Liquidity and optimal consumption with random income

Master’s Thesis in Financial Mathematics

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Preface

The world financial crisis revealed the significance of several problems that were well known on the global markets before but seemed unimportant and were regarded as the aspects that could have only a weak, if even noticeable, impact on the general situation. One of the problems that captured the interest of the practitioners was the problem of illiquidity which attracted our interest as well.

While working on the problem we have found out that the management of the portfolio that has an illiquid asset in some aspects is really close to the problem of the optimal consumption in the case of a random income. Therefore, we have decided to put this mathematical model into the focus of our attention.

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Abstract

In the first part of our work we focus on the model of the optimal consumption with a random income. We provide the three dimensional equation for this model, demonstrate the reduction to the two dimensional case and provide for two different utility functions the full point-symmetries' analysis of the equations. We also demonstrate that for the logarithmic utility there exists a unique and smooth viscosity solution the existence of which as far as we know was never demonstrated before. In the second part of our work we develop the concept of the empirical liquidity measure. We provide the retrospective view of the works on this issue, discuss the proposed definitions and develop our own empirical measure based on the intuitive mathematical model and comprising several features of the definitions that existed before. Then we verify the measure provided on the real data from the market and demonstrate the advantages of the proposed value for measuring the illiquidity.
Chapter 1

Introduction

Our work is devoted to the problems of illiquidity that arises especially when there is a high possibility of a default and in the periods of the so-called "liquidity shocks". This problem is known on the financial markets since the moment they started to function and became extremely important now when the financial community all over the world tries to learn a lesson out of the recent crisis. The most general understanding of the term "illiquidity" can be given by the simplest common sense: "The asset is illiquid if you can not sell it whenever you want" (a more detailed understanding would be given later). So, asset is illiquid because you have to hold it till the moment when you find a buyer; however, during this time the price of your asset changes which should be taken into consideration while you are establishing your investment strategy and your consumption plan. The problem of the liquid capital allocation for the portfolio that has such kind of an asset is quite complicated and is extremely challenging. The first analytical solution as far as we know was provided only in 2008 by C. Tebaldi and E.S. Schwartz see [36].

In order to find this analytical solution authors use a standard Black-Scholes market but also introduce an idea of an illiquid asset and generally work in the framework of the optimal consumption for the random income, which will be described in detail later. They define illiquid asset as an asset that can not be sold before our investment’s horizon, denoted as $T$, at which our asset generate a random cashflow. The model can be described by the following equations

\[
\begin{align*}
    dB_t &= rB_t dt, \\
    dS_t &= \alpha S_t dt + \sigma S_t dW^1_t, \\
    dH_t &= (\mu - \delta)H_t dt + \eta (\rho dW^1_t + \sqrt{1-\rho^2} dW^2_t), 
\end{align*}
\]
where $B$ is a riskless liquid asset, that could be regarded as a bank account, $S$ is a risky liquid asset that could be regarded as a stock, $r$ is a constant interest rate for the riskless liquid asset, $\alpha$ is a drift of a risky liquid asset, $\sigma$ is a standard deviation for the liquid asset, say, a stock, $\mu$ is a continuously compounded total expected rate of return for the illiquid asset, $\eta$ - its continuous standard deviation, $\delta$ is a dividend rate paid on the illiquid asset and $W^1_t$ and $W^2_t$ are independent standard Brownian motions. The value of the illiquid asset is correlated with the stocks. The correlation coefficient is denoted as $\rho$.

The solution is found in the following way. The authors formulate the stochastic optimization problem for the function depending on the initial amount of liquid and illiquid capital (denoted as $l$ and $h$ respectively) and on time. Then they obtain the corresponding Hamilton - Jacobi - Bellman equation (HJB). After that they do a number of the changes of variables, i.e. among the others the substitution $z = l/h$ that reduces the amount of variables. It turns out that the solution does not depend on the starting amount of liquid and illiquid wealth separately but on their ratio only. Then the authors do a duality transformation and find the solution of the obtained dual equation in the form of an infinite series.

Despite the fact that the analytical solution of such a deep and complicated problem itself is an extreme progress we have to admit that it is highly complicated. Therefore not only it is hardly to be used by the practitioners but also scarcely reveals any insight on the roots of the problem. However, the fact that authors reduce the amount of variables on the way to the solution gives the idea that the differential equation obtained has a Lie group of symmetries standing behind it that could be interesting. The Lie group analysis of the financial problems is rather new yet turned out to be really fruitful (see L. Borbag, [5], [6]). Moreover, quite often understanding of the underlying Lie group structure gives us a deeper understanding of the process that stands behind the problem. One of the aims of this work is to find the Lie algebra and the Lie group structure that are strictly one-to-one connected with the equation describing the model given above and to understand the general reasoning that stands behind these structures.

The recent financial crisis revealed also that though markets of defaultable (such as defaultable bonds) or illiquid (such as immobilities) assets are big and rapidly growing, the problem of the fair price valuation for such assets is still far from the acceptable solution and remains extremely actual for all market participants. As an illustration, according to [4], a market of collateralized debt obligations (or simply CDOs) reached his peak $180$ billion in 2007. This skyrocketing was fueled by the highest credit rating "AAA" assigned by one or more of the three recognized US credit rating agencies.
The modeling of the credit derivatives has been focused on the correlation between default times. Firstly, this approach was developed by Li [30], who offered one-factor copula functions for analyzing correlation between default times. However, despite that one-factor Gaussian copula model became an industry standard, traded prices of many credit derivatives can be realized only through implausibly high correlation parameter, the so-called correlation smile. From the other side, it seems that the price of the defaultable asset depends on the investor’s expectation and the utility function. One of the most promising attempt to incorporate effects of a risk aversion on valuation of single- and two-name credit derivatives was made by Sircar and Zariphopoulou [38]. The impact of the risk was introduced in terms of framework for utility indifference valuation. This framework in the simplest case postulates that any asset price satisfies the principle of equivalent utility, i.e. it’s equal the amount of money at which the buyer of the claim is indifferent, in terms of maximum expected utility, between holding or not holding the derivative. To evaluate indifference price, defined in this way, one typically need to address the stochastic optimization problem of utility maximization, both for holder and non-holder expected utility. Firstly, this idea was initiated by Hodges and Neuberger [19] to European claims. Following this way, Sircar and Zariphopoulou demonstrated that the utility valuation produces non-trivial yield spreads within even the simplest of intensity-based models of default. Thus, using this approach one is indeed able to incorporate the equity market information (growth rates, volatilities of the non-defaulted firms, etc.) as well as the investor’s risk aversion in a natural fashion.

Basing on these general facts and knowing that the model of Tebaldi Schwartz could be regarded as the problem of the optimal consumption for the random income we will work on this model and demonstrate the Lie point-symmetries for the equations in the case of different utility functions and prove the existence and uniqueness of the viscosity solution in the case of the logarithmic utility. Then we will look on this problem from the other point of view. We will describe the liquidity problem in detail and provide our alternative empirical measure the model for which was partly inspired by the ideas mentioned above. Then basing on the market data we carry out the research that could be more regarded as an econometrical one and show the advantages of the proposed definition.
Chapter 1. Introduction
Chapter 2
Indifference pricing

The valuation of an asset’s fair price has been the most fundamental task posed by financial markets, especially in the derivatives market. The key obstacle for a straightforward calculation is that one needs to take into account the uncertainty of future market movements and determine the risk premium for holding the asset, even if the probability distribution of the future cash flows is quite certain.

The first successful and revolutionary approach was developed by Black and Scholes [7] in 1973 for pricing of derivative securities. The essential point of Black-Scholes theory states that since underlying stock price is driven by the Brownian motion, one can perfectly hedge any terminal European claim, continuously adjusting the portfolio of stocks and risk-free bonds. The assumptions of this model seem to be natural and the consequences are very strict in terms of risk premiums. Indeed, the fair price as an initial portfolio capital appears to be independent upon investor’s risk attitude. Such strong fact of the market completeness comes from the uniqueness of the martingale measure that delivers a non-arbitrage price as an expectation of discounted terminal wealth.

However, in the case of an incomplete market the situation becomes dramatically harder. Due to the market incompleteness, the absence of the arbitrage gives only the interval of prices (so-called lower and upper price) and moreover, there is a martingale measure for any price in between. Obviously, the exact value for the price of the derivative now depends on investor’s willingness to bear additional risk rewarded with an extra premium. Thus, any pricing scheme must somehow evaluate the source of market frictions in terms of particular investor’s utility function and should be built around the specific source of the market incompleteness. From the wide amount of works
on this field we the mention classical stochastic volatility model of Hull and White \cite{23}, 1987 and the transaction costs model by Leland \cite{29} in 1985.

Rather than concentrating on internal market frictions, in this paper we will focus on trading of illiquid assets and defaultable securities, which leads to de facto incomplete environment. Observations on the real market of CDS (Credit Default Swaps) and more sophisticated credit derivatives as CDOs (Collateralized Debt Obligations) reveals the non-linear impact of investor’s default fears. For more information about credit derivatives see, for example, \cite{32}. As an illustration, high credit spreads of CDO’s senior tranches show surprisingly high investors expectations of the default of 15 – 30% of investor investment grade US firms over the next five years \cite{35}.

The static framework for the utility-based valuation was proposed by Buhlmann \cite{8} and also known as the principle of the equivalent utility. It postulates that the value of the claim should coincide with the expected utility of the payoff

$$\nu(C) = U^{-1}(\mathbb{E}_\mathbb{P}(U(C(Y)))),$$

where $U$ is increasing and concave utility function that represents investor’s risk attitude and $C(Y)$ is the payoff of the claim, written on the risky asset $Y$. The crucial point here that in contrast to the arbitrage-free price the certainty equivalent price is nonlinear with respect to $Y$ and uses the physical measure instead of the martingale one.

However, this framework handles only European-type claims, and should be modified for a dynamic context (e.g. if asset could default at the unpredictable time as in the case of credit derivatives). The most fruitful technique was introduced by Hodges and Neuberger \cite{19} in terms of so-called indifference price. The idea is that holder of the asset should be indifferent in terms of the maximum expected utility whether to buy the asset now at the specified price or just invest this money in his or her portfolio. Such definition catches the stochastic control problem inside, because one need to determine the optimal strategy for holder and non-holder and extract the price based on expected utility at the initial time. Functionally, this means that the asset price $p_0$ should solve the equation

$$M(0, x) = H(0, x - p_0),$$

where $M(t, x)$ is the terminal expected utility of the non-holder given that portfolio wealth is equal to $x$ at the time $t$, and $H(t, x)$ is the terminal expected utility but for the asset holder.
Chapter 3

Utility

The concept of the utility was firstly introduced by von-Neumann and Morgenstern in 1944 \[33\]. The fundamental theorem they proved states that if the certain axioms are satisfied any individual can be considered as a rational investor. Following von-Neuman and Morgenstern we assume that there is a so-called consumption set \( X \) with the convex hull (mixing) operation. This means that for any two goods \( A, B \in X \) there is a good \( C = pA + (1 - p)B \in X \) for any \( p \in [0, 1] \). Also, let us assume that the following four axioms hold

**Axiom 1. (Completeness)** For every two goods \( A \) and \( B \) either \( A < B \), \( A > B \) or \( A = B \).

This means that for the individual \( A \) must be always either worse than \( B \), better than \( B \), or equally good.

**Axiom 2. (Transitivity)** For every \( A, B \), and \( C \) with \( A \geq B \) and \( B \geq C \) we must have \( A \geq C \).

This means that the individual’s choice must be consistent. Together with the first axiom we get that the investor preferences induce the complete order on the set of goods.

**Axiom 3. (Independence)** Let \( A, B, C \in X \) with \( A \geq B \). Then for every \( t \in [0, 1] \), \( tA + (1 - t)C \geq tB + (1 - t)C \).

This axiom is the most controversial and requires the order relation to be stable under the mixing operation.

**Axiom 4. (Continuity)** Let \( A, B, C \in X \) with \( A > B > C \). Then there exists \( p \in (0, 1) \) such that \( B = pA + (1 - p)C \).
Von-Neumann and Morgenstern have shown that under the axioms (1) – (4) the individual’s choice can be represented by the utility function $U : X \to \mathbb{R}$ such that he or she always prefers $A$ over $B$ if and only if $U(A) > U(B)$. However, in financial markets the return of an investment is in general not known due to the uncertainty. Nevertheless, we assume that probability distributions of the future cash flows are known. In that case the consumption set $X$ is realized as set of random variables on some probability space $(\Omega, \mathcal{F}, P)$ and the mixing operation is just a simple linear combination of functions $\Omega \to \mathbb{R}$. In this settings the Von-Neumann-Mongerstern theorem states that an individual chooses the lottery not by the highest expected value, but by the highest expected utility. In other words, holds the following theorem

**Theorem 1.** For any ordered set $X$ of random variables on which axioms (1) – (4) are satisfied there exists the increasing utility function $U : \mathbb{R} \to \mathbb{R}$ such that

$$A > B \iff E[U(A)] > E[U(B)],$$

and

$$A = B \iff E[U(A)] = E[U(B)]$$

for any $A, B \in X$.

The economical interpretation of the utility function is the measure of the personal investor’s attitude to the possible risk. The increasing property means that the investor prefers more than less wealth. Hereafter we consider an agent who dislikes the risk: with respect to a random return $X$, he prefers to get with certainty the expectation $E[X]$ of this return. This means that the utility function satisfies the Jensens inequality

$$U(E[X]) \geq E[U(X)],$$

which holds true only for concave functions. From the other hand, it’s obvious that any utility function is invariant under the affine transformations because the induced order remains unchanged. Taking this into account, Arrow, [2] and Pratt, [34] defined the normalized measure of an absolute risk aversion as

$$ARA(x) = -\frac{U''(x)}{U'(x)}$$

and the measure of a relative risk as

$$RRA(x) = -\frac{xU''(x)}{U'(x)}.$$
Here and later we assume that $U$ is twice differentiable if it is not stated the opposite. Now we see that the Arrow-Pratt risk aversion measure gives the reasonable estimate of the risk premium that investors wants to get for bearing the risky asset. More formally, the risk premium is defined as $\pi(X)$ such that

$$U(E[X] - \pi(X)) = E[U(X)].$$  \hspace{1cm} (3.1)

The discounted value $\mathcal{E}(X) = E[X] - \pi(X)$ is called the certainty equivalent of $X$ and means the dollar price of the risky asset $X$ that investor with the utility function $U$ is ready to pay.

Assuming that the variance of the return is small enough, one can approximate the utility function by its Taylor expansion in the form

$$U(X) \approx U(\bar{X}) + (X - \bar{X})U'(\bar{X}) + \frac{1}{2}(X - \bar{X})^2U''(\bar{X}),$$

where $\bar{X}$ is a mean value of $X$. The expectation of this value is then

$$E[U(X)] \approx U(\bar{X}) + Var(X)\frac{U''(\bar{X})}{2}.$$  

Substituting $E[U(X)]$ to (3.1), we get the following approximation of the risk premium in terms of the risk aversion coefficient

$$\pi(X) \approx -\frac{U''(\bar{X})}{2U'(X)}Var(X) = \frac{1}{2}ARA(X)Var(X).$$

Based on the Arrow-Pratt risk aversion measures there is a classification of the most commonly used utility functions:

- **CARA** (Constant Absolute Risk Aversion) means the utility function with the constant risk aversion. Obviously, it must be in the form

  $$U(x) = C - e^{\alpha x}.$$  

  Usually one takes constant $C$ equal to zero or one.

- **HARA** (Hyperbolic Absolute Risk Aversion). It is the most general class of utility functions that are usually used in practice. As follows from the name, in this case absolute risk aversion is assumed to be a hyperbolic function

  $$-\frac{U''(x)}{U'(x)} = \frac{1}{Ax + B}.$$  

  The general solution is

  $$U(x) = \frac{(x - C)^{1-R}}{1 - R},$$
where \( R = 1/A \) and \( C = -B/A \). Note, that this leads to CARA if \( A = 0 \) since \( ARA(x) \) becomes equal to \( 1/B \).

- **DARA/IARA** (*Decreasing/Increasing Absolute Risk Aversion*) states for increasing/decreasing \( A(x) \). The most common DARA function is \( \log(x) \), and IARA functions are usually taken quadratic.

The similar definitions can be constructed based on the relative risk aversion function, which leads to the CRRA, DRRA and IRRA utility functions.

*Kramkov and Schachermayer* [28] have proved that for the correctness of the problem of maximizing the expected utility of the terminal wealth in the general incomplete market one need to pose the asymptotic conditions on the utility function \( U \). It appears, that except the standard Inada conditions

\[
U(0) = \lim_{x \to 0} U'(x) = \infty, \quad (3.2)
\]

\[
U'(\infty) = \lim_{x \to \infty} U'(x) = 0, \quad (3.3)
\]

it is necessary and sufficient that

\[
AE(U) = \limsup_{x \to \infty} \frac{xU'(x)}{U(x)} < 1. \quad (3.4)
\]

Here the correctness is understood as the finiteness of the maximal terminal wealth \( E(U(X_T)) \), given by the corresponding value function

\[
u(x) = \sup_{X \in \mathcal{X}(x)} E(X_T).
\]

The set \( \mathcal{X}(x) \) represents all almost surely positive integrable processes with the initial endowment \( X_0 \) equal to \( x \).
Chapter 4

Optimal consumption with a random income

In this chapter we consider the intertemporal investment and consumption problem with the presence of the random income. From the large strand of literature on this topic we mention only a few results most relevant to our case.

1. *Merton* (1971), [31] in his pioneering work solved the optimal consumption problem with no random income and main classes of utility functions. He studied the case of HARA, logarithmic and exponential utility functions and both finite and infinite time horizons.

2. *Karatzas, Lehoczsky, Sethi, Shreve* (1986), [25] gave the explicit formula for the general case of Merton’s problem for complete markets. They were provided for the both finite and infinite horizons and $C^1$ utility functions $U : (0, \infty) \to \mathbb{R}$ with

$$\lim_{z \downarrow 0} U'(z) \geq 0, \lim_{z \to \infty} U'(z) = 0.$$ 

3. *Duffie and Zaripopolou* (1993) in [16] proved the existence and uniqueness of the viscosity solution of the associated HJB equation for the class of concave functions. They considered infinite time horizon and posed the following conditions on the utility function $U(c)$

(a) $U$ is strictly concave, $C^2(0, +\infty)$,

(b) $U(c) \leq M(1 + c)^\gamma$, with $0 < \gamma < 1, M > 0$,

(c) $U(0) \geq 0, \lim_{c \to 0} U'(c) = +\infty, \lim_{c \to \infty} U'(c) = 0$. 

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5. Cuoco (1997) generalized this result in [12] and proved the existence of the optimal consumption under the general integrability assumptions on the price processes and the following conditions on the utility function

(a) There exists at least one consumption plan $c_t$ with $\int_0^\infty U(c_t)dt > -\infty$,

(b) $U$ is bounded from above or $U(c) \leq M(1 + c^\gamma)$, with $0 < \gamma < 1, M > 0$.

He developed the martingale methods, proposed by Karatzas et al. in [25].

6. Tebaldi and Shwartz (2006) [36] provide the analytical series representation for the value function in the case of HARA utility and both finite and infinite time horizon.

The proof of the smoothness of the viscosity solution in [15] heavily relies on the reduction of the initial HJB equation to the ODE. After the reduction the main result follows from the uniform convergence of the classical solutions of the uniformly elliptic equation to the viscosity solution that is unique as shown in [16]. In the next chapters we provide the full study of such reductions using the Lie symmetries analysis and provide the similar reduction in the case of the logarithmic utility.

**Economic Setting.** We assume that the economical setting of the model consists of the riskless bond, the risky asset and the non-traded asset that generates the stochastic labor income as dividends. This economical setting is similar to one introduced by Tebaldi and Shwartz in [36] but without the instantaneous selling of the non-traded asset at the final time $T$ and defined as follows.

- A risk-free bank account $B_t$ with the interest rate $r$

  $$dB_t = rB_t dt, \quad t \leq T,$$

  where we assume $r$ to be constant for simplicity.
The stock price $S_t$, which follows the geometrical Brownian motion
\[ dS_t = S_t(\alpha dt + \sigma dW^1_t), \quad t \leq T, \]
with the continuously compounded rate of return $\alpha > r$ and the standard deviation $\sigma$.

An illiquid asset $H_t$ that cannot be traded up to the time $T$ and whose notional price is correlated with the stock price. It follows
\[ \frac{dH_t}{H_t} = (\mu - \delta)dt + \eta(\rho dW^1 + \sqrt{1 - \rho^2}dW^2), \quad t \leq T. \quad (4.1) \]
Here $\mu$ is the expected rate of return on the risky illiquid asset, $\delta$ is the rate of dividend paid by the illiquid asset, $\eta$ is the continuous standard deviation of the rate of return, and $\rho$ is the correlation coefficient between the stock index and the illiquid risky asset.

Given the filtration $\{\mathcal{F}_t\}$ generated by the Brownian motion $W = (W^1, W^2)$ we assume that the consumption process is an element of the space $\mathcal{L}_+$ of non-negative $\{\mathcal{F}_t\}$-progressively measurable processes $c_t$ such that $E(\int_0^\tau c_t dt) < \infty$ for any $\tau \in [0, T]$. The investor wants to maximize the overall utility given by
\[ \mathcal{U}(c) := E \left[ \int_0^T e^{-\kappa \tau} U(c(\tau)) d\tau \right]. \quad (4.2) \]
The wealth process is fed by the holdings in bond, stock and (random) dividends from the non-traded asset and covers the consumption stream. Thus, it follows
\[ dL_t = (rL_t + \delta H_t + \pi_t(\alpha - r) - c_t)dt + \pi_t \sigma dW^1_t. \quad (4.3) \]
We define the set of admissible policies $\mathcal{A}(l, h, t)$ for the state equation (4.3) as the set of pairs $(c_t, \pi_t)$ such that
1. $c_t$ belongs to $\mathcal{L}_+$ and $\int_t^s e^{-\kappa \tau} U(c(\tau)) d\tau < \infty$ for any $t \leq s \leq T$,
2. $\pi_t$ is $\{\mathcal{F}_t\}$-progressively measurable and $\int_t^s (\pi_\tau)^2 d\tau < \infty$ for any $t \leq s \leq T$,
3. $L_t$ defined by the stochastic differential equation (4.3) and initial conditions $L_t = l > 0$, $H_t = h > 0$ is a.s. positive for any $t \leq \tau \leq T$. 
HJB equation. Finite Time Horizon. The Bellman’s linear programming principle is one of the most powerful tool in the optimal control theory. In our case it leads to the Hamilton-Jacobi-Bellmann (or simply HJB) equation that describes the optimal investment policy. Here we formally derive the HJB equation for our stochastic optimization problem assuming that the value function is finite and twice differentiable. Now, according to the linear programming principle, the value function $V(l,h,t)$ satisfies the equation

$$0 = \max_{(\pi,c)\in A(l,h,t)} \left[ AV(l,h,t) + e^{-\kappa t}U(c(t)) \right],$$

where $A$ is the backward evolution operator of the system that acts at the arbitrary function $f(l,h,t)$ as

$$Af(l,h,t) = \lim_{\varepsilon \to 0^+} \varepsilon^{-1} \left[ E_{th}f(l(t+\varepsilon),h(t+\varepsilon),t+\varepsilon) - f(l,h,t) \right]. \quad (4.4)$$

Here $E_{th}$ denotes the expectation given that $L_t = l$ and $H_t = h$. To calculate the explicit form of the evolution operator we use Ito’s formula to express the infinitesimal change of the function $f$ driven by the processes $H_t$ and $L_t$. We obtain

$$E_{th}f(l(t+\varepsilon),h(t+\varepsilon),t+\varepsilon) - f(l,h,t) =$$

$$E_{th} \left[ f(l,h,t) + \int_t^{t+\varepsilon} df \right] - f(l,h,t) =$$

$$\int_t^{t+\varepsilon} E_{th} \left[ \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial l} dL + \frac{\partial f}{\partial h} dH + \frac{1}{2} \frac{\partial^2 f}{\partial l^2} (dL)^2 \right. +$$

$$\left. \frac{1}{2} \frac{\partial^2 f}{\partial h^2} (dH)^2 + \frac{\partial^2 f}{\partial h \partial l} (dL)(dH) \right].$$

We have moved the expectation under the integral since we assume that $f$ belongs to $\mathcal{L}_s(0,T)$ together with all derivatives up to the order two. Taking into account (8.1) and (4.3) we finally obtain

$$Af = \lim_{\varepsilon \to 0^+} \varepsilon^{-1} \int_t^{t+\varepsilon} E_{th} \left[ \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial l} dL + \frac{\partial f}{\partial h} dH + \frac{1}{2} \frac{\partial^2 f}{\partial l^2} (dL)^2 + \frac{1}{2} \frac{\partial^2 f}{\partial h^2} (dH)^2 + \frac{\partial^2 f}{\partial h \partial l} (dL)(dH) \right] =$$

$$f_t + f_l [(r l + \delta h - c) + \pi(\alpha - \rho)] + f_h(h(\mu - \delta))$$

$$+ \frac{1}{2} f_{ll}(\pi)^2 \sigma^2 + \frac{1}{2} f_{hh} \eta^2 h^2 + f_{lh} \eta \rho \sigma h \quad (4.5)$$
In the final line we exploited the relations for the differentials
\[(dL)^2 = (\pi)^2 \sigma^2 dt,\]
\[(dH)^2 = \eta^2 H^2 dt,\]
\[(dL)(dH) = \eta \rho \pi \sigma H dt,\]
and the fact that the expectation of any integral with respect to the Brownian motion is equal to zero.

Summing up and collecting the terms in (4.5) we get the HJB-equation for the value function
\[V_t(l, h, t) + \frac{1}{2} \eta^2 h^2 V_{hh}(l, h, t) + (rl + \delta h)V_l(l, h, t) + (\mu - \delta)hV_h(l, h, t) + \max_{\pi} G[\pi] + \max_{c \geq 0} H[c] = 0,\]
\[(4.6)\]
\[G[\pi] = \frac{1}{2} V_{ll}(l, h, t) \pi \sigma^2 + V_{hl}(l, h, t) \eta \rho \pi \sigma h + \pi (\alpha - r)V_l(l, h, t),\]
\[(4.7)\]
\[H[c] = -cV_l(l, h, t) + e^{-\kappa t} U(c).\]
\[(4.8)\]

Assuming the smoothness of the value function \(V\) the optimality conditions (4.7) and (4.8) give the optimal policy \((c^*_t, \pi^*_t)\) in a form
\[c^*_t(l, h, t) = \arg \max_{c \geq 0} (-cV_l + e^{-\kappa t} U(c)),\]
\[(4.9)\]
\[\pi^*_t(l, h, t) = \arg \max_{\pi} \left( \frac{1}{2} \pi^2 V_{ll} \sigma^2 + \pi (V_{hl} \eta \rho \pi \sigma h + (\alpha - r)V_l) \right)\]
\[= -\frac{\eta \rho \pi h V_{lh} + (\alpha - r)V_l}{\sigma^2 V_{ll}}.\]
\[(4.10)\]

Such feedback formulae define the \(\{\mathcal{F}_t\}\)-measurable functions of consumption and investment at the time \(t\) with the given current wealth \(l\) and income rate \(h\).

**HJB equation. Infinite Time Horizon.** In the case of the infinite time horizon the investor maximizes the utility
\[\mathcal{U}(c) := E \left[ \int_0^\infty e^{-\kappa \tau} U(c(\tau)) d\tau \right].\]
\[(4.11)\]

The set of admissible policies \(\mathcal{A}(l, h)\) is defined as follows
1. \( c_t \) belongs to \( L_+ \) and \( \int_0^t e^{-\kappa \tau} U(c(\tau))d\tau < \infty \) for any \( t \geq 0 \),

2. \( \pi_t \) is \( \{\mathcal{F}_t\}\)-progressively measurable and \( \int_0^t (\pi_\tau)^2 d\tau < \infty \) for any \( t \geq 0 \),

3. \( L_\tau \) defined by the stochastic differential equation \( (4.3) \) and initial conditions \( L_0 = l > 0 \), \( H_0 = h > 0 \) is a.s. positive for any \( t \geq 0 \).

Since the value function becomes time independent, \( (4.6) \) reduces to

\[
\frac{1}{2} \eta h V_{hh}(l,h) + (rl + \delta h)V_l(l,h) + (\mu - \delta) h V_h(l,h) + \max_{\pi} G[\pi] + \max_{c \geq 0} H[c] = \kappa V(l,h),
\]

\[
G[\pi] = \frac{1}{2} V_h(l,h) \pi^2 \sigma^2 + V_{hl}(l,h) \eta \rho \pi \sigma h + \pi (\alpha - r) V_l(l,h),
\]

\[
H[c] = -c V_l(l,h) + U(c).
\]

The optimal policies \( (4.9) \) and \( (4.10) \) become

\[
c_\star(l,h) = \arg \max_{c \geq 0} (-c V_l + U(c)),
\]

\[
\pi_\star(l,h) = \arg \max_{\pi} \left( \frac{1}{2} \pi^2 V_{\eta \sigma}^2 + \pi \left( V_{hl} \eta \rho \sigma h + (\alpha - r) V_l \right) \right)
= -\frac{\eta \rho \sigma h V_{lh} + (\alpha - r) V_l}{\sigma^2 V_{ll}}.
\]
Chapter 5

Logarithmic utility function and the infinite time horizon

In this chapter we examine the optimal consumption problem introduced before in the case of the logarithmic utility and the infinite time horizon. Despite that we know from the more general theorems from [16] and [12] that the optimal strategy does exist and the value function is the viscosity solution of the HJB equation, we still can not present the optimal policy in a feedback form (4.16) and (4.17) because the value function is not a priori smooth \(^1\). Here we fill this gap proving that in the case at hand the value function is twice differentiable. As far as we know this fact was not explicitly addressed before, though the structure of the our proof is similar to the paper [15] where the smoothness was proved for the HARA utility case.

\(^1\)The original theorem should be slightly modified due to the assumption \(U(0) = 0\). Following the lines of the theorem it is easy to see that this difficulty can be overcome by constraining the optimal consumption \(C_t \geq \varepsilon\) for sufficiently small \(\varepsilon\), as we define for the dual problem. As we prove the smoothness of the value function and the optimality of (5.9) we conclude that such constraint does not have any impact on the value function due to the stability property of the viscosity solutions and can be replaced with the usual one \(C_t \geq 0\)
Reduction of the HJB equation. At first, we rewrite the HJB equation derived in the previous chapter for the logarithmic utility

$$\frac{1}{2}\eta^2 h^2 V_{hh}(l, h) + (r l + \delta h) V_l(l, h) + (\mu - \delta) h V_h(l, h) + \max \pi G[\pi] + \max_{c \geq 0} H[c] = \kappa V(l, h),$$

(5.1)

$$G[\pi] = \frac{1}{2} V_h(l, h) \pi^2 \sigma^2 + V_{hl}(l, h) \eta \rho \pi \sigma h + \pi (\alpha - r) V_l(l, h),$$

(5.2)

$$H[c] = -c V_l(l, h) + \log(c).$$

(5.3)

The starting point of the proof is the reduction of the original two-dimensional PDE (5.1) to the ODE that comes from the observation originally made by Davis and Norman [13].

Lemma 1.

(i) The value function $V(l, h)$ is strictly concave.

(ii) $V(l, h)$ is homothetic in the following sense

$$V(\lambda l, \lambda h) = V(l, h) + \frac{\log \lambda}{\kappa},$$

(5.4)

where $\kappa$ is a discount factor, defined above.

Proof.

(i) Let $(c_1, \pi_1)$ and $(c_2, \pi_2)$ be optimal policies for $V(l_1, h_1)$ and $V(l_2, h_2)$ respectively. Due to the linearity of (4.3)

$$(\tilde{c}, \tilde{\pi}) = (\lambda c_1 + (1 - \lambda)c_2, \lambda \pi_1 + (1 - \lambda)\pi_2)$$

is an optimal policy for the initial conditions

$$(\tilde{l}, \tilde{h}) = (\lambda l_1 + (1 - \lambda)l_2, \lambda h_1 + (1 - \lambda)h_2)$$

for any $\lambda \in [0, 1]$. Using the concavity of the utility function we have

$$V(\tilde{l}, \tilde{h}) = \sup_{A(\tilde{l}, \tilde{h})} U(c) \geq U(\tilde{c}) > \lambda U(c_1) + (1 - \lambda) U(c_2)$$

$$= \lambda V(l_1, h_1) + (1 - \lambda) V(l_2, h_2).$$

(ii) Like in Chapter 4, the optimal policy for the initial conditions $(\lambda l, \lambda h)$ is $\lambda(c, \pi)$ because (4.3) is linear. Substituting $\lambda c$ to (4.2) and expanding $\log(\lambda c) = \log \lambda + \log c$ we obtain the property (5.4) for the value function $V(l, h)$.
Basing on this lemma, we present $V(l, h)$ in a form

$$V(l, h) = W(z) + \frac{\log h}{\kappa},$$

with $z = l/h$. The derivatives in (4.12) transform as follows

$$V_l = \frac{W_z}{h}, \quad V_{ll} = \frac{W_{zz}}{h^2}, \quad V_h = \frac{1}{h} \left(-zW_z + \frac{1}{\kappa}\right),$$

$$V_{hh} = \frac{1}{h^2} \left(2zW_z + z^2W_{zz} - \frac{1}{\kappa}\right), \quad V_{hl} = -\frac{1}{h^2} (W_z + zW_{zz}).$$

and we rewrite (5.1) correspondingly in the form

$$\frac{\eta^2}{2} z^2 W'' + \max_\pi \left[ \frac{1}{2} \pi^2 \sigma^2 W' + \pi (-W' + zW'') \eta \rho \sigma + (\alpha - r) W' \right]$$

$$+ \max_{c \geq -\delta} [-cW_z + \log(c + \delta)] = \kappa W - \frac{1}{\kappa} \left(\mu - \delta - \frac{\eta^2}{2}\right). \quad (5.5)$$

To eliminate the free term in (5.5), we shift the function $v = W - K$ to the constant

$$K = \frac{1}{\kappa^2} \left(\mu - \delta - \frac{\eta^2}{2}\right)$$

and finally obtain

$$\frac{\eta^2}{2} z^2 v'' + \max_\pi \left[ \frac{1}{2} \pi^2 \sigma^2 v' + \pi (-v' + zv'') \eta \rho \sigma + (\alpha - r) v' \right]$$

$$+ \max_{c \geq -\delta} [-cv_z + \log(c + \delta)] = \kappa v. \quad (5.6)$$

Assuming that $v$ is smooth and strictly concave, we perform the maximization of the quadratic part, obtaining

$$\kappa v = \frac{1}{2} \eta^2 (1 - \rho^2) z^2 v'' - \frac{k_1^2}{2\sigma^2} \left(\frac{v'}{v''}\right)^2 + k_2 v' + \max_{c \geq -\delta} [-cv_z + \log(c + \delta)], \quad (5.7)$$

where

$$k_1 = \alpha - r - \eta \rho \sigma, \quad k_2 = \eta^2 + r - (\mu - \delta), \quad k = \frac{\rho \eta k_1}{\sigma} + k_2.$$

Now, we can reconstruct the original value function $V(l, h)$ for $l, h > 0$ as

$$V(l, h) = K + \frac{\log h}{\kappa} + v \left(\frac{l}{h}\right), \quad (5.8)$$
and the optimal policies

\[ c^*(l, h) = \arg \max_{c \geq 0} (-cV_l + \log c) = \frac{h}{v'(l/h)}, \] (5.9)

\[ \pi^*(l, h) = \arg \max_{\pi} \left( \frac{1}{2} \pi^2 V_{hh} \sigma^2 + \pi (V_h \eta \rho \sigma h + (\alpha - r)V_l) \right) \]
\[ = \frac{-\eta \rho \sigma h V_{lh} + (\alpha - r)V_l}{\sigma^2 V_l} = \frac{\eta l}{\sigma} - h \frac{k_1}{\sigma^2} v'(l/h), \] (5.10)

It turns out that the limit value \( h = 0 \) coincides with the classical solution without any stochastic income, provided by Merton [31]:

\[ V(l, 0) = \frac{1}{\kappa^2} \left[ r + \frac{1}{2} \frac{(\alpha - r)^2}{\sigma^2} - \kappa \right] + \log(\kappa l), \] (5.11)

with a constant consumption rate and the constant wealth fraction invested in the stock.

\[ c^*(l, 0) = \kappa l, \] (5.12)
\[ \pi^*(l, 0) = \frac{(\alpha - r)}{\sigma^2}. \] (5.13)

Summing up, we announce the main result of this Chapter.

**Theorem 2.** Suppose \( r - (\mu - \delta) > 0 \) and \( k_1 \neq 0 \).

(i) There is the unique \( C^2(0, +\infty) \) solution \( v \) of the ODE (5.7) in a class of concave functions.

(ii) The value function is given by (5.8) for \( h, l > 0 \) and by (5.11) for \( h = 0, l > 0 \).

(iii) If the ratio between the stochastic income and the total wealth tends to zero, the policies \((\pi^*, c^*)\) given by (5.9), (5.10) tend to the classical Merton’s policies (5.12), (5.13).

(iv) Policies (5.9) and (5.10) are optimal.

**The dual optimization problem.** In this paragraph we introduce the dual optimization problem with a synthetic asset such that the optimization equation formally coincides with (5.7). It turns out that the regularity of the dual problem proves the regularity of the original one due to the uniqueness of
the viscosity solution. Let us consider the investment-consumption problem with the wealth process $Z_t$ defined by

\[
Z_t = (kZ_t + k_1\pi_t - c_t)dt + \sigma\pi_t\sigma dW_t^1 + \eta Z_t\sqrt{1 - \rho^2}dW_t^2, \quad (5.14)
\]

We define the set of admissible controls $\hat{A}(z)$ as the set of pairs $(c, \pi)$ such that

1. There exists an a.s. positive solution $Z_t$ of the stochastic differential equation (5.14).
2. $c_t \geq -\delta$.
3. $c$ and $\pi$ satisfy the integrability conditions for $A$ (see Chapter 4, page 13).

The investor wants to maximize the total utility given by

\[
\hat{U}(c) = E\left[\int_0^\infty e^{-\kappa\tau} \log(\delta + c(\tau))d\tau\right]
\]

and the value function $w$ is defined as

\[
w(z) = \sup_{(\pi, c) \in \hat{A}(z)} \hat{U}(c).
\]

The associated HJB equation is

\[
\kappa w = \frac{1}{2}\eta^2(1 - \rho^2)z^2 w'' + \max\pi \left[\frac{1}{2}\sigma^2 \pi^2 w'' + k_1\pi w'\right] + \max_{c \geq -\delta} \left[-cw' + \log(\delta + c + \delta)\right],
\]

Now, it is easy to see that (5.15) reduces to (5.7) assuming that $w$ is smooth. Thus, if we prove that $w$ is smooth and concave, we will get the desired result for $v$ as well.

**Viscosity solutions of the HJB equation** In this paragraph we analyze the equation (5.15) using the general theory of *viscosity solutions*. The framework of viscosity solutions generalize the classical solutions of ODE to non-smooth functions. We will concentrate only on results and estimates that are specific for the logarithmic utility, referring to [15] and [16], where general theorems cover our case. We use the following definition of the viscosity solution, as in [15]. Consider a nonlinear second-order ODE

\[
F(x, u, u_x, u_{xx}) = 0 \text{ in } \Omega,
\]

where $\Omega$ is an open subset of $\mathbb{R}$ and $F: \Omega \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous and elliptic, i.e. $\frac{\partial}{\partial a} F(x, t, p, a) > 0$. 
Chapter 5. Logarithmic utility function and the infinite time horizon

Definition 1. A continuous function $u : \Omega \to \mathbb{R}$ is a constrained viscosity solution of (5.16) if

- $u$ is a viscosity subsolution, i.e. for any $\phi \in C^2(\bar{\Omega})$ and any local maximum point $x_0 \in \Omega$ of $u - \phi$ holds $F(x_0, u(x_0), \phi_x(x_0), \phi_{xx}(x_0)) \leq 0$

- $u$ is a viscosity supersolution, i.e. for any $\phi \in C^2(\bar{\Omega})$ and any local minimum point $x_0 \in \Omega$ of $u - \phi$ holds $F(x_0, u(x_0), \phi_x(x_0), \phi_{xx}(x_0)) \geq 0$

To avoid the difficulties with a non-integrable zero-consumption rate we assume $\Omega = (\varepsilon, +\infty)$ and $\bar{\Omega} = [\varepsilon, +\infty)$ with $\varepsilon > 0$. Without loss of generality making $\varepsilon$ sufficiently small we will finally extend $\Omega$ to $(0, +\infty)$. At first we prove some analytical properties of value function $w$.

Lemma 2. The value function $w$ is finite.

Proof. The function

$$W^+(z) = C_1 \log(z + C_2), \quad z \geq 0$$

is a supersolution of (5.15) for sufficiently large $C_1, C_2 > 0$, so $w \leq W^+$ due to verification arguments, (see [15] for details) and thus finite.

Lemma 3. The value function $w$ is concave, strictly increasing and continuous at $\Omega$.

Proof. The concavity and monotonicity follow from the linearity of the state equation (5.14) similarly to Lemma 1. The continuity easily comes from the concavity.

Now, we are ready to use the general theorems that state the existence and uniqueness of the viscosity solution in our case.

Theorem 3. The value function $w$ is a constrained viscosity solution of (5.15) in $\Omega$.

This fact comes from the general theory of stochastic optimization and viscosity solutions, see [17].

Theorem 4. The value function $v$ is the unique constrained viscosity solution of the HJB equation (5.7) on $\bar{\Omega} \times \bar{\Omega}$ in a class of concave functions.

The proof is presented in [16].

Theorem 5. The value function $w$ is the unique $C^2(\Omega)$ solution of the (5.15) in a class of concave functions.
The proof of this theorem follows the lines of the Theorem 5 in [15] since it relies only on the local ellipticity of (5.15) and the boundness given by Lemma 2. The main idea is to consider the optimization problem with the artificial constraints $-N \leq \pi_t \leq N$ on the arbitrary interval $[z_1, z_2]$ that makes the equation (5.15) uniformly elliptic. Due to the uniqueness of the viscosity solution at the one hand and the smoothness of the classical solution of the uniformly elliptic equation at the other hand we get the smoothness of the solution of the constrained problem. After that it turns out that the artificial constraint $-N \leq \pi_t \leq N$ can be eliminated and the interval can be extended to the whole area $\Omega$.

**Asymptotic behavior of the value function.** In this section we examine the asymptotic behavior of the value function $V$ and show that as $l/h \to \infty$ it becomes the classical Merton solution.

**Theorem 6.** There is a positive constant $C_1$ such that

$$M + \frac{\log(\kappa l)}{\kappa} \leq V(l, h) \leq M + \frac{\log(\kappa(l + C_1\delta h))}{\kappa},$$

(5.17)

where

$$M = \frac{1}{\kappa^2} \left[ r + \frac{1}{2} \frac{(\alpha - r)^2}{\sigma^2} - \kappa \right]$$

is a constant from the Merton’s formula (5.11).

**Proof.** The left-hand inequality is obvious since any strategy $(c, \pi)$ for the classical problem with $L_0 = l, H_0 = 0$ is admissible for the problem with any non-zero initial endowment as well. To prove the right-hand inequality, let us consider a fictitious investment-consumption problem without any stochastic income but with an additional synthetic asset with the price process $S'$

$$dS'_t = \alpha' S'_t + \sigma' S'_t dW_t, \quad t \geq 0$$

$$S'_0 = s', \quad s' > 0.$$ 

The constants $\alpha'$ and $\sigma'$ will be defined later. Next, we define the initial wealth equivalent of the stochastic income defined by

$$f(h) = \delta E_h \left[ \int_0^\infty e^{-rt} \xi_t H_t dt \right],$$

where

$$\xi_t = \exp \left( -\frac{1}{2} (\theta^2_1 + \theta^2_2) + \theta_1 W^{(1)}_t + \theta_2 W_t \right),$$
\[ \theta_1 = (\alpha - r)/\sigma_1 \text{ and } \theta_2 = (\alpha' - r)/\sigma_1'. \]

It turns out that with the properly chosen \( \alpha' \) and \( \sigma' \), \( f(h) < C_1 \delta h \). Moreover, the stochastic income rate \( H_t \) can be replicated with the self-financing strategy on the market \( (B_t, S_t, S_t') \) with the additional initial endowment \( f(y) \). This fact is well known from the martingale-based studies of the consumption-investment problem, primarily [25] and [22]. Thus, any total utility generated by the strategy \( (c, \pi) \in A(l, h) \) can be attained in the settings of a classical Merton’s problem with the initial wealth \( l + f(h) < l + C_1 \delta h \). This actually gives the right-hand inequality in (5.17).

From this theorem we immediately get that \( V(l, h) \) behaves as a classical Merton’s solution (5.11) as \( \delta \to 0 \) or \( l/h \to \infty \).

**Corollary 1.** \( V_\delta(l, h) \) converges locally uniformly to \( M + \log(\kappa l)/\kappa \) as \( \delta \to 0 \).

**Corollary 2.** \( V(l, h) = M + \log(\kappa l)/\kappa + O(1/z) \) as \( z = l/h \to \infty \). Also,

\[
w(z) = (M - K) + \frac{\log(\kappa z)}{\kappa} + O(1/z), \tag{5.18}
\]

**Proof.** Indeed,

\[
\left| V(l, h) - M - \frac{\log(\kappa l)}{\kappa} \right| < \left| \frac{1}{\kappa} \left[ \log(\kappa(l + \delta C_1 h)) - \log(\kappa l) \right] \right| = O \left( \frac{1}{z} \right).
\]

The second formula immediately follows from the form of \( V(l, h) \).

Finally, we verify that the optimal policies given by (5.9) and (5.10) asymptotically give the Merton strategy (5.12), (5.13).

**Lemma 4.** For the value function \( w(z) \) holds

\[
w'(z) = \frac{1}{\kappa z} + o \left( \frac{1}{z} \right), \quad z \to \infty. \tag{5.19}
\]

**Proof.** Consider the function \( w_\lambda \) defined as

\[
w_\lambda(z) = w(\lambda z) - \frac{\log(\lambda)}{\kappa},
\]

so that \( w_\lambda \) solves (5.15) but with the term

\[
F(w_z) = \max_{c \geq -\delta} \left[ -cw_z + \log(c + \delta) \right]
\]

replaced by

\[
F_\lambda(w_z) = \max_{c \geq -\delta/\lambda} \left[ -cw_z + \log(c + \delta/\lambda) \right] .
\]
Then, by Corollary 1 $w_\lambda$ converges locally uniformly to the Merton’s value function
\[ v(z) = (M - K) + \frac{\log(\kappa z)}{\kappa}. \]
We note that $v$ solves the (5.15) with $\delta = 0$ that is delivered by
\[ F_\infty(\cdot) = \lim_{\lambda \to \infty} F_\lambda(\cdot). \]

Thus, because of concavity of $w_\lambda$ the uniform convergence of $w_\lambda$ to $v$ implies the convergence of derivatives, so
\[ \lim_{\lambda \to \infty} w'_\lambda(z) = v'(z) = \frac{1}{\kappa z}. \]
Hence,
\[ \lim_{\lambda \to \infty} w'_\lambda(1) = \lim_{\lambda \to \infty} \lambda v'(\lambda) = \frac{1}{\kappa}, \]
that proves the lemma. \hfill \Box

**Theorem 7.** The following asymptotic formulae hold for the optimal policies (5.9) and (5.10) as $z = l/h \to \infty$.

\[ \frac{c^*}{l} \sim \frac{1}{\kappa}, \quad \frac{\pi^*}{l} \sim \frac{\alpha - r}{\sigma^2}. \]

**Proof.** The relation (5.20) immediately follows from Lemma 4. For the second part, we rewrite (5.10) in a form
\[ \frac{\pi^*}{l} = \frac{\eta \rho}{\sigma} - \frac{k_1 z v'(z)}{\sigma^2 z^2 v''(z)}. \]

To calculate the limit value of $z^2 v''(z)$ we rewrite (5.7) as a quadratic equation with respect to $w_z z$. Since $w_z z < 0$ we choose the negative root and obtain
\[ w''(z) = \frac{-B - \sqrt{B^2 - 4AC}}{2A}, \]
where
\[ A = \frac{1}{2} t^2 (1 - \rho)^2 z^2, \]
\[ B = k(z w') - 1 - (M - C) \kappa + o(1), \]
\[ C = -\frac{k_1^2}{2 \sigma^2} (w')^2. \]
Expanding all constants and using $zw' = 1/\kappa + o(1)$ we finally get

$$z^2w''(z) = \frac{(\alpha - r)l}{\sigma^2} + o(1).$$

\[\square\]

**Verification theorem.** To verify that (5.9) and (5.10) are optimal one need to check that

**Theorem 8.** The process $Z_t$ given by (5.14) is uniquely well defined.

**Proof.** It suffices to show that the coefficients of the stochastic differential for $Z$ grow at most quadratically. The quadratic growth follows from the fact that $c^*/z$ and $\pi^*/z$ are bounded and continuous, as shown in Theorem 7. Thus,

$$(c^*)^2 + (\pi^*)^2 < C_1(1 + z^2).$$

\[\square\]

To prove that $(c^*, \pi^*)$ are optimal we use the standard dynamic programming principle. Following [15] for the optimal policy $C^*$ we have

$$E \left[ \int_0^T e^{-\kappa t} \log(C^*_t) dt \right] = E[e^{-\kappa T}V(L_T, H_T)] + V(l, h)$$

for every $T > 0$. By monotone convergence and finiteness of $V(l, h)$ we only need to have that

$$\lim_{T \to \infty} E[e^{-\kappa T}V(H_T, L_T)] = 0.$$  

We know that

$$V(H_T, L_T) \leq \log(H_T + C_1L_T) < C + H_T + C_1L_T$$

and

$$e^{-\kappa T}E[H_T] = he^{-(\kappa - \mu)T} \to 0 \text{ as } T \to \infty.$$  

According to Theorem 7 there are constants $C_1$ and $C_2$ such that $c^*(H_t, L_t) > C_1L_t + C_2H_t$, so

$$E \left[ \int_0^\infty e^{-\kappa t} (C_1L_t + C_2H_t) dt \right] \leq U(C^*) < \infty.$$  

Since

$$E \left[ \int_0^\infty e^{-\kappa t} (C_2H_t + C_2H_t) dt \right] < \infty$$
there exists a sequence $T_n \to \infty$, such that

$$\lim_{n \to \infty} E[e^{-\kappa T_n} L_{T_n}] = 0,$$

and, consequently,

$$\liminf_{T \to \infty} E[e^{-\kappa T} V(H_T, L_T)] \leq \lim_{T \to \infty} E[C + C_1 H_T] e^{-\kappa T} + E[e^{-\kappa T_n} L_{T_n}] = 0.$$

This completes the proof of the main Theorem 2.
CHAPTER 5. LOGARITHMIC UTILITY FUNCTION AND THE INFINITE TIME HORIZON
Chapter 6

Point Symmetries

This section of our work is connected with the point symmetries and the corresponding reductions of the equation obtained in Chapter 4 and Chapter 5.

The method for calculating the point symmetries and obtaining the corresponding reductions is the same for both equations. We want to find the Lie point symmetry group $G(\mathbb{R}^2)$ for the given equation. In order to do that we will use the standard techniques and notations that are well known and could be found, for example, in the book by Ibragimov [24].

Let us denote the left hand side of equation as $\Delta$. It is a well known fact that the action of the second prolongation of the infinitesimal generator $V$ of the corresponding group $G(\mathbb{R}^2)$ on the equation calculated on the solution manifold is zero

$$pr^{(2)}V \Delta(s, u, u_s, u_{ss}, \ldots)|_{\Delta=0} = 0. \tag{6.1}$$

The second prolongation of the infinitesimal generator $V$ of the corresponding group $G(\mathbb{R}^2)$ in our case has the following form

$$pr^{(2)}V = \xi \frac{\partial}{\partial s} + \tau \frac{\partial}{\partial t} + \varphi_s \frac{\partial}{\partial u} + \varphi_t \frac{\partial}{\partial u_t} + \varphi_s \frac{\partial}{\partial u_s} + \varphi_{ss} \frac{\partial}{\partial u_{ss}}, \tag{6.2}$$

where

$$\varphi_s = \varphi_s + (\varphi_u - \xi_s)u_s - \tau_s u_t - \xi_u u_s^2 - \tau_u u_su_t,$$

$$\varphi_t = \varphi_t + (\varphi_u - \tau_t)u_t - \xi_t u_s - \tau_u u_t^2 - \xi_u u_su_t,$$

$$\varphi_{ss} = \varphi_{ss} + (2\varphi_{su} - \xi_{ss})u_s - \tau_{ss} u_t + (\varphi_{uu} - 2\xi_{su})u_s^2 - 2\tau_{su} u_su_t - \xi_{uu} u_s^3 -$$

$$- \tau_u u_su_s^2 + (\varphi_{uu} - 2\xi_u)u_{ss} - 2\tau_{su} u_su_{st} - 3\xi_u u_su_{ss} - \tau_u u_t u_{ss} - 2\tau_u u_su_{st}.$$

Due to the fact that $u, u_s, u_{ss}, u_t$ are linearly independent we can satisfy equation (6.1) only by assuming that all the coefficients in front of the monomials
are equal to zero. Therefore we obtain the system of the equations that determines us all possible symmetries.

Let us start with the equation that was obtained by Tebaldi and Schwartz in [36] and is a particular case of the equation (4.6) that was discussed in Chapter 4.

The equation goes as follows

\[ u_t + k_4 s^2 u_{ss} - k_3 \frac{(u_s)^2}{u_{ss}} + k_2 s u_s + k_1 u - \frac{(u_s)b}{b} + \delta u_s = 0, \quad (6.3) \]

where \( k_1 \ldots k_4 \) are the constants that are determined by the economical setting (see page 12) and \( b \) is inversely proportional to the risk-aversion of the investor.

Using the approach that we have described above we can obtain the following system of the determining equation.
Table 6.1: The determining system of the equations in the case of HARA-utility

<table>
<thead>
<tr>
<th>Equation</th>
<th>Description</th>
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<tr>
<td>$u_4 u_{ss}^2$</td>
<td>$k_4 \xi_u$</td>
</tr>
<tr>
<td>$u_4 u_{st} u_{ss}$</td>
<td>$k_4 \tau_u$</td>
</tr>
<tr>
<td>$u_4 u_{s}$</td>
<td>$k_4 \tau_s$</td>
</tr>
<tr>
<td>$u_4 u_{ss}$</td>
<td>$(k_2 x + \delta) \tau_u$</td>
</tr>
<tr>
<td>$u_4 u_{st}$</td>
<td>$(k_2 x + \delta) \xi_u$</td>
</tr>
<tr>
<td>$u_4 u_{ss}$</td>
<td>$\tau_u$</td>
</tr>
<tr>
<td>$u_4 u_{s}$</td>
<td>$\xi_{uu} + (k_2 x + \delta) \tau_u$</td>
</tr>
<tr>
<td>$u_4 u_{ss}$</td>
<td>$(k_2 x - \delta) \tau_u + 4 \xi_u + 4 k_4 s^2 \tau_{su}$</td>
</tr>
<tr>
<td>$u_4 u_{ss}$</td>
<td>$(k_2 x - \delta) \xi_u + 3(2 k_2 s + \delta) \xi_u - 2 k_4 s^2 (\varphi_{uu} - 2 \xi_{su})$</td>
</tr>
<tr>
<td>$u_4 u_{ss}$</td>
<td>$2 k_4 \xi + 2 k_4 s^2 (\varphi_u - 2 \xi) - 2 k_4 s^2 \tau_u \tau_t$</td>
</tr>
<tr>
<td>$u_4 u_{ss}$</td>
<td>$k_2 \xi + (k_2 s + \delta) (\varphi_u - \xi) - \xi_t + (k_2 s + \delta) (\varphi_u - 2 \xi_t)$</td>
</tr>
<tr>
<td>$u_4 u_{ss}$</td>
<td>$-(k_2 s + \delta) \tau_u \tau_t + 2 k_4 s^2 (2 \varphi_{su} - \xi ss)$</td>
</tr>
<tr>
<td>$u_4 u_{ss}$</td>
<td>$(k_2 s + \delta) \tau_u - 2 \varphi_u + \tau_t + 2 \xi_u - \tau_u \tau_t - 2 k_4 s^2 \tau_{su}$</td>
</tr>
<tr>
<td>$u_4 u_{ss}$</td>
<td>$2 k_3 \xi_u + (k_2 s + \delta) (\varphi_{uu} - 2 \xi_{su})$</td>
</tr>
<tr>
<td>$u_4 u_{ss}$</td>
<td>$2 k_3 \tau_u + (\varphi_u - 2 \xi_{su}) - 2 (k_2 s + \delta) \tau_{su}$</td>
</tr>
<tr>
<td>$u_4 u_{ss}$</td>
<td>$\tau_s$</td>
</tr>
<tr>
<td>$u_4 u_{ss}$</td>
<td>$k_1 \xi_u$</td>
</tr>
<tr>
<td>$u_4 u_{ss}$</td>
<td>$k_1 \varphi + (k_2 s + \delta) \varphi_s + \varphi_t + 2 k_4 s^2 \varphi_{ss}$</td>
</tr>
<tr>
<td>$u_4 u_{ss}$</td>
<td>$2 k_3 (\varphi_u - \xi) - (k_2 s + \delta) (2 \varphi_{su} - \xi ss)$</td>
</tr>
<tr>
<td>$u_4 u_{ss}$</td>
<td>$2 k_3 \tau_u + (2 \varphi_{su} - \xi ss) - (k_2 s + \delta) \tau_{ss}$</td>
</tr>
<tr>
<td>$u_4 u_{ss}$</td>
<td>$k_1 (\varphi_{uu} - 2 \xi_{ss})$</td>
</tr>
<tr>
<td>$u_4 u_{ss}$</td>
<td>$k_1 (\varphi_u - 2 \xi_s) - k_1 \tau_u \tau_t$</td>
</tr>
<tr>
<td>$u_4 u_{ss}$</td>
<td>$2 k_3 \varphi_s - (k_2 s + \delta) \varphi_{ss}$</td>
</tr>
<tr>
<td>$u_4 u_{ss}$</td>
<td>$\varphi_{ss}$</td>
</tr>
<tr>
<td>$u_4 u_{ss}$</td>
<td>$k_1 (2 \varphi_{su} - \xi ss)$</td>
</tr>
</tbody>
</table>

In Table 6.1 of the monomials and their coefficients we have omitted the monomials for which the coefficients simply coincide. Obviously they would not add any additional restrictions on $\xi, \tau$ and $\varphi$. This was done just in order to make this step of our calculations more tractable and clear. Also we did not mention the monomials that had $u_s$ in the power of $b$ or some integer plus $b$. Let us remember that in our notation $b = \frac{1}{\gamma}$, where $\gamma$ is the measure of the risk aversion of the client. This monomials form a special set of the equations. If $b$ is not integer then these equations give us some extra restrictions on the $\xi, \tau$ and $\varphi$, however, we can see that if $b = 2, 3, 4$ these monomials do not form some independent equations but enter the
corresponding monomials in the system written above. This makes the case of an integer \( b \) rather interesting. Later we show that in this case equation (6.3) has additional symmetries and admit corresponding reductions. Now let us just show below the table of the "\( b \)-controlled" monomials.

| \( u_s^{b+3} \) | \( \frac{1}{b} \xi uu \) |
| \( u_s e^{t+2} u_t \) | \( \frac{1}{b} \tau_s \) |
| \( u_s^{b+2} \) | \( \frac{1}{b} (2 \xi su - \varphi_{uu}) \) |
| \( u_s^{b+1} \) | \( \frac{1}{b} (\xi ss - 2 \varphi_{su}) \) |
| \( u_s e^{t+1} u_t \) | \( \frac{1}{b} \tau_u \) |
| \( u_t u_s^{b+1} \) | \( \frac{2}{b} \tau_{su} \) |
| \( u_s u_s^{b+1} \) | \( \frac{3}{b} \xi u \) |
| \( u_t u_s u_s^{b} \) | \( \tau_u \) |
| \( u_s u_s u_s^{b} \) | \( \frac{b+1}{b} \varphi_u - \frac{b+2}{b} \xi s - \frac{1}{b} \xi u \tau_t \) |
| \( u_s u_s^{b} \) | \( \frac{1}{b} \tau_s \) |
| \( u_t u_s^{b} \) | \( \frac{1}{b} \tau_{ss} \) |
| \( u_s^{b} \) | \( \frac{1}{b} \varphi_s \) |
| \( u_t u_s u_s^{b-1} \) | \( \tau_s \) |
| \( u_s u_s u_s^{b-1} \) | \( \varphi_s \) |

Solving the obtained systems we come to the following conclusions:

- If \( b \) is not equal to 2, 3, 4 then the infinitesimal generator \( U(\cdot) \) for the corresponding equation in the most general form looks as follows

\[
U = c_1 \frac{\partial}{\partial t} + c_2 e^{-k_1 t} \frac{\partial}{\partial u}, \quad c_1, c_2 = \text{const}
\]

The algebra \( L \) of the infinitesimal generators of the group \( G(\mathbb{R}^2) \) in this case is two dimensional

\[
L_2 = \langle U_1, U_2 \rangle,
\]

where \( U_1 = \frac{\partial}{\partial t}, \ U_2 = e^{-k_1 t} \frac{\partial}{\partial u} \) with the commutation relation \( [U_1, U_2] = -k_1 U_2 \).

This form of the infinitesimal generator holds even if we vary the constants \( k_1, k_2, k_3 \) and \( k_4 \). Some additional symmetries do exist in the trivial case \( k_4 = 0 \) but they do not have any interest for us, because this condition reduces our illiquidity problem to the well known case when we do not have any illiquid asset under our management. Moreover, the form of the infinitesimal generator that is given above also holds for the cases of an integer \( b \) if our illiquid asset pays out some dividends (\( \delta \neq 0 \)).

Additional symmetries exist only if \( b \in \mathbb{N}, 2 \leq b \leq 4 \) and \( \delta = 0 \).
• If \( b = 2 \), \( k_i \neq 0, i = 1, \ldots, 4 \) and \( \delta = 0 \) then the general form of the infinitesimal generator is

\[
U = c_1 \frac{\partial}{\partial t} + c_2 s \frac{\partial}{\partial s} + 2c_2 u \frac{\partial}{\partial u} + c_3 e^{-k_1 t} \frac{\partial}{\partial u},
\]

where \( c_1, c_2, c_3 \) are arbitrary constants. The algebra \( L \) of the infinitesimal generators of the group \( G(\mathbb{R}^2) \) in this case is three dimensional

\[
L = \langle U_1, U_2, U_3 \rangle,
\]

where \( U_1 = \frac{\partial}{\partial s}, U_2 = e^{-k_1 t} \frac{\partial}{\partial u} \) and \( U_3 = s \frac{\partial}{\partial s} + u \frac{\partial}{\partial u} \). The commutator table looks as follows

Table 6.2: The commutator table for Lie algebra \( L_3 \)

<table>
<thead>
<tr>
<th></th>
<th>( U_1 )</th>
<th>( U_2 )</th>
<th>( U_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( U_1 )</td>
<td>0</td>
<td>(-k_1 U_2)</td>
<td>0</td>
</tr>
<tr>
<td>( U_2 )</td>
<td>( k_1 U_2 )</td>
<td>0</td>
<td>( U_2 )</td>
</tr>
<tr>
<td>( U_3 )</td>
<td>0</td>
<td>(-U_2)</td>
<td>0</td>
</tr>
</tbody>
</table>

• If \( b = 3 \), \( k_i \neq 0, i = 1, \ldots, 4 \) and \( \delta = 0 \) then the infinitesimal generator takes the form

\[
U = c_1 \frac{\partial}{\partial t} + c_2 e^{-k_1 t} \frac{\partial}{\partial u},
\]

where \( c_1, c_2 \) are arbitrary constants. The Lie algebra \( L \) is exactly the same as in (6.5).

• If \( b = 4 \), \( k_i \neq 0, i = 1, \ldots, 4 \) and \( \delta = 0 \) then we have to do with the three dimensional Lie algebra with the infinitesimal generator

\[
U = c_1 \frac{\partial}{\partial t} + 3c_2 s \frac{\partial}{\partial s} + 4c_2 u \frac{\partial}{\partial u} + c_3 e^{-k_1 t} \frac{\partial}{\partial u}, \quad c_1, c_2, c_3 = \text{const}
\]

The algebra \( \widetilde{L}_3 \) of the infinitesimal generators of the group \( G(\mathbb{R}^2) \) in this case is

\[
\widetilde{L}_3 = \langle U_1, U_2, U_3 \rangle,
\]

where \( U_1 = \frac{\partial}{\partial t}, U_2 = e^{-k_1 t} \frac{\partial}{\partial u} \) and \( U_3 = 3s \frac{\partial}{\partial s} + 4u \frac{\partial}{\partial u} \).

The case of logarithmic utility function. Now let us look on equation (5.5) that was obtained in Chapter 5. Exactly the same technique gives us the following table of monomials
Table 6.3: The determining system of the equations in the case of the logarithmic utility

<table>
<thead>
<tr>
<th>$w_z$</th>
<th>$k_1\varphi + (k_2z + \delta)\varphi_z + \varphi_t - \varphi_w + \xi_z + k_3z^2\varphi_{zz}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$w_t$</td>
<td>$(k_2z + \delta)\varphi_z + \varphi_w - \tau t + \tau w - k_3z^2\tau_{zz}$</td>
</tr>
<tr>
<td>$w_z^2$</td>
<td>$-(k_2z + \delta)\tau_w - \xi_w - 2k_3z^2\tau_{zw}$</td>
</tr>
<tr>
<td>$w_zw_t$</td>
<td>$-(k_2x + \delta)\xi_w + k_3z^2(\varphi_{ww} - 2\xi_{zw})$</td>
</tr>
<tr>
<td>$w_z^2$</td>
<td>$k_4\varphi_w$</td>
</tr>
<tr>
<td>$w_z^2$</td>
<td>$k_4\xi_w$</td>
</tr>
<tr>
<td>$\frac{1}{w_z}$</td>
<td>$\varphi_z$</td>
</tr>
<tr>
<td>$\frac{u_s}{w_z}$</td>
<td>$\tau_z$</td>
</tr>
<tr>
<td>$\frac{w_s}{w_z}$</td>
<td>$k_4(2\varphi_{zw} - \xi_{zz})$</td>
</tr>
<tr>
<td>$\frac{w_t^2}{w_z}$</td>
<td>$k_4(\varphi_{ww} - 2\xi_{zw})$</td>
</tr>
</tbody>
</table>

If $k_i \neq 0$, $i = 1, \ldots, 4$, then the infinitesimal generator $U(.)$ for the corresponding equation in the most general form looks exactly as for the corresponding case for the HARA-utility

$$U = c_1 \frac{\partial}{\partial t} + c_2 e^{-k_1t} \frac{\partial}{\partial u}, \quad c_1, c_2 = \text{const}$$

The algebra $L$ of the infinitesimal generators of the group $G(\mathbb{R}^2)$ in this case is also two dimensional

$$L_2 = \langle U_1, U_2 \rangle,$$

where $U_1 = \frac{\partial}{\partial t}$, $U_2 = e^{-k_1t} \frac{\partial}{\partial u}$ with the commutation relation $[U_1, U_2] = -k_1U_2$.

As well as for the HARA utility, if $k_i \neq 0$, $i = 1, \ldots, 4$ and $\delta = 0$ then we have to do with the three dimensional Lie algebra with the infinitesimal generator

$$U = c_1 \frac{\partial}{\partial t} + c_2s \frac{\partial}{\partial s} - c_2 \frac{\partial}{k_1 \partial u} + c_3 e^{-k_1t} \frac{\partial}{\partial u}, \quad c_1, c_2, c_3 = \text{const}$$

The algebra $L_3$ of the infinitesimal generators of the group $G(\mathbb{R}^2)$ in this case is

$$\widehat{L}_3 = \langle U_1, U_2, U_3 \rangle,$$

where $U_1 = \frac{\partial}{\partial t}$, $U_2 = e^{-k_1t} \frac{\partial}{\partial u}$ and $U_3 = \frac{s}{\partial w} - \frac{1}{k_1} \frac{\partial}{\partial w}$. The commutator table looks as follows...
Table 6.4: The commutator table for Lie algebra $L_3$ \((6)\)

<table>
<thead>
<tr>
<th></th>
<th>$U_1$</th>
<th>$U_2$</th>
<th>$U_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$U_1$</td>
<td>0</td>
<td>$-k_1U_2$</td>
<td>0</td>
</tr>
<tr>
<td>$U_2$</td>
<td>$k_1U_2$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$U_3$</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Let us briefly summarize the results obtained in this Chapter. As we can see both equations have a lot in common. We have shown that the algebra of the infinitesimal generators is usually two-dimensional ($L_2$) and becomes three-dimensional ($L_3, \tilde{L}_3, \hat{L}_3$) only in one particular case, when $\delta = 0$. These third operators are different for different utility-functions. Their existence in this case of $\delta = 0$ is inevitable and could be predicted because when $\delta = 0$ our problem becomes the well-known Merton-type problem without any random income.
Chapter 7

Illiquidity

As we have mentioned in Introduction, the understanding of the liquidity or, correspondingly, the illiquidity of a given asset is still a matter of a debate. The intuitive idea that is standing behind is quite clear but the definition that could be not only mathematically correct and correspond to the given model but also could make sense from the practical point of view is still discussed.

The problem of the optimal portfolio allocation for the investor with a portfolio that includes an illiquid asset is very natural for the modern markets, yet, as far as we know, the first analytical solution for it was given only in 2006 by Tebaldi and Schwartz in [36]. The solution of the problem was obtained in the framework of the following idea: they set up an investment horizon, denoted as $T$, and assume that the investor sells the asset that is regarded as illiquid right in the end of the investment horizon and could not sell it in any moment of time, $t$, that is smaller than $T$. This definition simplifies the problem but does not seem very practical. Of course it can be applicable in some special cases (i.e. when an investor regards his human capital as an illiquid asset and knows exactly the day of his retirement) but generally we can not fix the time when the opportunity for trading the illiquid asset occurs.

Let us now look on the history of the problem retrospectively in order to have a general understanding of the approaches to the illiquidity that are popular among the practitioners and researchers in the field of finance.

The first definition of illiquidity was given by Keynes, [26] as early as 1930: an asset is more liquid if it is ”more certainly realizable at short notice without loss”. This compact definition was used for more than 50 years as the only one in the scientific literature. It could be explained by the fact that the researchers assumed the problem to be too complex and not that important for the applications while the practitioners on the market were using their
intuitive understanding of the issue.

This intuitive understanding that could be regarded as a classical one for the brokers on the market can be found in any text book. Let us for example look in the "Wall Street Words: An A to Z Guide to Investment Terms for Today’s Investor" by Scott [37]: Illiquid asset is an asset that is difficult to buy or sell in a short period of time without its price being affected. It is much more flexible then the definition of Keynes. Though it gives us the practical understanding of the matter it can hardly be used in the mathematical applications because of its’ flexibility. However this definition gives us several important aspects of the term that deserve separate attention. First of all, the time aspect: difficult to buy or sell in a short period of time. This time scale is given to us by the model we work with, for example for Tebaldi and Schwartz any time $t$, that is $t < T$, is regarded as a "short" one; however, this scale doesn’t have to be deterministic, moreover, on the real modern markets even the most liquid assets do need some time to be traded. The second aspect is the price of an asset. Naturally, the buyer and the seller can have different view of the market and therefore can adjust a different price as a fair one for the same asset. This "misunderstanding" can postpone the deal.

Both of this two factors that are mentioned above are of the crucial importance for our future work. The first factor is defined by the framework of the market model while the second one depends on the investors’ utility function. In 1998 Froot and Stein (see [18]) in a model that has very strong implications for a risk management up to now give us the following definition of the illiquid asset: illiquid financial asset is an asset which, because of its information-intensive nature, cannot be frictionlessly traded in the capital markets. Despite the fact that the paper was criticized in particular for this "stricktness", that could be the source of the unusual consequences of the provided model (see, for example, [20]) it is quite widely cited in top Finance journals. For example in 2007 Cao and Teiletche, [9] were using the same definition for the problem of the alternative assets (the assets that are different from core assets such as money market, bonds and equities, say, factories, immobilities, etc.). Between 2000 and 2005, the amount of these assets under the management of the hedge funds industry doubled and reached the bns $1,000 threshold. One of the major features of alternative investments is that they are less liquid than standard ones. Cao and Teiletche in their paper give the idea of the management strategy for the portfolio that is comprised of these illiquid assets; however that work neither provides a full mathematical model for the portfolio of a kind nor gives us an analytical solution of the problem.

The first work that intended to tract the problem of illiquidity from the
mathematical point of view and to give a definition that would correspond to the "empirical" understanding of this property was done in 1986 by Lippman and McCall, [27]. They defined the environment that was characterized by four objects in the discrete time framework: \( c_i, T_i, X_i \) and \( \beta \).

Here \( c_i \) is a cost of owning/operating the asset during the period number \( i \) as well as the cost of the attempt to sell the asset. Offer arrives at each time in the set \( \{ S_i : i = 1, 2, \ldots \} \) of arrival times. The random variables \( S_i \) satisfy

\[
S_i = \sum_{j=1}^{i} T_j,
\]

where the integer valued random variables \( T_i \geq 0 \) need not to be either independent nor identically distributed.

\( X_i \) are positive independent identically distributed random variables that correspond to the price offered in the \( i \)th moment. All the expenditures are discounted at the rate \( \beta \) so that a present value of a dollar received in period \( i \) is \( \beta^i \). The discounted net receipts \( R(\tau) \) associated with a stopping time \( \tau \) is given by

\[
R(\tau) = \beta^\tau Y_{N(\tau)} - \sum_{j=1}^{\tau} \beta^j c_i
\]

\[
Y_i = \begin{cases} 
X_i, & \text{if recall is not allowed} \\
\max(X_1, \ldots, X_i), & \text{if recall is allowed},
\end{cases}
\]

where \( N(\tau) = \max\{n : S_n \leq \tau\} \) is the random number of offers that the seller observes when employing the decision rule \( \tau \) and the random variable \( Y_{N(\tau)} \) is the size of the accepted offer. Consequently, the seller chooses a stopping rule \( \tau^* \) in the set \( T \) of all stopping rules such that

\[
E[R(\tau^*)] = \max\{E[R(\tau)] : \tau \in T\}.
\]

Obviously, the time it takes to realize the asset’s value and to convert the asset into cash is the random variable \( \tau^* \). Lippman and McCall, [27] proposed to regard the expectation of this variable, \( E[\tau^*] \) as the measure of an asset’s illiquidity. With an increase in \( E[\tau^*] \) corresponding to a decrease in liquidity.

We pay so much attention to this approach because this work turned out to be a milestone in the area and the majority of the papers concerning illiquidity either use this approach directly or try to improve and broaden it. However, it is focused on the time aspect of the illiquidity problem, while we have already figured out that there is another aspect of illiquidity that is connected with the price of an asset and the inevitable loss that is associated with the immediate need to sell an illiquid asset.

This aspect was described in 1994 by Hooker and Kohn, [21]. The authors
decided to look on the money aspect and to use the same framework (known as a search problem) but to make emphasis on the affection of the price by our intention to sell an asset. They introduce an index of liquidity, $\lambda(I_t)$, as

$$
\lambda(I_t) = \frac{V(I_t) - L(I_t)}{V(I_t)}
$$

where $V(I_t)$ is the value of the asset under optimal sale, as a function of the information set $I_t$ and $L(I_t)$ is a loss from immediate sale of the asset. Since $\lambda$ depends on the information set $I_t$ they call this index the conditional liquidity of the asset. The authors also introduce the expected liquidity of an asset, $\Lambda$, which, obviously, is

$$
\Lambda = E[\lambda(I_t)].
$$

In their work Hooker and Kohn, [21] showed how their approach could be implemented on the market and give impressive results. They also gave a very clear and intuitive example in order to show that their approach has more economic sense then the time-approach that we want to give here as well.

Example 1. Suppose the current price of an asset is $100, and the value of optimal sale is $100.01. The expected time to sale when following the optimal policy is 100 years. According to Lippman and McCall’s definition this asset is highly illiquid. According to Hooker and Kohn it is almost perfectly liquid.

Judging from the other point of view it is easy to construct an example for which Lippman and McCall’s approach would give a better description of the real situation on the market.

Example 2. You have a factory which you want to sell. You have estimated the price of it and wait for a reasonable offer. Buyers when knowing that you want to have a deal get interested but do their own estimation. The offers they give come with a big time lag but could be really close to your estimated ”optimal” price, therefore, according to Hooker and Kohn the asset could look rather liquid while our common sense tells us that it is not.

As we have already mentioned there are numerous other approaches to the illiquidity that are based on the common sense and the empirical facts and estimations. Most of them have a lot in common with one of the definitions mentioned before. For example the idea that becomes more and more popular among the practitioners since the beginning of 2000s and was proposed by
several scientists separately (see Bangia et al. (1999), [3] or Coppejans et al. (2000) [10]). These are the approaches that could be in some sense regarded as the generalizations of the approach of Hooker and Kohn, [21], because they are estimating the liquidity of the assets through the bid-ask spread that could be treated in some sense as a ratio between the price by which we can sell the asset and the price by which we desire to sell it.

The last but not the least work which we need to mention is the paper of E. Acar, R. Adams and R. Williams, [1]. The authors propose to measure the illiquidity as ”the ratio between volume and the distance moved by the market”. They derive this concept from two basic ideas:

- The perfect liquidity indicator would assess the probable cost of the execution in the market.
- The perfect liquidity indicator would assess whether the markets were liable to anomalous moves.

They measure volume and price movement as evolving time series and then basing on the empirical data they verify their approach and make several conclusions out of which we would like to mention the following peculiarity: ”An increase in liquidity corresponds to an increase in volume but a decrease in volatility if anything”, [1].

This interesting fact that was found empirically would be interesting for us in the next chapter, so we mention it here.
Chapter 8

Compound liquidity indicator

In this Chapter we will use the ideas of Lippman and McCall, [27] and Hooker and Kohn, [21] that were mentioned in the previous Chapter 7. We will try to build a "comprised" measure of illiquidity that would take into consideration the time aspect as well as the aspect of price. Then we will show how our measure corresponds to the empirical data in order to verify the proposed model. In order to use our verified measure in the application to the model of Tebaldi and Schwartz, [36], we will use similar concept and notation yet will try to be as general in our reasoning as possible.

Let us look on the asset which illiquidity we want to estimate. Let us also assume that the price that we consider to be "fare" is developing according to the geometric Brownian motion

\[ dS_t = S_t(\alpha dt + \sigma dW^1_t). \]

Suppose that we continuously have the offer-price by which we can sell our asset immediately. This price is "given" to us by the market and is correlated with the moves of the "fair" price

\[ dH_t = H_t(\mu dt + \eta(\rho dW^1_t + \sqrt{1-\rho^2} dW^2_t)). \]  (8.1)

Here \( \rho \) is a correlation between two prices, \( \alpha \) and \( \mu \) are the drifts and \( \sigma \) and \( \eta \) are the volatilities. Typically the price that is proposed to us is lower than the one that we demand, so \( S_t > H_t \), unless the asset is perfectly liquid and could be sold immediately without any losses. Now let us estimate the \( E[\tau^*] \) that is the measure of illiquidity according to Lippman and McCall, [27].

This \( \tau^* \) would be the time when \( S_{\tau^*} = H_{\tau^*} \) because at this moment we would be able to sell our asset without losses. The calculations show us that

\[ E[\tau^*] = \frac{1}{\mu - \alpha + \frac{\sigma^2}{2} - \frac{\eta^2}{2}} \ln \frac{H_0}{S_0} \]  (8.2)
where $S_0$ and $H_0$ are the fair and offered prices in the moment of our liquidity estimation.

Now let us use the approach of Hooker and Kohn, [21] in the same framework. We can calculate the illiquidity as follows

$$\Lambda = E \left[ \frac{H_t}{S_t} \right].$$

Substituting the formulae for the prices we obtain

$$\Lambda = \frac{H_0}{S_0} e^{(\mu - \alpha + \sigma^2 - \eta \sigma^2) t}. \quad (8.3)$$

As long as this is a time-dependent expression we can substitute the value of the expectation form the equation (8.2) and, therefore, estimate the average losses that we will have to take when selling our asset in the average time that is enough for the perfect sale. In another words our strategy is as follows: we wait till the time $T = E[\tau^*]$ if before this time there is an offer such that $S_t = H_t$, we sell our asset. If there is no such offer we sell our asset at the moment $T$ anyway and bear the losses that we have to bear. Substituting (8.2) into (8.3) we obtain a new measure of illiquidity that we will denote as $\Upsilon$

$$\Upsilon = \left( \frac{H_0}{S_0} \right)^{\frac{1}{2}(\sigma + \eta)^2 - \eta \sigma (1+\rho)}. \quad (8.4)$$

This new measure of illiquidity has several advantages that are visible from the most general point of view.

- It takes into consideration both aspects of illiquidity (time and price).
- It does not depend on the drifts of the prices, which are difficult for the estimation on the real markets.
- It does depend on the ratio of the offered price and the price of a seller. This ratio in particular case could be regarded as the bid-ask ratio which allows our model to comprise the ideas to measure illiquidity as a function of this order-book value.
Let us now have a close look on the results represented in Table 8.1. In the first column we see the name of the equity, the second one shows the volume of trades, the third one shows an average time for an order to be executed (Lippman-measure) and the fourth one gives the ratio between the price of a transaction and the ask price (Hooker-measure). In the last column we show our measure of the liquidity. The sign "UNDEF" denotes the cases of weak trade for which the correlation between bid and ask turns out to be almost equal to one and the power of the ratio is very close to 0. In this cases upsilon is almost equal to one though from the most general point of view it is clear that this assets are by far not the most liquid ones. This is a disadvantage of the model, of course. Yet the stock market was used just in order to verify the proposed measure and to understand its' advantages and disadvantages better, so this fact is also important and useful.

First of all as we could expect the stocks turn out to be highly liquid assets (the upsilon is very close to 1). Yet even in the case of this highly liquid equities the liquidity measure varies and this differences clearly illustrate the advantages of our approach. Let us look closer on some symbols now.

The first three lines are a very interesting illustration of the fact that upsilon-illiquidity comprises the factors of time and price and "feels" them both while "Lippman-type" or "Hooker-type" do not. According to Hooker ATVI, CEPH and DISH are almost equally liquid (though ATVI is formally a bit more liquid, CEPH is in the middle and DISH is a bit less). Yet we see that Lippman-measure shows us that ATVI is indeed the most liquid one out of three while there is a huge difference between the liquidity of DISH and CEPH. DISH is traded much faster and the volume of trades is way bigger. Upsilon-liquidity shows us that. All three assets are highly liquid yet DISH and ATVI differ only by $8 \times 10^{-5}$ while CEPH turns out to be $5 \times 10^{-4}$ less liquid. All this perfectly fits into our understanding of liquidity.

Another three interesting symbols could be BRCM, ALTR and AMAT. The Lippman-measure for all three is very close $\sim 30,5$ - AMAT being a bit more liquid, ALTR and BRCM - a bit less. Yet we see a difference in the Hooker-measure which shows that BRCM is more liquid then ALTR. Upsilon-liquidity makes this difference more clear. It could also be seen that the assets with the bigger volumes of trade give the higher values for the upsilon-liquidity which corresponds to the results of E. Acar, R. Adams and R. Williams, that were mentioned in the chapter. Also the formula obtained for the upsilon-liquidity shows the connection between volatility and the liquidity indicator that explains the results obtained by the authors empirically.
Table 8.1: The example of different liquidity measures calculated empirically. Lippman-measure - the average time for a trade in seconds, Hooker-measure and upsilon - non-dimensional values that are closer to 1, if the asset is more liquid.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Transactions</th>
<th>Lippman-measure</th>
<th>Hooker-measure</th>
<th>Upsilon</th>
</tr>
</thead>
<tbody>
<tr>
<td>DISH</td>
<td>71</td>
<td>44.9</td>
<td>0.9971</td>
<td>0.9999</td>
</tr>
<tr>
<td>CEPH</td>
<td>20</td>
<td>68.1</td>
<td>0.9975</td>
<td>0.9994</td>
</tr>
<tr>
<td>ATVI</td>
<td>44</td>
<td>38.4</td>
<td>0.9979</td>
<td>0.9997</td>
</tr>
<tr>
<td>TEVA</td>
<td>23</td>
<td>99.8</td>
<td>0.99596</td>
<td>0.9986</td>
</tr>
<tr>
<td>PAYX</td>
<td>16</td>
<td>142.1</td>
<td>0.9848</td>
<td>0.9944</td>
</tr>
<tr>
<td>SPLS</td>
<td>46</td>
<td>24.8</td>
<td>0.9826</td>
<td>0.9999</td>
</tr>
<tr>
<td>CERN</td>
<td>22</td>
<td>48.0</td>
<td>0.9709</td>
<td>0.9712</td>
</tr>
<tr>
<td>CELG</td>
<td>11</td>
<td>73.7</td>
<td>0.9963</td>
<td>UNDEF</td>
</tr>
<tr>
<td>SYMC</td>
<td>58</td>
<td>31.1</td>
<td>0.9972</td>
<td>0.9984</td>
</tr>
<tr>
<td>EBAY</td>
<td>292</td>
<td>34.8</td>
<td>0.9991</td>
<td>0.9999</td>
</tr>
<tr>
<td>FLEX</td>
<td>84</td>
<td>25.2</td>
<td>0.9925</td>
<td>0.9978</td>
</tr>
<tr>
<td>ORCL</td>
<td>209</td>
<td>32.1</td>
<td>0.9991</td>
<td>0.99999</td>
</tr>
<tr>
<td>VRSN</td>
<td>633</td>
<td>32.4</td>
<td>0.9987</td>
<td>0.9998</td>
</tr>
<tr>
<td>LLTC</td>
<td>254</td>
<td>31.6</td>
<td>0.9980</td>
<td>0.9999</td>
</tr>
<tr>
<td>FLIR</td>
<td>8</td>
<td>205.1</td>
<td>0.9998</td>
<td>UNDEF</td>
</tr>
<tr>
<td>CTXS</td>
<td>61</td>
<td>40.7</td>
<td>0.9987</td>
<td>0.9997</td>
</tr>
<tr>
<td>MSFT</td>
<td>500</td>
<td>29.8</td>
<td>0.9990</td>
<td>0.99996</td>
</tr>
<tr>
<td>DELL</td>
<td>244</td>
<td>33.6</td>
<td>0.9982</td>
<td>0.99997</td>
</tr>
<tr>
<td>VRTX</td>
<td>15</td>
<td>57.6</td>
<td>0.9976</td>
<td>0.9573</td>
</tr>
<tr>
<td>CISCO</td>
<td>385</td>
<td>28.4</td>
<td>0.9988</td>
<td>0.99999</td>
</tr>
<tr>
<td>KLAC</td>
<td>423</td>
<td>27.7</td>
<td>0.9991</td>
<td>0.9999</td>
</tr>
<tr>
<td>ESRX</td>
<td>10</td>
<td>43.5</td>
<td>0.9999</td>
<td>UNDEF</td>
</tr>
<tr>
<td>FISV</td>
<td>65</td>
<td>52.8</td>
<td>0.9944</td>
<td>0.9981</td>
</tr>
<tr>
<td>LRCX</td>
<td>84</td>
<td>33.4</td>
<td>0.9961</td>
<td>0.99989</td>
</tr>
<tr>
<td>XNIX</td>
<td>363</td>
<td>28.6</td>
<td>0.9984</td>
<td>0.99987</td>
</tr>
<tr>
<td>BRCM</td>
<td>558</td>
<td>30.7</td>
<td>0.9981</td>
<td>0.9991</td>
</tr>
<tr>
<td>ALTR</td>
<td>254</td>
<td>30.8</td>
<td>0.9925</td>
<td>0.9989</td>
</tr>
<tr>
<td>AMAT</td>
<td>426</td>
<td>30.3</td>
<td>0.9991</td>
<td>0.99989</td>
</tr>
<tr>
<td>NTAP</td>
<td>232</td>
<td>29.3</td>
<td>0.9957</td>
<td>0.9996</td>
</tr>
<tr>
<td>AMGN</td>
<td>218</td>
<td>33.8</td>
<td>0.9985</td>
<td>0.99996</td>
</tr>
<tr>
<td>CHKX</td>
<td>273</td>
<td>29.2</td>
<td>0.9981</td>
<td>0.9999</td>
</tr>
<tr>
<td>SBUX</td>
<td>46</td>
<td>78.6</td>
<td>0.9935</td>
<td>0.9996</td>
</tr>
</tbody>
</table>
Summing up we want to say that in this chapter we have built an alternative liquidity indicator that as far as we know was not used before. This indicator comprises several popular ideas concerning the measures of illiquidity that were used before. Especially it is connected with the models of Lippman and McCall, [27] and Hooker and Kohn, [21]. The proposed measure does have economical sense and corresponds to the understanding of the liquidity as it is among the practitioners. Yet in some sense this measure turns out to be better than the ones that were proposed before because it distinguishes the assets that the older measures treat as almost equal in terms of liquidity. All these conclusions are made basing on the empirical calculations of the different liquidity indicators for the assets traded via Instinet. However, in the real world scenario it is quite difficult to estimate the volatility of the illiquid asset, that is a key component for the liquidity indicator $\Upsilon$. In the further research we are going to deal with this drawback considering more sophisticated price processes (e.g. Levy processes) and analyzing historical data of illiquid assets’ trades.
Bibliography


*Optimal policies and equilibrium prices with portfolio constraints and stochastic labor income*. Journal of Economic Theory, 72, 33 – 73.


*Hedging in incomplete markets with HARA utility*. Journal of Economic Dynamics and Control, 21, 753 – 782


Optimal replication of contingent claims under transaction costs. Review of Futures Markets, 8, 222 – 239.


The Pricing of Options on Assets with Stochastic Volatilities. The Journal of Finance, 42(2), 281 – 300


[26] J. Keynes (1930)


The asymptotic elasticity of utility functions and optimal investment in incomplete markets. The Annals of Applied Probability, 9, 904 – 950

[29] H. Leland (1985)
Option pricing and replication with transactions costs. The Journal of Finance 40(5), 1283 – 1301

Optimum consumption and portfolio rules in a continuous time model.
Journal of Economic Theory 3, 373 – 413.

[32] D. Mengle (Fourth Quarter 2007)


[34] J. W. Pratt (1964)

Correlation Trading Strategies.


[38] R. Sircar and T. Zariphopoulou (2007)
Utility Valuation of Credit Derivatives: Single and Two-Name Cases. Advances in mathematical finance, Applied and Numerical Harmonic Analysis, Part III, 279 – 301
Appendix

Here we demonstrate the code of the perl-programs that we used in order to obtain our data for Table 8.1

Listing 8.1: main.pl

```perl
package main;

require "OrderBook.pl";

# We do not print symbols with too few transactions
my $MINIMUM_TRANSACTIONS = 2;

use Data::Dumper;

my $order_book = OrderBook->new();
open( NASD, "Nasdaq_100.txt" );
open( OUT, "liquidity_measures.txt" );
open( ORDERS, "sample_data.txt" );
<NASD>;
<OUT>;

while (<NASD>) {
    chomp;
    my @ar = split (/\|/);
    $order_book->push_symbol( $ar[1] );
}
close(NASD);

while (<ORDERS>) {
    $order_book->push_string($_);
}

$order_book->calculate_spread_ratio_statistics();
for my $symbol ( keys %{$order_book} ) {
    if ( $order_book->{$symbol} {"trans_count"} >
```
$MINIMUM_TRANSACTIONS

)

{  
  print OUT $symbol, "$nTransactions:\n",  
  $order_book->$symbol["trans_count"]", "Lippman:\n",  
  $order_book->$symbol["lippman"], "Hooker:\n",  
  $order_book->$symbol["hooker"]", "Daddy:\n",  
  $order_book->$symbol["new_liquidity"]", "$n\n";
}

close (OUT);

close (ORDERS);
package OrderBook;

require "FIXQuote.pl";

use warnings;
use strict;
use Data::Dumper;

# Now we are interested only in messages of types
# mapped to 1. -1 means that we are going to omit
# messages of such type.

my $MESSAGE_TYPE_MARKER = "35";

my %MESSAGE_TYPES = {
    "U186" => -1,
    "U185" => -1,
    "U180" => -1,
    "U123" => 1,
    "U130" => -1,
    "U135" => -1,
    "U147" => -1,
    "U145" => -1,
};

sub new {
    my $class = shift;
    my %params = @_;  
    my ($self) = { }; 

    return ( bless( $self, $class ) );
}

sub push_string {
    my $self = shift;
    my $input_string = $_[0] ? $_[0] : return;
    my %fix_hash =
        split ( /[=,]/, $input_string ) ;  

    my $type =  
        $fix_hash{$MESSAGE_TYPE_MARKER}
? $fix_hash{MESSAGE_TYPE_MARKER}
  : return;

my $accept =
  exists $MESSAGE_TYPES{ $type }
? $MESSAGE_TYPES{ $type }
  : return;

#filter messages without quotes
if ( $accept != 1 ) {
  return;
}

my $quote = FIXQuote->new(%fix_hash);
if ( !( defined $quote->["SYMBOL"] ) ) {
  return;
}

#filter quotes with zero sizes and undefined spread
if ( !$quote->is_spread_well_defined() ) {
  return;
}

my $symbol = $quote->["SYMBOL"];

#keep info only for preloaded symbols
if ( !( defined $self->{ $symbol } ) ) {
  return;
}

my $time = $quote->exec_time();
my $bid = $quote->["BID_PRICE"];
my $ask = $quote->["ASK_PRICE"];
my $price = $quote->["EXECUTION_PRICE"];

$self->{ $symbol }{ "trans_count" }++;
$self->{ $symbol }{ "bid_sum" } += $bid;
$self->{ $symbol }{ "ask_sum" } += $ask;
$self->{ $symbol }{ "ask_sq" } += $ask * $ask;
$self->{ $symbol }{ "bid_sq" } += $bid * $bid;
$self->{$symbol}{"bid_times_ask"} += $ask * $bid;

$self->{$symbol}{"ex_time_sum"} += $time;
$self->{$symbol}{"ex_price_sum"} += $price;

}

sub push_symbol {
    my $self = shift;
    my $symbol = shift;
    $self->{$symbol}{"trans_count"} = 0;
    $self->{$symbol}{"bid_sum"} = 0;
    $self->{$symbol}{"ask_sum"} = 0;
    $self->{$symbol}{"ask_sq"} = 0;
    $self->{$symbol}{"bid_sq"} = 0;
    $self->{$symbol}{"bid_times_ask"} = 0;
    $self->{$symbol}{"ex_time_sum"} = 0;
    $self->{$symbol}{"ex_price_sum"} = 0;
}

sub calc_ask_avg {
    my $self = shift;
    my $symbol = shift;
    $self->{$symbol}{"ask_avg"} =
        1.0 * $self->{$symbol}{"ask_sum"} / $self->{$symbol}{"trans_count"};
}

sub calc_bid_avg {
    my $self = shift;
    my $symbol = shift;
    $self->{$symbol}{"bid_avg"} =
        1.0 * $self->{$symbol}{"bid_sum"} / $self->{$symbol}{"trans_count"};
}
sub calc_lippman {
    my $self = shift;
    my $symbol = shift;

    $self->{symbol}{"lippman"} =
    1.0 * $self->{symbol}{"ex_time_sum"} /
    $self->{symbol}{"trans_count"};
}

sub calc_hooker {
    my $self = shift;
    my $symbol = shift;

    $self->{symbol}{"ex_price_avg"} =
    1.0 * $self->{symbol}{"ex_price_sum"} /
    $self->{symbol}{"trans_count"};

    $self->{symbol}{"hooker"} =
    $self->{symbol}{"ex_price_avg"} /
    $self->{symbol}{"ask_avg"};
}

sub calc_ask_var {
    my $self = shift;
    my $symbol = shift;

    if ( $self->{symbol}{"trans_count"} < 1 ) {
        $self->{symbol}{"ask_var"} = 0;
        $self->{symbol}{"ask_std"} = 0;
    } else {
        $self->{symbol}{"ask_var"} =
        1.0 * $self->{symbol}{"ask_sq"} - 2 *
        $self->{symbol}{"ask_sum"} *
        $self->{symbol}{"ask_avg"} +
        $self->{symbol}{"trans_count"} *
        ( $self->{symbol}{"ask_avg"}**)2 ;

        # filter out too small variations
        if ( $self->{symbol}{"ask_var"} < 1.0e-10 ) {
            $self->{symbol}{"ask_var"} = 0;
            $self->{symbol}{"ask_std"} = 0;
        }
    }
}
```perl
else {
    $self->{symbol}{"ask_std"} =
        sqrt ( $self->{symbol}{"ask_var"} /
            ( $self->{symbol}{"trans_count"} - 1 ) );
}
}

sub calc_bid_var {
    my $self = shift;
    my $symbol = shift;
    if ( $self->{symbol}{"trans_count"} < 1 ) {
        $self->{symbol}{"bid_var"} = 0;
        $self->{symbol}{"bid_std"} = 0;
    } else {
        $self->{symbol}{"bid_var"} =
            1.0 * $self->{symbol}{"bid_sq"} - 2 *
                $self->{symbol}{"bid_sum"} *
                $self->{symbol}{"bid_avg"} +
                $self->{symbol}{"trans_count"} *
                ( $self->{symbol}{"bid_avg"}**2 );
    }
    # filter out too small variations
    if ( $self->{symbol}{"bid_var"} < 1.0e-10 ) {
        $self->{symbol}{"bid_var"} = 0;
        $self->{symbol}{"bid_std"} = 0;
    } else {
        $self->{symbol}{"bid_std"} =
            sqrt ( $self->{symbol}{"bid_var"} /
                ( $self->{symbol}{"trans_count"} - 1 ) );
    }
}

sub calc_corr {
    my $self = shift;
    my $symbol = shift;
```
my $s_a = $self->{symbol}{"ask_sum"};
my $s_b = $self->{symbol}{"bid_sum"};
my $s_ab = $self->{symbol}{"bid_times_ask"};
my $av_a = $self->{symbol}{"ask_avg"};
my $av_b = $self->{symbol}{"bid_avg"};
my $var_a = $self->{symbol}{"ask_var"};
my $var_b = $self->{symbol}{"bid_var"};
my $N = $self->{symbol}{"trans_count"};

my $denom = 1.0 * $s_ab - $s_a * $av_b - $s_b * $av_a + $N * $av_a * $av_b;
my $numer = sqrt($var_a * $var_b);

if ($numer == 0) {
    $self->{symbol}{"bid_ask_corr"} = "UNDEFINED";
} else {
    $self->{symbol}{"bid_ask_corr"} = $denom / $numer;
}

sub calc_daddy_liquidity_measure {
    my $self = shift;
    my $symbol = shift;

    my $std_a = $self->{symbol}{"ask_std"};
    my $std_b = $self->{symbol}{"bid_std"};
    my $r = $self->{symbol}{"bid_ask_corr"};

    if ( abs($std_a * $std_b) < 1.0e-20 ) {
        $self->{symbol}{"new_liquidity"} = "UNDEFINED";
    } else {
        my $ratio = 1.0 * $self->{symbol}{"bid_avg"} / $self->{symbol}{"ask_avg"};
        my $power = ($std_a + $std_b)**2 - ( 1.0 + $r ) * $std_a *
```perl
$std_b;
$self->{symbol}{"new_liquidity"} = $ratio**$power;
}

sub calc_statistics {
}

sub calculate_spread_ratio_statistics {
  my $self = shift;
  my $symbol = shift;

  foreach my $symbol ( keys %{$self} ) {
    if ( $self->{symbol}{"trans_count"} != 0 ) {
      $self->calc_ask_avg($symbol);
      $self->calc_bid_avg($symbol);
      $self->calc_ask_var($symbol);
      $self->calc_bid_var($symbol);
      $self->calc_corr($symbol);
      $self->calc_lippman($symbol);
      $self->calc_hooker($symbol);
      $self->calc_daddy_liquidity_measure($symbol);
    }
  }
}

return 1;
```
package BookTypes;

sub new {
    my $class = shift;
    my %params = @_;

    bless {
        "U123" => {
            name => "Instinet Top of the Book",
            handler =>
        },
        "U130" => { name => "Island execution message" },
        "U135" => { name => "Island Top of the Book" },
        "U147" => { name => "REDI Top of the Book" },
        "U145" => { name => "ARCA Top of The Book" },
        "U180" => { name => "Nasdaq Market Maker Short Quote" },
        "U185" => { name => "NASDAQ Level I Quote" },
        "U186" => {
            name => "NASDAQ Short Trade Report",
            handler => sub { return 1 }
        },
    };
}
package FIXQuote;

use Time::Local;

sub new {
    my $class = shift;
    my %params = @_;

    my $ask = $params{"208"};
    if ( index( $ask, ";" ) >= 0 ) {
        $ask = substr $ask, 0, index( $ask, ";" );
        print "Ask: ", $ask, " old: ", $params{"208"}, \n;
    }

    my $bid = $params{"206"};
    if ( index( $bid, ";" ) >= 0 ) {
        $bid = substr $bid, 0, index( $bid, ";" );
        print "Bid: ", $bid, " old: ", $params{"206"}, \n;
    }

    bless {
        "SYMBOL" => $params{"201"},
        "CURRENT_UNIXTIME" => $params{"122"},
        "TRANSACTION_TIME" => $params{"60"},
        "CURRENT_TIME" => $params{"51"},
        "HHMMSS" => $params{"52"},
        "BID_PRICE" => $bid,
        "EXECUTION_PRICE" => $params{"31"},
        "EXECUTION_SIZE" => $params{"32"},
        "BID_SIZE" => $params{"207"},
        "ASK_PRICE" => $ask,
        "ASK_SIZE" => $params{"209"},
        "MESSAGE_TYPE" => $params{"35"}
    }, $class;
}

sub spread {
    my $self = shift;
    return $self->{"ASK_PRICE"} - $self->{"BID_PRICE"};
}
sub spread_mean_value {
    my $self = shift;
    return 0.5 *
        ( $self->{"BID_PRICE"} + $self->{"ASK_PRICE"} );
}

sub exec_time {
    my $self = shift;
    return $self->{"ORDER_TIME"};
}

sub is_spread_well_defined {
    my $self = shift;
    if ( !( defined $self->{"BID_PRICE"} ) ) { return 0 }
    if ( $self->{"BID_PRICE"} == 0 ) { return 0 }
    if ( !( defined $self->{"ASK_PRICE"} ) ) { return 0 }
    if ( $self->{"ASK_PRICE"} == 0 ) { return 0 }
    if ( !( defined $self->{"EXECUTION_PRICE"} ) ) { return 0 }
    if ( $self->{"EXECUTION_PRICE"} == 0 ) { return 0 }
    if ( $self->{"EXECUTION_PRICE"} < $self->{"EXECUTION_PRICE"} ) { return 0 }
    if ( $self->{"EXECUTION_PRICE"} < $self->{"BID_PRICE"} ) { return 0 }
    if ( !( defined $self->{"EXECUTION_SIZE"} ) ) { return 0 }
    if ( $self->{"EXECUTION_SIZE"} == 0 ) { return 0 }
    if ( !( defined $self->{"TRANSACTION_TIME"} ) ) { return 0 }
    if ( !( defined $self->{"CURRENT_TIME"} ) ) { return 0 }
    eval {
        my ( $year, $mon, $day, $hour, $min, $sec ) =
            $self->{"TRANSACTION_TIME"} = "
}
my $ex_time = timegm( $sec, $min, $hour, $day, $mon, $year );

my $curr_time = timegm( $sec, $min, $hour, $day, $mon, $year );
$self->{"ORDER_TIME"} = $curr_time - $ex_time;

if ($@) {
  return 0;
}

if ( ! ( defined $self->{"ORDER_TIME"} ) ) { return 0 }
if ( ! ( defined $self->{"BID_SIZE"} ) ) { return 0 }
if ( ! ( defined $self->{"ASK_SIZE"} ) ) { return 0 }

if ( ! ( defined $self->{"SYMBOL"} ) ) { return 0 }
return ( ( $self->{"BID_SIZE"} != 0 )
  && ( $self->{"ASK_SIZE"} != 0 ) );

return 1;