|\mathscr{F}_t}}{d\mathsf{P}_{|\mathscr{F}_t}}$ of the restrictions of Q and P to $(\Omega, \mathscr{F}_t)$.
Girsanov theorem (Theorem 2) implies, that the density process in this case has the following form

$$Z_t = \exp \left\{ - \int_0^t \varphi_s d\widehat{W}_s - \int_0^t \frac{\varphi_s^2}{2} ds \right\}. \tag{5.17}$$

To define an exact form of the density process Z_t, we use the fact, that process $E_t Z_t$ should be a martingale with respect to the physical measure. We take the differential $d\big(E_t Z_t\big)$ and obtain

$$\begin{aligned} d\Big(E_t Z_t\Big) &= Z_t dE_t + E_t dZ_t + d\langle E, Z\rangle_t \\ &= E_t Z_t \Big(\big(hdt + \zeta d\widehat{W}_t\big) - \varphi_t d\widehat{W}_t - \varphi_t \zeta dt \Big) \\ &= E_t Z_t \Big(\zeta - \varphi_t\Big) d\widehat{W}_t + E_t Z_t \Big(h - \varphi_t \zeta\Big) dt. \end{aligned} \tag{5.18}$$

Relying on the the knowledge that the drift coefficient of the process equals zero, but E_t and Z_t are positive, we conclude, that the condition is satisfied

only when $\varphi_t = \frac{h}{\zeta}$.
The density process has the form

$$Z_t = \exp\Big\{ -\frac{h}{\zeta}\widehat{W}_t - \frac{h^2}{2\zeta^2}t \Big\}. \tag{5.19}$$

To make the process E_t a martingale with respect to the measure Q we apply Girsanov theorem (Theorem 2) to (5.9)

$$\begin{aligned} E_t &= E_0 \exp\Big\{ ht - \frac{\zeta^2}{2}t + \zeta\widehat{W}_t \Big\} \\ &= E_0 \exp\Big\{ -\frac{\zeta^2}{2}t + \zeta\Big(\widehat{W}_t + \frac{h}{\zeta}t\Big) \Big\}, \end{aligned} \tag{5.20}$$

make the substitution

$$W_t^* = \widehat{W}_t + \frac{h}{\zeta}t, \tag{5.21}$$

obtain the process in the integral form

$$E_t = E_0 \exp\Big\{ -\frac{\zeta^2}{2}t + \zeta W_t^* \Big\} \tag{5.22}$$

and in differential form

$$dE_t = E_t\Big(\zeta W_t^*\Big). \tag{5.23}$$

5.2.2 The free boundary problem

Solving the optimal stopping problem (5.14), we use the standard arguments to formulate the free-boundary problem

$$L_E V(x) = rV(x) \quad \text{for} \quad x > b, \tag{5.24}$$
$$V(x) = \big(K - x\big)^+ \quad \text{for} \quad x = b, \tag{5.25}$$
$$V'(x) = -1 \quad \text{for} \quad x = b, \tag{5.26}$$
$$V(x) > \big(K - x\big)^+ \quad \text{for} \quad x > b, \tag{5.27}$$
$$V(x) = \big(K - x\big)^+ \quad \text{for} \quad 0 < x < b, \tag{5.28}$$
$$V(x) \leq K. \tag{5.29}$$

L_E is an infinitesimal generator for the diffusion process, given by

$$L_E G = \lim_{t \downarrow 0} \frac{\mathsf{E}[G(E_t)] - G(x)}{t}. \tag{5.30}$$

Applying the Itô formula to the function $G(E_t)$ with the process dE_t, given by (5.23), we obtain

$$dG(E_t) = E_t\zeta\frac{\partial}{\partial x}G(E_t)d\widehat{W}_t + \frac{E_t^2\zeta^2}{2}\frac{\partial^2}{\partial x^2}G(E_t)dt. \tag{5.31}$$

Then we place the result into (5.30)

$$L_E G(E_t) = \lim_{t\downarrow 0}\frac{\mathsf{E}[G(x) + \int_0^t \frac{E_s^2\zeta^2}{2}\frac{\partial^2}{\partial x^2}G(E_s)ds] - G(x)}{t}. \tag{5.32}$$

Finally, we can argue, that

$$L_E = \frac{E_t^2\zeta^2}{2}\frac{\partial^2}{\partial x^2}. \tag{5.33}$$

Now we can solve (5.24)

$$\frac{E_t^2\zeta^2}{2}\frac{\partial^2}{\partial x^2}V(x) = rV(x). \tag{5.34}$$

This is the Caushy-Euler equation, so we look for the solution in the form

$$V(x) = x^p. \tag{5.35}$$

As the solution of the quadratic equation, we obtain two roots

$$\begin{cases} p_1 = \frac{\zeta^2+\sqrt{\zeta^4+8r\zeta^2}}{\zeta^2}, \\ p_2 = \frac{\zeta^2-\sqrt{\zeta^4+8r\zeta^2}}{\zeta^2}. \end{cases} \tag{5.36}$$

Thus the general solution is

$$V(x) = C_1 x^{\frac{\zeta^2+\sqrt{\zeta^4+8r\zeta^2}}{\zeta^2}} + C_2 x^{\frac{\zeta^2-\sqrt{\zeta^4+8r\zeta^2}}{\zeta^2}}. \tag{5.37}$$

Because of (5.29), $C_1 = 0$ and the solution can be written as

$$V(x) = C_2 x^{\frac{\zeta^2-\sqrt{\zeta^4+8r\zeta^2}}{\zeta^2}}. \tag{5.38}$$

To find the coefficient C_2 and the boundary b, we use the conditions (5.25)

$$V(b) = (K-b)^+ \tag{5.39}$$

and (5.26)

$$V'(b) = -1. \tag{5.40}$$

After we substitute $V(x)$, we have the set of equations

$$\begin{cases} C_2 b^p = K - b, \\ pC_2 b^{p-1} = -1, \end{cases} \tag{5.41}$$

where $p = \frac{\zeta^2 - \sqrt{\zeta^4 + 8r\zeta^2}}{\zeta^2}$. Solving it, we get

$$\begin{cases} b = \frac{Kp}{p-1}, \\ C_2 = -K\left(\frac{p-1}{Kp}\right)^p. \end{cases} \tag{5.42}$$

The solution of the option pricing problem, presented by (5.24–5.29), can be written as

$$V(x) = \left[\begin{array}{l} K - x, x < b, \\ C_2 x^p, x \geq b. \end{array}\right. \tag{5.43}$$

Then the optimal stopping moment is

$$\tau^* = \inf_t \left\{ t : E_t \leq \frac{Kp}{p-1} \right\}. \tag{5.44}$$

5.2.3 The hedging strategy against the American put option on futures adjusted for the index level

Now we introduce the wealth process and define the strategy for the perpetual American put option.
Using (5.14), we define the wealth process

$$\begin{aligned} \vartheta(x) &= \sup_{\tau \geq 0} \mathsf{E}_x \Big[e^{-r(\tau+t)} f(E_{\tau+t}) \Big] = \sup_{\tau \geq 0} \mathsf{E}_x \Big[\mathsf{E}_x [e^{-r(\tau+t)} f(E_\tau)\theta_t] \Big] \\ &= \sup_{\tau \geq 0} \mathsf{E}_x \Big[E_{X_\tau} [e^{-r(\tau+t)} f(E_\tau)] \Big] = \mathsf{E}_x \Big[e^{-rt} V(X_t) \Big]. \end{aligned} \tag{5.45}$$

We designate $e^{-rt}V(X_t)$ as H, apply the Itô formula to H and get

$$dH = \Big(-re^{-rt} + \frac{1}{2} E_t^2 \zeta^2 \frac{\partial^2 V}{\partial X^2} e^{-rt} \Big) dt + E_t \zeta e^{-rt} \frac{\partial V}{\partial E} d\widehat{W}_t. \tag{5.46}$$

To make the process a martingale, we state, that $-re^{-rt} + \frac{1}{2} E_t^2 \zeta^2 \frac{\partial^2 V}{\partial X^2} e^{-rt} = 0$, and get

$$dH = \frac{\partial V}{\partial E} \frac{dE_t}{B_t}. \tag{5.47}$$

The portfolio is presented in the following way

$$V(E_t) = \gamma_t E_t + \theta_t B_t. \tag{5.48}$$

From (5.47) it obviously follows

$$\begin{cases} \gamma_t = \frac{\partial V}{\partial E}, \\ \theta_t = e^{-rt}\Big(V(E_t) - \gamma_t E_t\Big). \end{cases} \tag{5.49}$$

The hedging strategy (γ_t, θ_t) is defined by

$$\begin{cases} \gamma_t = -Kp\Big(\frac{p-1}{Kp}\Big)^p E_t^{p-1}, \\ \theta_t = e^{-rt}\Big(V(E_t) + Kp\Big(\frac{p-1}{Kp}\Big)^p E_t^p\Big). \end{cases} \tag{5.50}$$

As far as we can not buy the index E_t, we use the representation (5.10) to construct the hedging strategy, that contains two futures contracts and the bank account

$$\begin{aligned} V(E_t) &= \gamma_t\Big(\xi_1 F^I(t,T_1) + \xi_2 F^I(t,T_2)\Big) + \theta_t B_t \\ &= \gamma_t \xi_1 F^I(t,T_1) + \gamma_t \xi_2 F^I(t,T_2) + \theta_t B_t. \end{aligned} \tag{5.51}$$

Finally, we have the hedging strategy

$$\begin{cases} \gamma_t^1 = \gamma_t \xi_1, \\ \gamma_t^2 = \gamma_t \xi_2, \\ \theta_t = e^{-rt}\Big(V(E_t) - \gamma_t^1 F^I(t,T_1) + \gamma_t^2 F^I(t,T_2)\Big). \end{cases} \tag{5.52}$$

5.3 The American put option on the futures index

In this Section we present the American put option with the finite horizon. As far as there is no closed-form solution for this type of the problem, we formulate the free-boundary problem and provide the hedging strategy.
We follow Peskir and Shiryaev [6] to define the price of the American put option on the finite horizon

$$V(t,x) = \sup_{0 \le \tau \le T-t} \mathsf{E}_{t,x}\Big[e^{-r\tau}\big(K - E_{t+\tau}\big)\Big], \tag{5.53}$$

where τ is the stopping time.
Let us divide the space into two sets to introduce the stopping time for the finite time interval. The continuation set for the problem is

$$C = \{(t,x) \in [0,T) \times (0,\infty) : V(t,x) > G(t,x)\}, \tag{5.54}$$

the stopping set is

$$D = \{(t,x) \in [0,T) \times (0,\infty) : V(t,x) = G(t,x)\}, \tag{5.55}$$

consequently, the stopping time is

$$\tau_D = \inf\{0 \le s \le T-t : E_{t+s} \in D\}. \tag{5.56}$$

In this case, the time dependence arises, so the infinitesimal operator from the previous section (5.33) transforms into

$$\frac{\partial}{\partial t} + \frac{E_t^2 \zeta^2}{2} \frac{\partial^2}{\partial x^2}. \tag{5.57}$$

The free-boundary problem for the unknown function V and the unknown boundary b, in the continuation set

$$\frac{\partial V}{\partial t} + L_E V = rV, \tag{5.58}$$

$$V(t,x) > \big(K-b\big)^+, \tag{5.59}$$

at the boundary

$$V(t,x) = \big(K-b\big)^+, \tag{5.60}$$

$$\frac{\partial V}{\partial t} = -1, \tag{5.61}$$

in the stopping set

$$V(t,x) = \big(K-b\big)^+. \tag{5.62}$$

According to Peskir and Shiryaev [6], the function $V(t,x)$ and the boundary b have the following properties

- $V(t,x)$ is continuous on $[0,T) \times (0,\infty)$,
- $V(t,x)$ is $C^{1,2}$ on the sets C and D,
- along $x, V(t,x)$ is decreasing and convex with $V_x(t,x) \in [-1,0]$,

- along t, decreasing with $V(T,x) = \big(K - x\big)^+$,
- along t, $b(t)$ is increasing and continuous with $0 < b(0+) < K$ and $b(T-) = K$.

To find the hedging strategy, we differentiate the wealth process by the Itô formula

$$d\big(e^{-rt}V(E_t)\big) = \Big(- re^{-rt} + \frac{1}{2}E_t^2\zeta^2\frac{\partial^2 V}{\partial x^2}e^{-rt}\Big)dt + E_t\zeta e^{-rt}\frac{\partial V}{\partial x}d\widehat{W}_t. \quad (5.63)$$

To make the process a martingale, we state, that $-re^{-rt} + \frac{1}{2}E_t^2\zeta^2\frac{\partial^2 V}{\partial x^2}e^{-rt} = 0$, and get

$$d\big(e^{-rt}V(E_t)\big) = \frac{\partial V}{\partial E}\frac{dE_t}{B_t}. \quad (5.64)$$

The hedging strategy against the expiring American put option is

$$\begin{cases} \gamma_t = \frac{\partial V}{\partial E}, \\ \theta_t = e^{-rt}\Bigg(V(E_t) - \gamma_t E_t\Bigg), \end{cases} \quad (5.65)$$

and expressed through the traded futures

$$\begin{cases} \gamma_t^1 = \gamma_t\xi_1, \\ \gamma_t^2 = \gamma_t\xi_2, \\ \theta_t = e^{-rt}\Bigg(V(E_t) - \gamma_t^1 F^I(t,T_1) + \gamma_t^2 F^I(t,T_2)\Bigg), \end{cases} \quad (5.66)$$

where $\xi_i,\ i = 1,2$ are given by (5.13).

Chapter 6

The Investment Problem

In this Chapter we consider the optimal investment problem. Hence we turn from the problem of the issuer of securities to the problem of his counterpart. The holder of securities invests its capital in a such way to maximize the expected utility function of the terminal wealth.
The Chapter consists of two sections. In the first one we consider the methodology of the construction of the optimal investment strategy. In the second Section we investigate the case of logarithmic utility function. We solve the problem for two securities — the bank account and the American put option on finite horizon — and show the solution for the case of three assets — a risk-free one, the option and the spot assets.

6.1 The methodology of the investment problem

First of all, we introduce a framework of the investment problem. The preferences of the security holer are determined by a continuously differentiable utility function $U : \mathbb{R}^1_+ \to \mathbb{R}^1$ such, that

$$U'(0+) = \lim_{x \downarrow 0} U'(x) = \infty, \tag{6.1}$$

$$U'(\infty) = \lim_{x \uparrow \infty} U'(x) = 0. \tag{6.2}$$

In the (B,S,F)-market we want to find such an optimal investment strategy π^*, that for a given function U

$$\mathsf{E}U\big(X_T^{\pi^*}(x)\big) = \sup_{\pi \in SF} \mathsf{E}U\big(X_T^{\pi}(x)\big) = u(x), \tag{6.3}$$

where $u(x)$ is a price function of optimization problem (6.3).
According to [3], a well-known method of examining the expected utility problem is the use of duality relationships. Thus we have to introduce a conjugate function of $U(x)$

$$V(y) = \sup_{x>0}[U(x) - xy], \; y > 0, \tag{6.4}$$

which is the Legendre transform of a function $-U(-x)$. Thus the function $V(y)$ is strictly decreasing, convex, continuously differentiable and satisfies the following conditions

$$V'(0) = -\infty, \quad V'(\infty) = 0, \tag{6.5}$$

$$V(0) = U(\infty), \quad V(\infty) = U(0), \tag{6.6}$$

$$U(x) = \inf_{y>0}[V(y) + xy], \; x > 0. \tag{6.7}$$

The following statement is true

$$I(x) = \big(U'(x)\big)^{-1} = -V'(x). \tag{6.8}$$

As we know, the process of discounted capital of self-financing strategy (x, γ_t, θ_t) can be presented in the following way

$$Y_t(x) = x + \int_0^t \gamma_s dX_s, \tag{6.9}$$

where γ is an amount of stocks in the self-financing strategy. We designate the set of such process as $\mathscr{X}(x)$. Due to (6.9), we transform the problem (6.3) into

$$\mathsf{E}U\big(Y_T^*(x)\big) = \sup_{Y\in\mathscr{X}} \mathsf{E}U\big(Y_T^\pi(x)\big). \tag{6.10}$$

Q is a unique martingale measure, and Z_t^* is a density process. Now we can define a differentiable function

$$v(y) = \mathsf{E}V(yZ_T^*). \tag{6.11}$$

Relying on Melnikov, Volkov, Nechaev [8]

Theorem 3 *Let the (B,S)-market to be complete. Then the following statements are correct*

1. *The function* $u(x) < \infty \ \forall X > 0$ *is continuously differentiable and strictly concave in* $(0, x_0)$*, the function* $v(Y) < \infty$ *for sufficiently large* y *is continuously differentiable and strictly convex in* (y, ∞)*. And functions* u *and* v *are conjugate in the following sense*

$$v(y) = \sup_{x>0} \big(u(x) - xy\big), \quad y > 0, \tag{6.12}$$

$$u(x) = \inf_{y>0} \big(v(y) + xy\big), \quad x > 0, \tag{6.13}$$

and $u'(0) = \infty, \quad v'(\infty) = 0.$

2. *If in the region* $x < x_0$ *and* $y < y_0 \quad y = u'(x)$*, then the optimal for 6.10 terminal capital is an integrable* $\mathscr{F}_T$*-measurable random variable and is defined by*

$$Y_T^*(x) = I(yZ_T^*). \tag{6.14}$$

6.2 The investment problem for the case of logarithmic utility function

Below we construct the optimal investment strategy for two assets — the bank account and the American put option with finite horizon. At the end of the Section we show, how the solution changes, if we introduce additionally the spot assets. For the case of simplicity we consider a logarithmic utility function

$$U(x) = \ln x. \tag{6.15}$$

Using (6.15) and (6.8), we state, that $I(x) = \frac{1}{x}$. In the case of logarithmic utility function the problem (6.4) transforms into

$$V(y) = \sup_{x>0} [\ln x - xy]. \tag{6.16}$$

The necessary condition for the supremum has the following form

$$\frac{\partial}{\partial x}(\ln x - xy) = 0 \tag{6.17}$$

and is achieved at the point $x = \frac{1}{y},\ y > 0$. From (Theorem 3) $y = U'(x) = \frac{1}{x}$, therefore

$$V(y) = \ln \frac{1}{y} - 1 = -\ln y - 1. \tag{6.18}$$

We apply the investment problem theory to the American put option with finite horizon. To simplify the calculations, we consider the capital X_t with two assets — a risk-free bank account and a futures index as a risk assets. After we find an optimal investment strategy, we are able two represent the strategy for two futures contracts as risk assets. As there are no initial costs to construct a portfolio of futures, the capital for the portfolio of the futures index and the bank account has a form

$$X_t = \theta_t B_t, \tag{6.19}$$

while its differential is given by

$$dX_t = \theta_t dB_t + \gamma_t dE_t. \tag{6.20}$$

The underlying assets price process is defined by the differential equation (5.8)

$$dE_t = E_t(hdt + \zeta d\widehat{W}_t).$$

We use the form of the density process given by (5.19) and obtain

$$\begin{aligned} v(y) &= \mathsf{E}V(yZ_T^*) = -\mathsf{E}\ln(yZ_T^*) - 1 = -\ln y - \mathsf{E}\ln Z_T^* - 1 \\ &= -\ln y - 1 + \frac{h^2}{2\zeta^2}T. \end{aligned} \tag{6.21}$$

As $\frac{\partial}{\partial x}(\ln x - xy) = 0$ at the point $x = \frac{1}{y},\ y > 0$, we compute a price function

$$u(x) = \inf_{y>0}\big(v(y) + xy\big) = \ln x - 1 + \frac{h^2}{2\zeta^2}T + 1 = \ln x + \frac{h^2}{2\zeta^2}T. \tag{6.22}$$

Discounted optimal capital $\frac{X_T^{\pi^*}(x)}{B_T}$ is equal to

$$\frac{X_T^{\pi^*}}{B_T} = I(yZ_T^*) = \frac{1}{yZ_T^*} = \frac{x}{Z_T^*} = x\exp\left\{\frac{h}{\zeta}\widehat{W}_T + \frac{h^2}{2\zeta^2}T\right\}. \tag{6.23}$$

Let us consider the following ratio

$$a_t^* = \frac{\gamma_t^* E_t}{X_t^{\pi^*}(x)}, \tag{6.24}$$

where γ_t^* is an amount of index in the optimal portfolio $\pi^* = (\theta^*, \gamma^*)$. For the sake of simplicity, we assume, that a_t^* is constant a^*.
We use the fact, that the optimal capital $X_t^{\pi^*}$ is built on the base of the self-financing strategy π^*

$$\begin{aligned} dX_t^{\pi^*} &= \theta_t^* dB_t + \gamma_t^* dE_t \\ &= \theta_t^* rB_t dt + \gamma_t^* E_t(hdt + \zeta d\widehat{W}_t) \\ &= X_t^{\pi^*} rdt + a^* X_t^{\pi^*}(hdt + \zeta d\widehat{W}_t) \\ &= X_t^{\pi^*}(r + a^* h)dt + X_t^{\pi^*} a^* \zeta d\widehat{W}_t. \end{aligned} \tag{6.25}$$

Using the Itô formula, we get the expression for the optimal discounted capital

$$\frac{X_T^{\pi^*}}{B_T} = x \exp\left\{a^* hT - \frac{1}{2}(a^*)^2\zeta^2 T + a^*\zeta\widehat{W}_T\right\}. \tag{6.26}$$

Comparing (6.23) and (6.26), we get, that $a^* = \frac{h}{\zeta^2}$. Therefore $\gamma_t^* = \frac{hX_t^{\pi^*}}{\zeta^2 E_t}$ and $\theta_t^* = \frac{X_t^{\pi^*} - \gamma_t^* E_t}{B_t^*}$. The solution of the problem for a portfolio with two futures contracts and a bank account has the following form

$$\begin{cases} \gamma_t^* = \frac{hX_t^{\pi^*}}{\zeta^2(\xi_1 F_1 + \xi_2 F_2)}, \\ \theta_t^* = \frac{X_t^{\pi^*} - \gamma_t^*(\xi_1 F_1 + \xi_2 F_2)}{B_t^*}. \end{cases} \tag{6.27}$$

If we add to the portfolio X_t the spot assets, which follows standard geometrical Brownian motion in the form (2.27), we obtain

$$X_t = \theta_t B_t + \delta_t S_t, \tag{6.28}$$

and in differential form

$$dX_t = \theta_t dB_t + \delta_t dS_t + \gamma_t dE_t. \tag{6.29}$$

In this case we need to introduce two proportions, $a_t^* = \frac{\gamma_t^* E_t}{X_t^{\pi^*}(x)}$ and $b_t^* = \frac{\delta_t^* S_t}{X_t^{\pi^*}(x)}$. Following the same logic in the calculations as in the previous case, we obtain the optimal ratios

$$a^* = \frac{h}{\zeta^2} \text{ and } b^* = \frac{\mu - r}{\sigma^2}. \tag{6.30}$$

Therefore the solution of the problem for a portfolio with one spot assets, two futures contracts and a bank account has the following form

$$\begin{cases} \gamma_t^* = \frac{hX_t^{\pi^*}}{\zeta^2(\xi_1 F_1 + \xi_2 F_2)}, \\ \delta_t^* = \frac{(\mu - r)X_t^{\pi^*}}{\sigma^2 S_t}, \\ \theta_t^* = \frac{X_t^{\pi^*} - \gamma_t^*(\xi_1 F_1 + \xi_2 F_2) - \delta_t^* S_t}{B_t^*}. \end{cases} \tag{6.31}$$

Chapter 7

Conclusions

We have developed the model for the futures contract pricing. The main feature of our model is that it takes into account the nature of the futures as a separate contract, which has its own source of uncertainty. We have modeled the additional source of risk by the independent Brownian motion, corrected to maturity structure by the time-dependent parameter of the volatility.
Based on the knowledge that some economy processes depend heavily on the oil and gas prices, we have constructed the futures contract adjusted for the inflation level.
For these two types of futures we have derived the explicit formulæ of the European call options. Both of them are similar and both are expressed through the Black-Scholes formula. All parameters are the same, except the volatility. Therefore we have concluded that the option with the higher volatility parameter is more expensive. We found the condition, under which the second option is cheaper. It implies, that the hedging portfolio requires rarer rebalancing. We have calculated the corresponding hedging strategies. As we have shown, the hedging portfolio can be presented both through the pair: the underlying futures and the risk-free assets, — or through the triplet: two traded futures contracts and the risk-free assets.
Furthermore, we have constructed an index on the base of futures contracts without dependence on the time to maturity. This assets could be used both in perpetual contingent claims and in long-time-to-expiration derivatives on the short-term underlying assets. We have solved an optimal stopping problem for the inflation adjusted contract, and thus we have obtained the price of the perpetual American put option. In addition, we have formulated the problem for an expiring American put option. We have provided the corresponding hedging strategies for these contingent claims. As in the case with the European call options, we have demonstrated, that the hedging portfolios can be presented both through the pair: the underlying futures index and

the risk-free assets, — or through the triplet: two traded futures contracts and the risk-free assets.

Notation

$(\Omega, \mathscr{F}, \{\mathscr{F}_t\}, \mathsf{P})$	A filtered probability space with set of outcomes Ω, sigma algebra $\mathscr{F}$, flow or filtration $\{\mathscr{F}_t\}$ and probability measure P.
$\mathsf{E}(X)$, $\mathsf{Var}(X)$	Expectation and variance of the random variable X with respect to the measure P.
$\mathsf{E}_{\mathsf{Q}}(X)$	Expectation of the random variable X with respect to the martingale measure Q.
(B, S, F)	A (B, S, F)-market consisting of a bank account ('risk-free' assets) $B = (B_t)_{t\geq 0}$, a stock ('risk assets') $S = (S_t)_{t\geq 0}$ and a futures contract with the expiration date $T_i \geq t$ ('risk assets') $F^{T_i} = (F_t^{T_i})_{t\geq 0},\ i = 1, \ldots, n$.
$\pi = (\theta, \delta, \gamma)$	An investment portfolio.
X^π	A capital of an investment portfolio π.
Z_t	A density process of Q with respect to P.
W_t^i	An i-th Brownian motion.
I_t	An index process.
E_t	An futures index process.
$\mathbb{C}$	An European call option price.
$\mathbb{C}_{\mathbb{BS}}(F_0, T, K, \sigma, r)$	The price of the European call option according to the Black-Scholes model with initial price of the underlying assets F_o, expiration date T, strike K, volatility σ, and risk-free interest rate r.
V	A value function.
G	A gain function (contingent claim).
τ	A stopping time.
L_E	An infinitesimal generator for the diffusion process.
U	A utility function.

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