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Pricing and Hedging of Defaultable Models

Master's Thesis in Financial Mathematics

Magdalena Antczak Marta Leniec

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School of Information Science, Computer and Electrical Engineering
Halmstad University

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Magdalena Antczak
Marta Leniec

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Supervisor: Prof. Lioudmila Vostrikova-Jacod
Examiner: Prof. Ljudmila A. Bordag
External referees: Prof. Mikhail Babich

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Department of Mathematics, Physics and Electrical engineering
School of Information Science, Computer and Electrical Engineering
Halmstad University

Preface

This thesis has been prepared at the University of Angers under the supervision of Professor Lioudmila Vostrikova-Jacod. We would like to thank her for help in understanding the defaultable framework and useful remarks. The conversations at the Faculty and seminars were priceless. We also want to express our sincere gratitude to Professor Ljudmila Bordag for organizing our Erasmus in France.

Abstract

Modelling defaultable contingent claims has attracted a lot of interest in recent years, motivated in particular by the Late-2000s Financial Crisis. In several papers various approaches on the subject have been made. This thesis tries to summarize these results and derive explicit formulas for the prices of financial derivatives with credit risk. It is divided into two main parts. The first one is devoted to the well-known theory of modelling the default risk while the second one presents the results concerning pricing of the defaultable models that we obtained ourselves.

Contents

Chapter 1

Introduction

In the world of finance, it is crucial to consider the models based on the fact that the companies may default. Hearing the word 'default' one can imagine the biggest defaults in the history of economy like that of Lehman Brothers in 2008. However, the exact definition of a default explains it only as a failure to meet debt obligations such as loans or bonds. The debtor is in default when he is either unable or unwilling to pay the debt. One has to distinguish the default from a state of being unable to pay the debts precisely which is called insolvency. The company is insolvent when it is unable to pay debts as they fall due (cash flow insolvency) or when the liabilities exceed the assets (balance sheet insolvency). It is worth mentioning that the insolvency can lead to a bankruptcy which is the process of legally defining a financial situation as insolvent. While modelling credit risk, one usually takes under consideration the company's default in general, without looking into the causes and hence distinguishing between being unable or unwilling to pay the debts.

In the world of mathematics, the default appears as default time which is a strictly positive random variable. One can define this random variable in many ways. However, the most common one is the first time of crossing a barrier by a certain process, e.g. a stock price process of a company (see a Figure 9.1).

Modelling of the default event can be done in two manners. The first one is called structural approach. It assumes that default time τ is a stopping time in the assets filtration $\mathbb{F}$. The second one, called reduced-form approach, is based on the assumption that τ is a stopping time in a larger filtration and may no longer be measurable with respect to the prices filtration. In our thesis, we focus on the last approach.

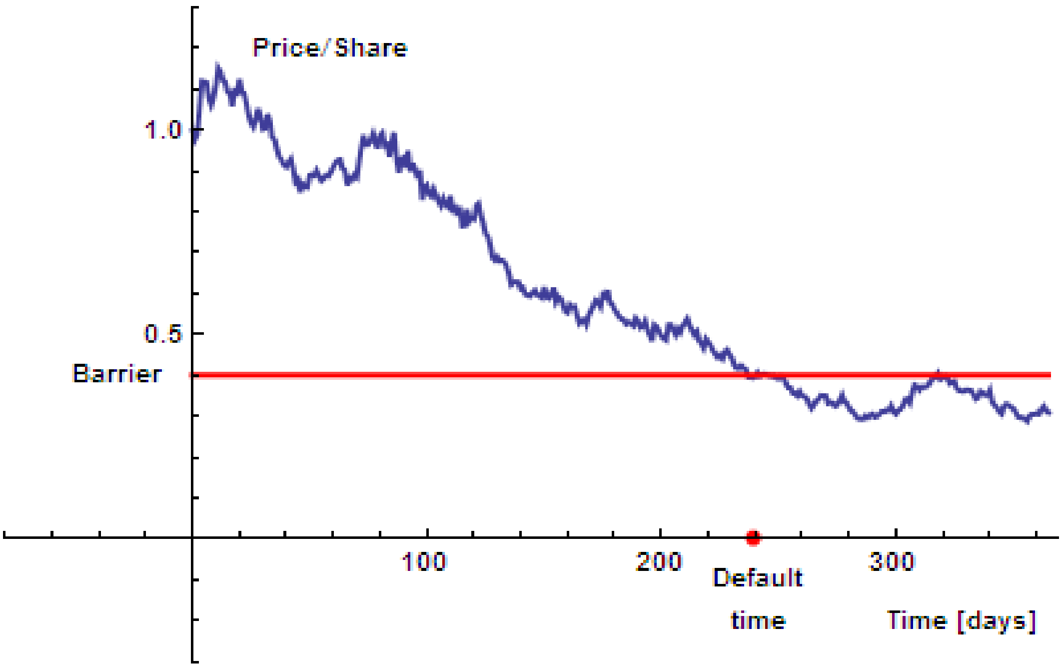

Figure 1.1: An example of a defaultable company stock price process.

We consider a non-defaultable world which consists of riskless and risky assets. A filtration generated by the prices of those assets is denoted by $\mathbb{F}$ and called the reference filtration. It represents the information available to the regular investor in a non-defaultable world. However, when we take under consideration a possibility of a default we have to introduce default time τ and create a defaultable framework which may consists of default-free and defaultable assets, e.g. stock of the company that may default. We have to study different types of information flows available to agents trading in a defaultable market. On the one hand, the regular investors add the information about default to $\mathbb{F}$ when it occurs, i.e. they work in a progressive enlargement setting. On the other hand, we shall examine also the insider, i.e. the agent who possesses information about default time from the beginning. The information accessible to this agent is represented by a filtration $\mathbb{F}$ initially enlarged by a positive random variable τ. In our thesis, we explore the special theory which establishes methods of enlarging the reference filtration by the additional information, namely Carthaginian Enlargement of Filtrations (see [2]).

We distinguish two methods of modelling default time in a reduced-form approach, namely the intensity (see [1]) and the conditional density-based approach (see [2] and [3]). They are used to establish the expectation and projection tools which are necessary for pricing an hedging of financial derivatives. An intensity of default is simply a ratio of probability that default will appear in a infinitely small time interval (under the condition that there was no default before) and the time step. However, to determine the conditional density of default, we need to assume that the conditional law of τ is equivalent to the law of τ.

In the first chapter, we study some basic results concerning probability spaces and filtrations, as well as stochastic processes, in particular a Brownian motion. We introduce some facts concerning stopping times and martingales.

In the second chapter, we introduce crucial assumptions related to the filtered probability space involving default time and all the price processes. Then, we introduce the law of τ and we give a definition of a default process. We determine the form of a random variable measurable with respect to the σ-algebra generated by that process and give some properties of the corresponding filtration.

Third chapter is devoted to the intensity approach in the filtration generated by the default process. In this framework, we give tools to compute expectations with respect to the σ-algebra generated by this process. Then, we value under the physical measure defaultable zero-coupon bond at time t in the case of zero and non-zero spot rate for the agent whose information flow is the filtration mentioned above. Finally, we give formulas and properties of the survival and hazard function and we represent once again the defaultable zero-coupon bond value using these functions.

In the fourth chapter, we present firstly the theory of Carthaginian Enlargement of Filtrations and hence, the methods to enlarge reference filtration with an additional information. Secondly, we represent random variables with respect to the corresponding σ-algebras. Then, we introduce the crucial assumption that states that the conditional law of default time τ is equivalent to the law of τ. In addition, we present the density hypothesis which allows to express the distribution of τ conditioned on the information from the reference filtration in terms of the conditional density process and the law of τ. We show that under the additional assumption concerning the law of τ, namely the property of being non-atomic, default time avoids stopping times from the reference filtration. The second important part of this chapter is devoted to introducing the so-called decoupling measure which makes

τ and the underlying risky assets independent. We consider some properties of the new measure and establish the expectation tools using obtained independence. What is more, we establish the form of the survival process under the physical and decoupling measure. Finally, we prove that initially enlarged filtration inherits right-continuity from the reference filtration.

Fifth chapter presents some results obtained in the initially enlarged filtration, i.e. the expectation tools and the characterization of martingales from the enlarged filtration in terms of martingales from the reference filtration. We finish the chapter with establishing the conditions for the absence of arbitrage in the enlarged filtration.

In the sixth chapter we examine the progressive enlargement framework. We begin with the intensity-based approach and assume that a price process follows the log-normal distribution and the reference filtration is generated by a standard Brownian motion. Firstly, we establish some expectation tools. Secondly, we introduce a hazard process in terms of the results obtained from the expectation tools. Then, we introduce the intensity in the progressively enlarged filtration. We continue the chapter by studying the hypothesis that martingales from the reference filtration remain martingales in the enlarged filtration, namely $\mathcal{H}$-hypothesis which is strongly related to the absence of arbitrage. We finish the intensity-based approach part with demonstrating the value of the default information, i.e. the difference between the price of a defaultable contingent claim for an agent who possesses the information about the default when it occurs and the one who does not have this information. In the second part of this chapter, we analyse the density-based approach. We begin with establishing the projection of random variables on the progressively enlarged filtration and we obtain the Radon-Nikodým on this filtration. We continue with examining the relation between the density hypothesis and the $\mathcal{H}$-hypothesis and finish with the martingales characterization.

The seventh chapter consists of our own results. We calculate the price of the option written on a investment consisting of both, default-free and defaultable assets. We consider a default-free market consisting of one riskless asset and one risky asset and a defaultable market created by adding one defaultable asset to the preceding model. We define a reference filtration as a filtration generated by a price process of a default-free asset. We define default time τ as the first time when defaultable asset's price crosses a certain barrier from interval $(0, 1)$ and we establish distribution of τ. We consider two agents trading in a defaultable market, a regular investor who observes only a price process of a default-free asset and a special agent who has additional information concerning default time τ from the beginning, i.e. its distribution. We put an accent on the fact that the defaultable market is arbitrage-free and incomplete for the regular investor and hence, we find it interesting to calculate the price of the option for such an investor. We find a pricing measure using the connection between two well-known methods, the utility maximization and the f-divergence minimization.

Chapter 2

Stochastic background

In the Theory of Financial Markets pricing is based either on the stochastic or partial differential equations approach. We will focus on the former one. It is important to remind the most important definitions from the Theory of Stochastic Processes which will be used throughout our thesis.

2.1 The probability space and filtrations

While considering the randomness, it is necessary to introduce a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ which is a mathematical form essential for modelling the stock prices and default processes consisting of the states which occur with uncertainty. A non-empty sample space Ω is an universe of all possible random events ω. In our case it is a space of all possible scenarios that can happen on the financial market. For further calculations and reasoning it is crucial to use a certain type of collections of these events $\omega \in \Omega$. Let us denote $\mathcal{P}(\Omega)$ the set of all subsets of Ω. From the Theory of Probability we know how to treat the collections which are closed under countable unions and joints. Consequently, we introduce the most important algebraic structure, σ-algebra over Ω, as following.

Definition 2.1. Let Ω be a non-empty sample space. $\mathcal{F} \subset \mathcal{P}(\Omega)$ is called a σ-algebra over Ω, if

i) $\emptyset \in \mathcal{F}$,

ii) $F \in \mathcal{F} \Rightarrow F^C \in \mathcal{F}$,

iii) $\forall i \in I$, $F_i \in \mathcal{F} \Rightarrow \bigcup_{i \in I} F_i \in \mathcal{F}$, where $I \subset \mathbb{N}$.

$\mathbb{N}$ is a set of natural numbers.

From the De Morgan's laws we can easily combine ii) and iii) from the previous definition and get that the countable joints remain in the σ-algebra.

Remark 2.1. If $\mathcal{F}$ is a σ-algebra over Ω, then

i) $\Omega \in \mathcal{F}$,

ii) $\forall i \in I,\ F_i \in \mathcal{F} \Rightarrow \bigcap_{i\in I} F_i \in \mathcal{F}$.

Through equipping the sample space with the σ-algebra $\mathcal{F}$ we get a pair $(\Omega, \mathcal{F})$ called a measurable space. On such a space we can define a probability measure and obtain the probability space.

Definition 2.2. Let Ω be a non-empty sample space and $\mathcal{F}$ a σ-algebra over Ω. The pair $(\Omega, \mathcal{F})$ is called a measurable space.

In the Mathematical Finance, for pricing financial derivatives, one can use several probability measures calculated from the actual market movements. For instance, a martingale measure is based on the risk-neutrality approach. Accordingly, in pursuance of the previous notations and assumptions we can define a probability measure $\mathbb{P}$ on measurable space $(\Omega, \mathcal{F})$ defined on the set of events from Ω.

Definition 2.3. We call a function $\mathbb{P} : \mathcal{F} \to [0, 1]$ a probability measure on $(\Omega, \mathcal{F})$ if

i) $\mathbb{P}(\emptyset) = 0$,

ii) $\mathbb{P}(\Omega) = 1$,

iii) $\forall i \in I\ F_i \in \mathcal{F}$ are disjoint, i.e. $F_i \cap F_j = \emptyset$ if $i \neq j$ then

$$\mathbb{P}(\bigcup_{i\in I} F_i) = \sum_{i\in I} \mathbb{P}(F_i),$$

where $I \subset \mathbb{N}$.

Broadly speaking, a probability space is a measurable space such that the measure of the whole space is equal to one. In accordance with the previous suppositions we can define it more formally.

Definition 2.4. We call a triplet $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space where $\Omega \neq \emptyset$, $\mathcal{F}$ is a σ-algebra over Ω and $\mathbb{P}$ is a probability measure on $(\Omega, \mathcal{F})$.

In mathematics there are some sets which can be ignored. In the Theory of Probability we call them $\mathbb{P}$-negligible sets. They can be omitted when calculating integrals of measurable functions.

Definition 2.5. A set $A \in \mathcal{F}$ is called a $\mathbb{P}$-negligible set if $\mathbb{P}(A) = 0$.

In general, the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ does not have to contain all $\mathbb{P}$-negligible sets. However, it can be completed by incorporating all subsets of $\mathbb{P}$-negligible sets in a suitable manner.

Definition 2.6. A triplet $(\Omega, \mathcal{F}, \mathbb{P})$ is called a complete probability space if $\mathcal{F}$ contains all $\mathbb{P}$-negligible sets.

It is important in the Theory of Martingales to define the filtration on a measurable space $(\Omega, \mathcal{F})$. In the mathematical finance we understand the filtration as the information available up to and including each time t which is more and more precise as more data from the stock becomes accessible.

Definition 2.7. $\mathbb{F}$ is a filtration if $\mathbb{F}$ is a family of non-decreasing sub-σ-algebras $(\mathsf{P}F_t)_{t\geq 0}$ such that $\forall t \geq 0$ $\mathcal{F}_t \subset \mathcal{F}$ and $\forall 0 \leq s < t < \infty$ $\mathcal{F}_s \subset \mathcal{F}_t$.

Similarly as before, we define a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ also known as a stochastic basis or a probability space with a filtration of its σ-algebra.

Definition 2.8. We call the quadruple $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ a filtered probability space, where $\Omega \neq \emptyset$, $\mathcal{F}$ is a σ-algebra over Ω, $\mathbb{F}$ is a filtration and $\mathbb{P}$ is a probability measure.

For further considerations we introduce a complete filtered probability space.

Definition 2.9. $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ a complete filtered probability space if $\mathcal{F}$ contains all $\mathbb{P}$-negligible sets and $\forall t \geq 0$ $\mathcal{F}$ contains all $\mathbb{P}$-negligible sets.

2.2 Stochastic processes

In the study of stochastic processes there is an important reason to include σ-fields and filtrations because they are necessary to keep the track of the information. The relating to time feature of stochastic processes implies the flow of time. It means that at every moment $t \geq 0$ we can talk about the past, present and future as well as ask how much the observer of the process knows about them at present. We can compare this information with how much he knew in the past or will know in some certain time in the future.

In this chapter we give the definition of a stochastic process, a natural filtration and we distinguish three types of measurability.

Definition 2.10. A stochastic process $X = (X_t)_{t\geq 0}$ is a family of $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$-valued random variables X_t, where $\forall t \geq 0$ X_t is defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

We will assume that $d = 1$ in further considerations.

Given the stochastic process X the most intuitive and the simplest way to choose the filtration is to take the one generated by the stochastic process itself.

Definition 2.11. A natural filtration $\mathbb{F}^X$ of a process $X = (X_t)_{t\geq 0}$ is a filtration

$$\mathbb{F}^X = (\mathcal{F}_t^X)_{t\geq 0},$$

where

$$\mathcal{F}_t^X = \sigma(X_s, s \leq t)$$

is the smallest σ-algebra with respect to which X_s is measurable for every $s \in [0, t]$.

One can interpret set $A \in \mathcal{F}_t^X$ as follows. By the time t the observer knows if the set A has occurred or not.

To avoid problems with the measurability in the Theory of Lebesgue Integration, the probability measures are defined on σ-algebras and considered random variables are assumed to be measurable with respect to these σ-algebras.

X is a function of two variables (t, ω) and it is convenient to have the following definitions of the measurability.

Definition 2.12. The stochastic process $X = (X_t)_{t\geq 0}$ is called $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{F}$-measurable if for every $A \in \mathcal{B}(\mathbb{R})$, the set

$$\{(t, \omega) | t \in \mathbb{R}_+, \omega \in \Omega : X_t(\omega) \in A\}$$

belongs to the product σ-algebra $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{F}$.

One can be more precise and say that the stochastic process is $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{F}$-measurable if $\forall t \geq 0$ the mapping

$$(t, \omega) \mapsto X_t(\omega) : (\mathbb{R}_+ \times \Omega, \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{F}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$$

is measurable.

The concept of measurability presented in the previous definition is rather weak. Given the definition of the filtration we can define a stronger and more interesting concept.

Definition 2.13. A stochastic process X is $\mathbb{F}$-adapted if $\forall t \geq 0$ X_t is $\mathcal{F}$-measurable.

Certainly, every process X is adapted to its natural filtration $\mathbb{F}^X$. Furthermore, if $\mathbb{F}^X$ consists of all $\mathbb{P}$-negligible sets and a process Y is a modification of X then Y is also $\mathbb{F}$-adapted. We can extend the previous study with the definition of a progressive measurability as follows.

Definition 2.14. We say that a process X is progressively measurable if for every $A \in \mathcal{B}(\mathbb{R})$ the set

$$\{(s,\omega)|s \leq t, \omega \in \Omega : X_s(\omega) \in A\}$$

belongs to the product σ-algebra $\mathcal{B}([0,t]) \otimes \mathcal{F}_t$.

In other words, X is a progressively measurable stochastic process if $\forall s \geq 0$ the mapping

$$(s,\omega) \mapsto X_s(\omega) : ([0,t] \times \Omega, \mathcal{B}([0,t]) \otimes \mathcal{F}_t) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}_+))$$

is $\mathcal{B}([0,t]) \otimes \mathcal{F}_t$-measurable. Fr the further calculations it is necessary to introduce the following lemma.

Lemma 2.1. *Let Y be an integrable random variable defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $(A_i)_{i\in\mathbb{N}}$ be a sequence of disjoint sets such that $\bigcup_{i\in\mathbb{N}} A_i = \Omega$. Then*

$$\mathbb{E}_{\mathbb{P}}(Y) = \sum_{i\in\mathbb{N}} \mathbb{E}_{\mathbb{P}}(Y|A_i)\mathbb{P}(A_i). \tag{2.1}$$

2.3 The Brownian filtration

In this section we will remind the definition of a standard Brownian motion and make discussion about the Brownian filtration. In describing the Brownian motion we put an accent on the fact that it is important to distinguish different filtrations.

Definition 2.15. A standard, one-dimensional Brownian motion is a continuous adapted process $B = (B_t, \mathcal{F}_t)_{t\geq 0}$ defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with the properties that:

i) $B_0 = 0$ a.s.,

ii) for each $t \geq s \geq 0$, the increment $B_t - B_s$ is independent of $\mathcal{F}_s$,

iii) for each $t \geq s \geq 0$, the increment $B_t - B_s$ is normally distributed with mean 0 and variance $t - s$.

Consequently, the filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ is a part of the definition of a Brownian motion. However, if it is not precise which filtration we are dealing with but we know that B has stationary independent increments and that $B_t - B_s$ is normally distributed with mean 0 and variance $t - s$, then $B = (B_t, \mathcal{F}_t^B)_{t \geq 0}$ is a Brownian motion. $\mathbb{F}^B = (\mathcal{F}^B)_{t \geq 0}$ is Brownian motion's natural filtration. Moreover, it $\forall t$ $\mathcal{F}_t^B \subset \mathcal{F}_t$ and $B_t - B_s$ is independent of $\mathcal{F}_s$ then $(B_t, \mathcal{F}_t)_{t \geq 0}$ is also a Brownian motion. We mentioned before how to construct the natural filtration $\mathbb{F}^B = (\mathcal{F}^B)_{t \geq 0}$. We will study the definition of an augmented filtration.

Firstly, we denote by $\mathcal{F}^B$ a σ-algebra generated by a Brownian motion, i.e.

$$\mathcal{F}^B = \sigma(B_s, s \in \mathbb{R}_+).$$

We remind that $\mathcal{F}_t^B = \sigma(B_s, s \leq t)$. We consider the following definition of a collection of $\mathbb{P}$-negligible sets relative to a σ-algebra $\mathcal{F}$.

Definition 2.16. We say that $\mathcal{N}$ is a collection of $\mathbb{P}$-negligible sets relative to a σ-algebra $\mathcal{F}$ if for any set $A \in \mathcal{N}$ there exists a set $B \in \mathcal{N}$ such that $A \subset B$ and $\mathbb{P}(B) = 0$.

Let us denote by $\mathcal{N}$ a collection of $\mathbb{P}$-negligible sets relative to $\mathcal{F}_t^B$. We consider the following filtration.

Definition 2.17. In the previous notations we call $\tilde{\mathbb{F}^B} = (\tilde{\mathcal{F}}_t^B)_{t \geq 0}$ an augmentation of $\mathbb{F}^B$ where $\forall t$ $\tilde{\mathcal{F}}_t^B = \sigma(\mathcal{F}_t^B \cup \mathcal{N})$.

From this definition we also get a σ-algebra $\tilde{\mathcal{F}}_B$. We can easily consider the process B on the filtration $(\Omega, \tilde{\mathcal{F}}_B, \mathbb{P})$ and get that $(B_t, \tilde{\mathcal{F}}_t^B)_{t \geq 0}$ is a Brownian motion.

We can define the usual conditions for a filtration.

Definition 2.18. We say that the filtration $\mathbb{F}$ satisfies the usual conditions if it is complete and right-continuous.

Lemma 2.2. *The augmented filtration* $\tilde{\mathbb{F}^B} = (\tilde{\mathcal{F}}_t^B)_{t\geq 0}$ *satisfies the usual conditions.*

We will be only considering filtrations which satisfy the usual conditions.

2.4 Stopping times

In the Financial Mathematics it is essential to introduce the Stopping Times Theory.

Let us consider an American option. The buyer of such a financial derivative can decide when to exercise it. The choice of such a moment, let us call it τ, depends on the information about the stock price process up to time t. Then the value of an American call at τ is $(S_\tau - K)^+$. When the agent pricing the option knows which stopping time the buyer will follow the cost of such a financial derivative at time 0 will be $\mathbb{E}_{\mathbb{P}^*}(\exp(-r\tau)(S_\tau - K)^+)$, where $\mathbb{P}^*$ is the equivalent martingale measure. However, if we do not know which stopping time exactly will the observer use, he has to take the supremum. Accordingly, the price of the contingent claim at time 0 will be

$$\sup_\tau \mathbb{E}_{\mathbb{P}^*}(\exp(-r\tau)(S_\tau - K)^+).$$

It is crucial to consider the following definition of a random time.

Definition 2.19. A random time T is a strictly positive $\mathbb{P}$-a.s. random variable.

It is essential to define an $\mathbb{F}$-stopping time τ, which is an example of a random time.

Definition 2.20. A random variable τ such that

$$\tau : (\Omega, \mathcal{F}) \to (\mathbb{R}^+ \mathcal{B}(\mathbb{R}^+))$$

is called an $\mathbb{F}$-stopping time if $\forall t \geq 0$

$$\{\tau \leq t\} \text{ is } \mathcal{F}\text{-measurable.}$$

Definition 2.21. $X^T = X_{T \wedge t}$ is a process stopped at a stopping time T if

i) X is a stochastic process,

ii) T is a stopping time.

2.5 The Martingale Theory

In this section we present a fundamental characteristic which underlies many important results in Finance, namely a martingale property. Its motivation lies in the notion of a fair game. Broadly speaking, the martingale property states that tomorrow's price is expected to be today's and thus it is its best prediction. The martingale condition is assumed to be essential for an efficient market in which the information included in the past prices is fully reflected in the current prices. Furthermore, the Fundamental Theorem of Asset Prices states that if the market is arbitrage-free then discounted assets prices are martingales under a risk-neutral measure.

Here, we give a formal definition of a martingale and more general processes such as a submartingale and a supermartingale.

Definition 2.22. An adapted, integrable stochastic process $M = (M_t)_{t\geq 0}$ on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is a

i) martingale if $\mathbb{E}_{\mathbb{P}}(M_t|\mathcal{F}_s) = M_s \ \forall s \leq t$,

ii) submartingale if $\mathbb{E}_{\mathbb{P}}(M_t|\mathcal{F}_s) \geq M_s \ \forall s \leq t$,

iii) supermartingale if $\mathbb{E}_{\mathbb{P}}(M_t|\mathcal{F}_s) \leq M_s \ \forall s \leq t$.

Chapter 3

The default setting

3.1 Basic assumptions

We consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$, where $\mathbb{F}$ fulfills the usual conditions, i.e. it is right-continuous and complete, $\mathcal{F}_0$ is trivial σ-field. Let us specify that σ-algebra $\mathcal{F}_t$ represents a t-time information available to the agent in the default-free market.

We can define default time τ as a $\mathbb{R}^+$-valued finite random variable on

$$(\Omega, \mathcal{F}, \mathbb{P}).$$

Let us determine the distribution of τ as a càdlàg function F such that $F(t) = \mathbb{P}(\tau \leq t)$, where $F(0) = 0$ and $\lim_{s\to t} F(s) = \mathbb{P}(\tau < t) = F(t-)$. F defines a measure η which is the distribution of τ on $\mathbb{R}^+$, e.g.

$$\eta([a,b]) = F(b) - F(a-), \quad [a,b] \in \mathcal{B}(\mathbb{R}+)$$

and

$$\eta(du) = \mathbb{P}(\tau \in du).$$

Assumption 3.1. Let us assume that η is absolutely continuous with respect to the Lebesgue measure λ. Then, τ admits a Radon-Nikodým density f_τ such that

$$f_\tau = \frac{d\eta}{d\lambda}.$$

Moreover, if F is differentiable, then $f_\tau = F'$.

Remark 3.1. Let us now interpret $\mathbb{P}(\tau \in du)$. This is a probability of τ being in a small interval which we can denote also as $(u, u+du)$. We know that

$$\mathbb{P}(\tau \in (u, u+du)) = F(u+du) - F(u).$$

If F is continuously differentiable, then from the Taylor series we have that

$$F(u+du) = F(u) + F'(u)du$$

which gives us

$$F'(u)du = \mathbb{P}(\tau \in (u, u+du)) = \mathbb{P}(\tau \in du).$$

Then, since in this case the law η of τ has a density with respect to the Lebesgue measure, the equality above becomes

$$\mathbb{P}(\tau \in du) = f(u)du.$$

In addition,

$$\forall A \in \mathcal{B}(\mathbb{R}) \quad \mathbb{P}(\tau \in A) = \int_A \mathbb{P}(\tau \in du) = \int_A \eta(du) = \int_A f_\tau(u)du.$$

3.2 The default process

We define a default process indicating whether the default occured or not as $N = (N_t)_{t \geq 0}$ where $N_t = \mathbb{I}_{\{\tau \leq t\}}$ is *càd* and increasing. We denote $\mathbb{H} = (\mathcal{H}_t)_{t \geq 0}$ as a natural filtration generated by N, i.e. $\mathcal{H}_t = \sigma(N_u, u \leq t)$ and we complete $\mathbb{H}$ with all $\mathbb{P}$-negligible sets. The σ-algebra $\mathcal{H}_t$ represents the information generated by the observations of τ on the time interval $[0, t]$. It is necessary to mention two main properties of the filtration $\mathbb{H}$. First of all, $\mathbb{H}$ is the smallest filtration such that τ is $\mathbb{H}$-stopping time. Moreover, $\sigma(\tau) = \mathcal{H}_\infty$.

Let us now establish the form of an $\mathcal{H}_t$-measurable random variable with the following proposition.

Proposition 3.1. *A random variable U_t is $\mathcal{H}_t$-measurable if and only if it is of the form*

$$U_t(\omega) = \tilde{u}\mathbb{I}_{\{\tau(\omega) > t\}} + h(\tau(\omega))\mathbb{I}_{\{\tau(\omega) \leq t\}},$$

where h is a Borel function on $[0, t]$ and $\tilde{u}$ is constant.

Proof. We can base the proof on the fact that $\mathcal{H}_t$-measurable random variables are generated by random variables of the form $U_t^0(\omega) = h(t \wedge \tau(\omega))$, where h is a bounded Borel function on $\mathbb{R}^+$. Now we can specify $h(t \wedge \tau(\omega))$ on before the default set and after the default set, i.e.

$$h(t \wedge \tau(\omega)) = h(t \wedge \tau(\omega))\mathbb{I}_{\{\tau(\omega)>t\}} + h(t \wedge \tau(\omega))\mathbb{I}_{\{\tau(\omega)\leq t\}} =$$

$$= h(t)\mathbb{I}_{\{\tau(\omega)>t\}} + h(\tau(\omega))\mathbb{I}_{\{\tau(\omega)\leq t\}}.$$

For a fixed t, $h(t)$ is constant. We denote it as $\tilde{u}$ and we have

$$U_t(\omega) = \tilde{u}\mathbb{I}_{\{\tau>t\}} + h(\tau(\omega))\mathbb{I}_{\{\tau(\omega)\leq t\}}.$$

Since we use function h only on the set $\{\tau \leq t\}$ we can characterize the function h as a Borel function on $[0, t]$ without loss of generality. □

Chapter 4

The intensity-based approach in filtration $\mathbb{H}$

The intensity-based approach has a lot in common with the Reliability Theory. Clearly, default time is precisely expressed by the likelihood of the default event conditional on the information flow. These considerations help us to deliver the reduced form of a price for a defaultable contingent claim. Specifically, we assume that the agent pricing the contingent claim knows only time of default. The assumption of the agent's lack of knowledge about the price process is crucial for the first glance at the valuation.

Let τ, as defined before, be a positive random variable on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Firstly, we study the distribution function $F(t)$ of τ which is absolutely continuous with respect to the Lebesgue measure. In this case we can easily compute the intensity function which is a non-negative deterministic function defined as follows.

4.1 The $\mathbb{H}$-intensity of τ

In this section we give the definitions of $\mathbb{H}$-intensity of default time τ and deliver the expectation tools which are essential for pricing defaultable claims.

4.1.1 The intensity of default

Let us define more formally an intensity of default time.

Definition 4.1. An intensity of default time is **a ratio of the probability** that default will appear in a infinitely small time interval Δs, condition on that there was no default before, **and the time step** Δs, i.e.

$$\lambda_s = \lim_{\Delta s \to 0} \frac{\mathbb{P}(\tau \in (s, s + \Delta s) | \tau > s)}{\Delta s}.$$

Consequently, from the Reliability Theory, we can obtain the following form of the intensity.

Proposition 4.1. $\lambda_s = \frac{f_\tau(s)}{1-F(s)}$ *is the intensity function for a default time* τ.

Proof. Let us assume that the distribution function of τ is absolutely continuous with respect to the Lebesgue measure. From the definition we have that

$$\lambda_s = \lim_{\Delta s \to 0} \frac{\mathbb{P}(\tau \in (s, s + \Delta s) | \tau > s)}{\Delta s}.$$

Using the definition of the conditional probability we can write

$$\lambda_s = \lim_{\Delta s \to 0} \frac{\mathbb{P}(\{\tau \in (s, s + \Delta s)\} \cap \{\tau > s\})}{\mathbb{P}(\tau > s)\Delta s}.$$

We have that

$$\{\omega : \tau(\omega) \in (s, s + \Delta s)\} \cap \{\omega : \tau(\omega) > s\} = \{\omega : \tau(\omega) \in (s, s + \Delta s)\}.$$

Thus, we can write

$$\lambda_s = \lim_{\Delta s \to 0} \frac{\mathbb{P}(\{\tau \in (s, s + \Delta s)\})}{\mathbb{P}(\tau > s)\Delta s}.$$

From the definition of the distribution function $F(t)$ of τ and the fact that $F(t)$ is absolutely continuous it follows that

$$\lambda_s = \lim_{\Delta s \to 0} \frac{F(s + \Delta s) - F(s)}{\mathbb{P}(\tau > s)\Delta s} = \frac{f_\tau(s)}{1 - F(s)}.$$

□

Recall that we introduced the intensity function for default time τ. Since we defined the default process $N = (N_t)_{t\geq 0}$ with $N_t = \mathbb{I}_{\{\tau \leq t\}}$ and the filtration $\mathbb{H}$ is generated by the default process we can formulate the following definition.

Definition 4.2. An $\mathbb{H}$-adapted non-negative process

$$\lambda = (\lambda_t)_{t\geq 0}$$

is called an $\mathbb{H}$-intensity of τ if $(\mathbb{I}_{\{\tau \leq t\}} - \int_0^t \lambda_u \mathbb{I}_{\{\tau \geq u\}} du)_{t\geq 0}$ is an $(\mathbb{P}, \mathbb{H})$-martingale.

Here, we give a proposition which is essential in delivering the expectation tools for pricing defaultable claims.

Proposition 4.2. *Let ζ be an $\mathcal{F}$-measurable random variable, then*

$$\mathbb{E}_{\mathbb{P}}(\zeta|\mathcal{H}_t) = \mathbb{I}_{\{\tau>t\}}\frac{\mathbb{E}_{\mathbb{P}}(\zeta\mathbb{I}_{\{\tau>t\}})}{\mathbb{P}(\tau>t)} + \mathbb{I}_{\{\tau\le t\}}\mathbb{E}_{\mathbb{P}}(\zeta|\mathcal{H}_\infty).$$

Proof. Since ζ is an $\mathcal{F}$-measurable random variable, we can represent

$$\mathbb{E}_{\mathbb{P}}(\zeta|\mathcal{H}_t)$$

on two sets $\{\tau \le t\}$ and $\{\tau > t\}$ in the following way

$$\mathbb{E}_{\mathbb{P}}(\zeta|\mathcal{H}_t) = \mathbb{I}_{\{\tau>t\}}\mathbb{E}_{\mathbb{P}}(\zeta|\mathcal{H}_t) + \mathbb{I}_{\{\tau\le t\}}\mathbb{E}_{\mathbb{P}}(\zeta|\mathcal{H}_t).$$

Firstly, let us study the first term on the right-hand side of the last equation. Then using the properties of conditional probability with respect to σ-algebra $\mathcal{H}_t$ we have

$$\mathbb{I}_{\{\tau>t\}}\mathbb{E}_{\mathbb{P}}(\zeta|\mathcal{H}_t) = \mathbb{I}_{\{\tau>t\}}\mathbb{I}_{\{\tau>t\}}\frac{\mathbb{E}_{\mathbb{P}}(\zeta\mathbb{I}_{\{\tau>t\}})}{\mathbb{P}(\tau>t)} + \mathbb{I}_{\{\tau>t\}}\mathbb{I}_{\{\tau\le t\}}\frac{\mathbb{E}_{\mathbb{P}}(\zeta\mathbb{I}_{\{\tau\le t\}})}{\mathbb{P}(\tau\le t)}.$$

We see that the second term on the right-hand side vanishes and we obtain

$$\mathbb{I}_{\{\tau>t\}}\mathbb{E}_{\mathbb{P}}(\zeta|\mathcal{H}_t) = \mathbb{I}_{\{\tau>t\}}\frac{\mathbb{E}_{\mathbb{P}}(\zeta\mathbb{I}_{\{\tau>t\}})}{\mathbb{P}(\tau>t)}.$$

Secondly, let us ponder the term

$$\mathbb{I}_{\{\tau\le t\}}\mathbb{E}_{\mathbb{P}}(\zeta|\mathcal{H}_t).$$

To prove that

$$\mathbb{I}_{\{\tau\le t\}}\mathbb{E}_{\mathbb{P}}(\zeta|\mathcal{H}_\infty) = \mathbb{I}_{\{\tau\le t\}}\mathbb{E}_{\mathbb{P}}(\zeta|\mathcal{H}_t)$$

we use the fact that

$$\mathcal{H}_\infty = \sigma(N_s, s\in\mathbb{R}_+)$$

and

$$\forall A\in\mathcal{H}_\infty \quad A\cap\{\tau\le t\}\in\mathcal{H}_t.$$

From the properties of the conditional expectation we have

$$\int_A \mathbb{E}_{\mathbb{P}}(\zeta\mathbb{I}_{\{\tau\le t\}}|\mathcal{H}_\infty)d\mathbb{P} = \int_A \zeta\mathbb{I}_{\{\tau\le t\}}d\mathbb{P},$$

which is also equal to

$$\int_{A\cap\{\tau\leq t\}} \zeta \mathbb{I}_{\{\tau\leq t\}} d\mathbb{P}.$$

Again, using the property of the conditional expectation we obtain

$$\int_A \mathbb{E}_{\mathbb{P}}(\zeta \mathbb{I}_{\{\tau\leq t\}}|\mathcal{H}_\infty) d\mathbb{P} = \int_{A\cap\{\tau\leq t\}} \mathbb{E}_{\mathbb{P}}(\zeta \mathbb{I}_{\{\tau\leq t\}}|\mathcal{H}_t) d\mathbb{P},$$

which can be written as

$$\int_A \mathbb{I}_{\{\tau\leq t\}} \mathbb{E}_{\mathbb{P}}(\zeta|\mathcal{H}_t) d\mathbb{P}.$$

Finally, we obtain the result

$$\int_A \mathbb{E}_{\mathbb{P}}(\zeta \mathbb{I}_{\{\tau\leq t\}}|\mathcal{H}_\infty) d\mathbb{P} = \int_A \mathbb{E}_{\mathbb{P}}(\mathbb{I}_{\{\tau\leq t\}}\zeta|\mathcal{H}_t) d\mathbb{P}.$$

□

We give a lemma concerning previously defined intensity of τ.

Lemma 4.1. *A process $\lambda = (\lambda_t)_{t\geq 0}$, where*

$$\lambda_t = \frac{f_\tau(t)}{1 - F(t)}$$

is an $\mathbb{H}$-intensity of τ.

Proof. The process

$$\lambda_t = \frac{f_\tau(t)}{1 - F(t)}$$

is deterministic and non-negative. Thus it is $\mathbb{H}$-adapted.
Now, we will check that $M =(M_t)_{t\geq t}$ with

$$M_t = \left(\mathbb{I}_{\{\tau\leq t\}} - \int_0^t \lambda_u \mathbb{I}_{\{\tau\geq u\}} du\right)$$

is a $(\mathbb{P}, \mathbb{H})$-martingale.
Let us assume that $s < t$.

We will show that

$$\mathbb{E}_{\mathbb{P}}(M_t - M_s|\mathcal{H}_s) = 0$$

Using the previous notations and the additive property of integrals we obtain

$$\mathbb{E}_{\mathbb{P}}(M_t - M_s|\mathcal{H}_s) = \mathbb{E}_{\mathbb{P}}(N_t - N_s|\mathcal{H}_s) - \mathbb{E}_{\mathbb{P}}\Big(\int_s^t \lambda_u \mathbb{I}_{\{\tau \geq u\}} du|\mathcal{H}_s\Big).$$

We will show that

$$\mathbb{E}_{\mathbb{P}}(N_t - N_s|\mathcal{H}_s) = \mathbb{E}_{\mathbb{P}}\Big(\int_s^t \lambda_u \mathbb{I}_{\{\tau \geq u\}} du|\mathcal{H}_s\Big).$$

Let us use the fact that

$$\mathbb{E}_{\mathbb{P}}(\mathbb{I}_{\{\tau>t\}}|\mathcal{H}_s) = \mathbb{P}(\tau > t|\mathcal{H}_s).$$

Then, we can rewrite the right-hand side of the last equality on two sets, $\{\tau \leq s\}$ and $\{\tau > s\}$, and use the definition of the conditional probability to obtain

$$\mathbb{P}(\tau > t|\mathcal{H}_s) = \mathbb{I}_{\{\tau>s\}} \frac{\mathbb{P}(\tau > t, \tau > s)}{\mathbb{P}(\tau > s)} + \mathbb{I}_{\{\tau \leq s\}} \frac{\mathbb{P}(\tau > t, \tau \leq s)}{\mathbb{P}(\tau \leq s)}.$$

We easily see that the second term on the right-hand side of the last equation vanishes. We have that

$$\{\tau > t\} \cap \{\tau > s\} = \{\tau > t\}.$$

Thus we have

$$\mathbb{E}_{\mathbb{P}}(\mathbb{I}_{\{\tau>t\}}|\mathcal{H}_s) = \mathbb{I}_{\{\tau>s\}} \frac{1 - F(t)}{1 - F(s)}.$$

We have that

$$\mathbb{E}_{\mathbb{P}}(N_t - N_s|\mathcal{H}_s) = \mathbb{I}_{\{\tau>s\}} \frac{F(t) - F(s)}{1 - F(s)}.$$

Let us denote

$$J = \int_s^t \lambda_u \mathbb{I}_{\{\tau \geq u\}} du.$$

Then we can write

$$J = \int_{s\wedge\tau}^{t\wedge\tau} \lambda_u du.$$

Knowing that

$$\lambda_t = \frac{f_\tau(t)}{1 - F(t)}$$

we get

$$J = \ln \frac{1 - F(s \wedge \tau)}{1 - F(t \wedge \tau)}.$$

As previously, we can study J on two sets, before and after the default and get

$$J = \mathbb{I}_{\{\tau > s\}} \ln \frac{1 - F(s \wedge \tau)}{1 - F(t \wedge \tau)} + \mathbb{I}_{\{\tau \leq s\}} \ln \frac{1 - F(s)}{1 - F(s)}.$$

Consequently, we get

$$J = \mathbb{I}_{\{\tau > s\}} \ln \frac{1 - F(s \wedge \tau)}{1 - F(t \wedge \tau)}.$$

Thus

$$J = J\mathbb{I}_{\{\tau > s\}}.$$

Now, we use the Proposition 4.2 and calculate the conditional expectation of J.

$$\mathbb{E}_{\mathbb{P}}(J|\mathcal{H}_s) = \mathbb{I}_{\{\tau > s\}} \frac{\mathbb{E}_{\mathbb{P}}(J\mathbb{I}_{\{\tau > s\}})}{\mathbb{P}(\tau > t)} + \mathbb{I}_{\{\tau \leq s\}} \mathbb{E}_{\mathbb{P}}(J|\mathcal{H}_\infty).$$

Due to the fact that

$$J = J\mathbb{I}_{\{\tau > s\}}$$

we get

$$\mathbb{E}_{\mathbb{P}}(J|\mathcal{H}_s) = \mathbb{I}_{\{\tau > s\}} \frac{\mathbb{E}_{\mathbb{P}}(J)}{\mathbb{P}(\tau > s)}.$$

Using the definition of J and λ_u we get

$$\mathbb{E}_{\mathbb{P}}(J|\mathcal{H}_s) = \mathbb{I}_{\{\tau > s\}} \frac{\mathbb{E}_{\mathbb{P}}(\int_s^t \lambda_u \mathbb{I}_{\{\tau \geq u\}} du)}{\mathbb{P}(\tau > s)}.$$

We can take the expectation operator inside the integral and get

$$\mathbb{I}_{\{\tau > s\}} \frac{\int_s^t \lambda_u \mathbb{E}_{\mathbb{P}}(\mathbb{I}_{\{\tau \geq u\}}) du}{1 - F(s)}.$$

From the fact that

$$\mathbb{E}_{\mathbb{P}}(\mathbb{I}_{\{\tau \geq u\}}) = \mathbb{P}(\tau \geq u)$$

we obtain

$$\mathbb{I}_{\{\tau > s\}} \frac{\int_s^t \lambda_u \mathbb{P}(\tau \geq u) du}{1 - F(s)}.$$

Consequently, from the form of the function λ_u we get

$$\mathbb{I}_{\{\tau>s\}} \frac{\int_s^t f_\tau(u)}{1-F(s)} du,$$

which is equal to

$$\mathbb{I}_{\{\tau>s\}} \frac{F(t)-F(s)}{1-F(s)}.$$

Finally,

$$\mathbb{E}_{\mathbb{P}}(N_t - N_s|\mathcal{H}_s) = \mathbb{E}_{\mathbb{P}}\Big(\int_s^t \lambda_u \mathbb{I}_{\{\tau\geq u\}} du|\mathcal{H}_s\Big) \Rightarrow \Big(\mathbb{I}_{\{\tau\leq t\}} - \int_0^t \lambda_u \mathbb{I}_{\{\tau\geq u\}} du\Big)_{t\geq 0}$$

is a $(\mathbb{P}, \mathbb{H})$-martingale.

□

Using those results we can value a defaultable zero-coupon bond which pays 1 if the default has not appeared before maturity time T. Let us consider a case when default time τ is exponentially distributed with a deterministic intensity function λ_s.

Proposition 4.3. *Expected value of this contingent claim for an agent who knows only that the default is exponentially distributed, is*

$$\mathbb{E}_{\mathbb{P}}(\mathbb{I}_{\{\tau>T\}}|\mathcal{H}_t) = \mathbb{I}_{\{\tau>t\}} \exp\Big(-\int_t^T \lambda_s ds\Big).$$

Proof. We use the Proposition 4.2. Firstly, we realize that $\mathbb{I}_{\{\tau>T\}}$ is an $\mathcal{H}_T$-measurable random variable. We have

$$\mathbb{E}_{\mathbb{P}}(\mathbb{I}_{\{\tau>T\}}|\mathcal{H}_t) = \mathbb{I}_{\{\tau>t\}} \frac{\mathbb{E}_{\mathbb{P}}(\mathbb{I}_{\{\tau>t\}}\mathbb{I}_{\{\tau>T\}})}{\mathbb{P}(\tau>t)}$$

Using the property that

$$\mathbb{E}_{\mathbb{P}}(\mathbb{I}_A) = \mathbb{P}(A)$$

and the fact that τ is exponentially distributed we obtain

$$\mathbb{E}_{\mathbb{P}}(\mathbb{I}_{\{\tau>T\}}|\mathcal{H}_t) = \mathbb{I}_{\{\tau>t\}} \exp\Big(-\int_t^T \lambda_s ds\Big).$$

□

4.1.2 The hazard function Γ

In this section we define a survival and hazard function which are frequently used further. We begin with the assumption necessary for those functions to be well defined.

Assumption 4.1. We assume that $\forall\ t \geq 0\ F(t) < 1$.

Definition 4.3. We say that $G(t) = 1 - F(t)$ is a survival function of τ if $F(t)\ \forall t \geq 0$ is a distribution function of τ.

From the Assumption above we have that $\forall t \geq 0\ G(t) : \mathbb{R} \to (0, 1]$ because $\forall t \geq 0\ F(t) : \mathbb{R} \to [0, 1)$. In the default framework we have that the survival function for τ is given by the following formula

$$G(t) = \mathbb{P}(\tau > t).$$

From the fact that $\forall t \geq 0\ G(t) > 0$ we can take a natural logarithm of $G(t)$ and define a hazard function for τ.

Definition 4.4. We call a function $\Gamma(t) = -\ln(G(t))$ a hazard function of τ, where $G(t)$ is a survival function for $\tau\ \forall t \geq 0$.

If $F(u)$ is differentiable we can approximate it by $dF(u) = F'(u)du$. With the analogical argumentation we get $d\Gamma(u) = \Gamma'(u)du$. We can write the hazard function in a form as follows.

Proposition 4.4.

$$\Gamma(t) = \int_0^t \frac{dF(s)}{G(s)}$$

is a hazard function $\forall t \geq 0$.

Proof. We have

$$\Gamma(t) = \int_0^t \frac{dF(s)}{G(s)} = \int_0^t \frac{dF(s)}{1 - F(s)}.$$

We can easily obtain the result after realizing that the nominator of the fraction inside the integral is a derivative of the denominator but without the minus sign. By the formula

$$\int_0^t \frac{dV(s)}{V(s)} = \ln(V(t)) - \ln(V(0))$$

and the fact that $G(0) = 1$ we end the proof. □

From this form of the hazard function it is obvious that $\Gamma(t)$ satisfies the following property.

Proposition 4.5. *The hazard function* $\Gamma(t)$ *of* τ *is increasing.*

Proof. From the definition of an integral and the fact that if the integrand does not change but we integrate on a larger interval the integral will be greater. More formally, $\forall s < t$

$$\Gamma(s) = \int_0^s \frac{dF(u)}{G(u)} < \int_0^t \frac{dF(s)}{G(s)} = \Gamma(t).$$

□

In the case when $F(t)$ is continuous and has a derivative $F'(t) = f_\tau(t)$ we can write the hazard function of τ as

$$\Gamma(t) = \int_0^t \frac{f_\tau(s)}{G(s)} ds.$$

Consequently, the derivative of $\Gamma(t)$ is

$$\Gamma'(t) = \Big(\int_0^t \frac{f_\tau(s)}{G(s)} ds\Big)' = \frac{f_\tau(t)}{G(t)}.$$

Definition 4.5. We will call the derivative of Γ an $\mathbb{H}$-generalized intensity of τ if

$$\Big(\mathbb{I}_{\{\tau \leq t\}} - \Gamma(t \wedge \tau)\Big)_{t \geq 0}$$

is a $(\mathbb{P}, \mathbb{H})$-martingale.

Let us introduce and prove the following proposition which is important for further calculations.

Proposition 4.6. *Let* $h(\tau)$ *be a Borel function (i.e.* $h(\tau)$ *is* $\sigma(\tau)$*-measurable random variable). Then*

$$\mathbb{E}_{\mathbb{P}}(h(\tau)|\mathcal{H}_t) = \mathbb{I}_{\{\tau > t\}} \frac{\mathbb{E}_{\mathbb{P}}(h(\tau)\mathbb{I}_{\{\tau > t\}})}{\mathbb{P}(\tau > t)} + \mathbb{I}_{\{\tau \leq t\}} h(\tau).$$

Proof. We mentioned before that $\sigma(\tau) = \mathcal{H}_\infty$. According to the Proposition 4.2 we have

$$\mathbb{E}_{\mathbb{P}}(h(\tau)|\mathcal{H}_t) = \mathbb{I}_{\{\tau \leq t\}} \mathbb{E}_{\mathbb{P}}(h(\tau)|\mathcal{H}_\infty) + \mathbb{I}_{\{\tau > t\}} \frac{\mathbb{E}_{\mathbb{P}}(h(\tau)\mathbb{I}_{\{\tau > t\}})}{\mathbb{P}(\tau > t)}.$$

From the fact that $h(\tau)$ is an $\mathcal{H}_\infty$-measurable random variable we get

$$\mathbb{E}_{\mathbb{P}}(h(\tau)|\mathcal{H}_t) = \mathbb{I}_{\{\tau > t\}} \frac{\mathbb{E}_{\mathbb{P}}(h(\tau)\mathbb{I}_{\{\tau > t\}})}{\mathbb{P}(\tau > t)} + \mathbb{I}_{\{\tau \leq t\}} h(\tau).$$

□

Let us study a zero-coupon defaultable contingent claim that pays $h(\tau)$ if the default has not appeared before the maturity time T. We assume that the spot rate $r(s) \equiv 0$. It is natural to reckon such a payoff because the agent pricing the claim knows that it is a defaultable one and he studies the payoff as a Borel function of τ. Here, we do not assume that the distribution function F of τ is absolutely continuous but we assume it is continuous.

Proposition 4.7. *The expected value of this derivative in the case of the knowledge only about the default time distribution is*

$$\mathbb{E}_{\mathbb{P}}(h(\tau)\mathbb{I}_{\{\tau>T\}}|\mathcal{H}_t) = \mathbb{I}_{\{\tau>t\}} \exp(\Gamma(t)) \int_T^\infty h(u)dF(u).$$

Proof. From the Proposition 4.6 we induce

$$\mathbb{E}_{\mathbb{P}}(h(\tau)\mathbb{I}_{\{\tau>T\}}|\mathcal{H}_t) = \mathbb{I}_{\{\tau>t\}} \frac{\mathbb{E}_{\mathbb{P}}(h(\tau)\mathbb{I}_{\{\tau>T\}}\mathbb{I}_{\{\tau>t\}})}{\mathbb{P}(\tau > t)} + \mathbb{I}_{\{\tau\le t\}}\mathbb{I}_{\{\tau>T\}}h(\tau).$$

The second term of the right-hand side of the equation above vanishes as well as the indicator $\mathbb{I}_{\{\tau>t\}}$ in the second term. From the definition of expected value we obtain

$$\mathbb{E}_{\mathbb{P}}(h(\tau)\mathbb{I}_{\{\tau>T\}}|\mathcal{H}_t) = \mathbb{I}_{\{\tau>t\}} \frac{\int_{\mathbb{R}} h(u)\mathbb{I}_{\{u>T\}}dF(u)}{1 - F(t)}.$$

Using the correlation between F and Γ we obtain

$$\mathbb{E}_{\mathbb{P}}(h(\tau)\mathbb{I}_{\{\tau>T\}}|\mathcal{H}_t) = \mathbb{I}_{\{\tau>t\}} \int_T^\infty h(u)\frac{1 - F(u)}{1 - F(t)}d\Gamma(u).$$

Substituting the terms with F by the terms with Γ we get

$$\mathbb{E}_{\mathbb{P}}(h(\tau)\mathbb{I}_{\{\tau>T\}}|\mathcal{H}_t) = \mathbb{I}_{\{\tau>t\}} \exp(\Gamma(t)) \int_T^\infty h(u)\exp(-\Gamma(u))d\Gamma(u).$$

Finally, after coming back to the terms with F we obtain

$$\mathbb{E}_{\mathbb{P}}(X(\tau)\mathbb{I}_{\{\tau>T\}}|\mathcal{H}_t) = \mathbb{I}_{\{\tau>t\}} \exp(\Gamma(t)) \int_T^\infty h(u)dF(u)$$

□

Now, let us derive a value similar to that one in the Proposition 4.3 but without any assumption about the distribution of τ except this one that the distribution is continuous. We consider a defaultable zero-coupon financial derivative which pays 1 if the default has not appeared before the maturity time T. We assume that the spot rate $r(s) \equiv 0$.

Proposition 4.8. *The expected value of the payoff for an agent who observes default when it occurs is*

$$\mathbb{E}_{\mathbb{P}}(\mathbb{I}_{\{\tau>T\}}|\mathcal{H}_t) = \mathbb{I}_{\{\tau>t\}} \exp(-[\Gamma(T) - \Gamma(t)]).$$

Proof. From the Proposition 4.7 we have

$$\mathbb{E}_{\mathbb{P}}(\mathbb{I}_{\{\tau>T\}}|\mathcal{H}_t) = \mathbb{I}_{\{\tau>t\}} \exp(\Gamma(t)) \int_T^\infty dF(u).$$

From the definition of the improper integral we induce

$$\mathbb{I}_{\{\tau>t\}} \exp(\Gamma(t)) \int_T^\infty dF(u) = \mathbb{I}_{\{\tau>t\}} \exp(\Gamma(t)) \lim_{v\to\infty} \int_T^v dF(u).$$

Then, after calculating the integral, taking the limit and writing F in terms of Γ, we obtain the result

$$\mathbb{E}_{\mathbb{P}}(\mathbb{I}_{\{\tau>T\}}|\mathcal{H}_t) = \mathbb{I}_{\{\tau>t\}} \exp(-[\Gamma(T) - \Gamma(t)]).$$

□

Let us assume that there exists a deterministic spot rate $r(s)$. Then the present value (at time t) of a zero-coupon bond which pays 1 when the default has not appeared before maturity time T is

$$\exp\Big(-\int_t^T r(s)ds\Big),$$

where $t \in [0, T]$. Let us study a firm which issues a zero-coupon bond which pays 1 at the maturity time T when the default has not appeared before T. On this financial market we have the following.

Proposition 4.9. *We assume that τ admits an $\mathbb{H}$-intensity λ_s. Then, the expected value at time t of described contingent claim calculated by an agent who has the information $\mathcal{H}_t$ is*

$$\mathbb{E}_{\mathbb{P}}(\exp(-\int_t^T r(s)ds)\mathbb{I}_{\{\tau>T\}}|\mathcal{H}_t) = \mathbb{I}_{\{\tau>t\}} \exp(-\int_t^T (r(s) + \lambda_s)ds).$$

Proof. We can take the deterministic part outside the integral and obtain after taking under consideration the Proposition 4.8 that the left-hand side is equal to

$$\exp\Big(-\int_t^T r(s)ds\Big)\mathbb{I}_{\{\tau>t\}}\exp\Big(-[\Gamma(T)-\Gamma(t)]\Big).$$

We can take

$$\exp\Big(-[\Gamma(T)-\Gamma(t)]\Big)$$

inside the integral and obtain

$$\mathbb{I}_{\{\tau>t\}}\exp\Big(-\int_t^T\Big(r(s)ds-[\Gamma(T)-\Gamma(t)]\Big)\Big).$$

From the fact that τ admits a $\mathbb{H}$-intensity λ_s and $\Gamma'(s)=\lambda_s$ we get

$$\mathbb{I}_{\{\tau>t\}}\exp\Big(-\int_t^T\Big(r(s)+\Gamma'(s))ds\Big)\Big).$$

Consequently,

$$\mathbb{E}_{\mathbb{P}}\Big(\exp\Big(-\int_t^T r(s)ds\Big)\mathbb{I}_{\{\tau>T\}}|\mathcal{H}_t\Big)=\mathbb{I}_{\{\tau>t\}}\exp\Big(-\int_t^T\Big(r(s)+\lambda_s\Big)ds\Big).$$

□

However, we should not treat the last result as an actual price for a defaultable zero-coupon bond. This is because we are calculating it under the initial measure $\mathbb{P}$. What is more, it is impossible to hedge this default. We can only use this value to see that the default might act as a change in the interest rate $r(s)$. The expected value calculated at time t of a contingent claim H under the condition that the default has not appeared before time T is

$$\mathbb{E}_{\mathbb{P}}\Big(H\exp\Big(-\int_t^T r(s)ds\Big)\mathbb{I}_{\{\tau>T\}}|\mathcal{H}_t\Big).$$

This was the case when H was dependent on τ.

Proposition 4.10. *If ζ is independent of default time τ then*

$$\mathbb{E}_{\mathbb{P}}\Big(\zeta\exp\Big(-\int_t^T r(s)ds\Big)\mathbb{I}_{\{\tau>T\}}|\mathcal{H}_t\Big)=\mathbb{I}_{\{\tau>t\}}\exp\Big(-\int_t^T(r(s)+\lambda_s\Big)ds\Big)\mathbb{E}_{\mathbb{P}}(\zeta).$$

Proof. We can take the exponent outside the expected value and obtain

$$\mathbb{E}_{\mathbb{P}}\Big(\zeta \exp\Big(-\int_t^T r(s)ds\Big)\mathbb{I}_{\{\tau>T\}}|\mathcal{H}_t\Big) = \exp\Big(-\int_t^T r(s)ds\Big)\mathbb{E}_{\mathbb{P}}\Big(\zeta\mathbb{I}_{\{\tau>T\}}|\mathcal{H}_t\Big).$$

Then, using the fact that $\mathbb{I}_{\{\tau>T\}}$ is $\mathcal{H}_t$-measurable we can also take the indicator function outside and from the independence ζ of τ, we obtain the independence ζ of $\mathcal{H}_t$ and get

$$\mathbb{E}_{\mathbb{P}}(\zeta)\exp\Big(-\int_t^T r(s)ds\Big)\mathbb{I}_{\{\tau>t\}}\exp\Big(-[\Gamma(T)-\Gamma(t)]\Big).$$

Finally, analogically to the proof of the Proposition 4.9, we obtain the result

$$\mathbb{E}_{\mathbb{P}}\Big(\zeta \exp\Big(-\int_t^T r(s)ds\Big)\mathbb{I}_{\{\tau>T\}}|\mathcal{H}_t\Big) = \mathbb{I}_{\{\tau>t\}}\exp\Big(-\int_t^T (r(s)+\lambda_s)ds\Big)\mathbb{E}_{\mathbb{P}}(\zeta).$$

□

Chapter 5

The Carthaginian enlargement of filtrations

5.1 Introduction

To add the information about the default to the filtration generated by the price process, we have to enlarge it by a positive random variable which is default time τ. It can be done in two different manners: initially, i.e. from the beginning with the corresponding information $\sigma(\tau)$ or progressively with $\sigma(\tau \wedge t)$. The procedure of enlargement lets us to obtain three nested filtrations, hence it was called Carthaginian Enlargement of Filtrations. The adjective "Carthaginian" was first introduced by Callegaro, Jeanblanc and Zargari (see [2]) and it refers to three levels of different civilizations which can be found at the archaeological site of Carthage.

The initially enlarged filtration $\mathbb{G}^\tau = (\mathcal{G}_t^\tau)_{t\geq 0}$ is generated by σ-algebras of the form $\mathcal{G}_t^\tau = \mathcal{F}_t \vee \sigma(\tau)$. More generally $\mathcal{G}_t^\tau = \mathcal{F}_t \vee \tilde{\mathcal{F}}$, where $\tilde{\mathcal{F}}$ is σ-algebra.

The progressively enlarged filtration $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$ is generated by σ-algebras of the form $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$, where $\mathbb{H}$ is the natural filtration of the default process $N = (N_t)_{t\geq 0}$ with $N_t = \mathbb{I}_{\{\tau \leq t\}}$. More generally $\mathcal{G}_t = \mathcal{F}_t \vee \tilde{\mathcal{F}}_t$, where $\tilde{\mathbb{F}} = (\tilde{\mathcal{F}}_t)_{t\geq 0}$ is the natural filtration generated by additional process. Usually we consider the right-continuous version of $\mathbb{G}$, namely

$$\forall t \geq 0 \quad \mathcal{G}_t = \mathcal{G}_{t+} = \bigcap_{s>t} \mathcal{F}_s \vee \mathcal{H}_s.$$

The three acquired filtrations represent different sources of information available to the investors. The Enlargement of Filtrations Theory plays very important role in modelling additional gain due to such asymmetric information as well as information itself.

In the previous chapter we introduced the intensity approach in filtration $\mathbb{H}$. Hereafter, some of the results for progressively enlarged filtration are also obtained using this approach. Nonetheless, the intensity process allows for a knowledge of the default conditional distribution only before the default. Thus, we have to consider density approach which gives the full characterization of the links between the default time and the filtration generated by the price process before and after the default.

5.2 General projection tools

Working in the initially enlarged filtration is easier since the whole information concerning the default is possessed by the insider from the beginning. However, we would like to represent the obtained results in terms of the progressively enlarged filtration so that they are accessible to the regular investor as well. Thus, we have to establish some projection tools. Let us introduce a following proposition determining a method of projecting martingale adapted to some arbitrary filtration on the smaller filtration.

Proposition 5.1. *[2] Let $\mathbb{K}$ and $\tilde{\mathbb{K}}$ be filtrations such that $\mathbb{K} \subset \tilde{\mathbb{K}}$ and let $\zeta = (\zeta_t)_{t\geq 0}$ be uniformly integrable $(\mathbb{P}, \mathbb{K})$-martingale.*

Then, there exists an $(\mathbb{P}, \tilde{\mathbb{K}})$-martingale $\tilde{\zeta} = (\tilde{\zeta}_t)_{t\geq 0}$ such that

$$\mathbb{E}_{\mathbb{P}}(\tilde{\zeta}_t|\mathcal{K}_t) = \zeta_t, \quad t \geq 0.$$

Proof. From ζ being a uniformly integrable $(\mathbb{P}, \mathbb{K})$-martingale it follows that $\mathbb{P}$-a.s.

$$\zeta_t = \mathbb{E}_{\mathbb{P}}(\zeta_\infty|\mathcal{K}_t).$$

We define $\tilde{\zeta}_t$ as $\mathbb{E}_{\mathbb{P}}(\zeta_\infty|\tilde{\mathcal{K}}_t)$. Let us check that it is a $(\mathbb{P}, \tilde{\mathbb{K}})$-martingale.
For any $s \leq t$ we have that

$$\mathbb{E}_{\mathbb{P}}(\tilde{\zeta}_t|\tilde{\mathcal{K}}_s) = \mathbb{E}_{\mathbb{P}}(\mathbb{E}_{\mathbb{P}}(\zeta_\infty|\tilde{\mathcal{K}}_t)|\tilde{\mathcal{K}}_s).$$

Applying the tower property we obtain $\mathbb{P}$-a.s.

$$\mathbb{E}_{\mathbb{P}}(\mathbb{E}_{\mathbb{P}}(\zeta_\infty|\tilde{\mathcal{K}}_t)|\tilde{\mathcal{K}}_s) = \mathbb{E}_{\mathbb{P}}(\mathbb{E}_{\mathbb{P}}(\zeta_\infty|\tilde{\mathcal{K}}_s)|\tilde{\mathcal{K}}_t) = \mathbb{E}_{\mathbb{P}}(\zeta_\infty|\tilde{\mathcal{K}}_s) = \tilde{\zeta}_s$$

and hence the martingale property.
Let us now prove that $\mathbb{E}_{\mathbb{P}}(\tilde{\zeta}_t|\mathcal{K}_t) = \zeta_t$. Indeed, from the uniform integrability and the tower property we obtain that

$$\zeta_t = \mathbb{E}_{\mathbb{P}}(\zeta_\infty|\mathcal{K}_t) = \mathbb{E}_{\mathbb{P}}(\mathbb{E}_{\mathbb{P}}(\zeta_\infty|\mathcal{K}_t)|\tilde{\mathcal{K}}_t) = \mathbb{E}_{\mathbb{P}}(\mathbb{E}_{\mathbb{P}}(\zeta_\infty|\tilde{\mathcal{K}}_t)|\mathcal{K}_t) = \mathbb{E}_{\mathbb{P}}(\tilde{\zeta}_t|\mathcal{K}_t).$$

□

5.3 Measurability properties in enlarged filtrations

Let us now introduce some important results on the characterization of the random variables measurable with respect to the filtrations $\mathbb{G}^\tau$ and $\mathbb{G}$. We begin with the representation of a $\mathcal{G}_t^\tau$-measurable random variable.

Proposition 5.2. *[2] A random variable Z_t is $\mathcal{G}_t^\tau$-measurable if and only if it is of the form*

$$Z_t(\omega) = z_t(\omega, \tau(\omega)),$$

where $\forall t \geq 0$ $z_t(\cdot, \tau(\cdot))$ is a $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^+)$-measurable random variable.

For the proof see [2].

Let us now give the analogous results about the representation of a $\mathcal{G}_t$-measurable random variable.

Proposition 5.3. *[2] A random variable X_t is $\mathcal{G}_t$-measurable if and only if it is of the form*

$$X_t(\omega) = \tilde{y}_t(\omega)\mathbb{I}_{\{\tau(\omega)>t\}} + \hat{z}_t(\omega, \tau(\omega))\mathbb{I}_{\{\tau(\omega)\leq t\}},$$

where $\tilde{y}_t$ is an $\mathcal{F}_t$-measurable random variable and $(\hat{z}_t(\omega, u)_{\omega\in\Omega, u\in\mathbb{R}})_{t\geq u}$ is a family of $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^+)$-measurable random variables.

Proof. $\mathcal{G}_t$-measurable random variables are generated by the random variables of the form $X_t^0(\omega) = y_t(\omega)h(t \wedge \tau(\omega))$, where y_t is an $\mathcal{F}_t$-measurable random variable and h is a Borel function on $\mathbb{R}^+$. Specifying $X_t^0(\omega)$ on before and after the default set we obtain

$$X_t^0(\omega) = y_t(\omega)h(t \wedge \tau(\omega))\mathbb{I}_{\{\tau(\omega)>t\}} + y_t(\omega)h(t \wedge \tau(\omega))\mathbb{I}_{\{\tau(\omega)\leq t\}},$$

which is equal to

$$y_t(\omega)h(t)\mathbb{I}_{\{\tau(\omega)>t\}} + y_t(\omega)h(\tau(\omega))\mathbb{I}_{\{\tau(\omega)\leq t\}}.$$

We can replace $y_t(\omega)h(t)$ with the $\mathcal{F}_t$-measurable random variable $\tilde{y}_t(\omega)$. What is more, it is well known that the measurable function of two variables can be approximated by the sum of the products of one variable measurable functions, i.e.

$$f(x,y) = \lim_{N\to\infty} \sum_{i=1}^{N} h_i(x) g_i(y).$$

where in this case $x \in \Omega$ and $y \in \mathbb{R}^+$. The random variable $y_t(\omega)h(\tau(\omega))$ is measurable with respect to the σ-algebra $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^+)$ and the sum of random variables of such form is also measurable with respect to $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^+)$. Then, by passing to the limit with $N \to \infty$, we obtain that the random variable $\hat{z}_t(\cdot, \tau(\cdot))$ which is an approximation of functions as in (5.3) is also an $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^+)$-measurable random variable. Finally we have that

$$X_t(\omega) = \tilde{y}_t(\omega)\mathbb{I}_{\{\tau(\omega)>t\}} + \hat{z}_t(\omega, \tau(\omega))\mathbb{I}_{\{\tau(\omega)\le t\}}.$$

□

5.4 The $\mathcal{E}$-hypothesis

Let us consider now the crucial assumption which will be in force throughout the rest of our thesis. It is called $\mathcal{E}$-hypothesis.

Hypothesis 5.1. (**$\mathcal{E}$-hypothesis**) We suppose that $\forall t \ge 0$, $\mathbb{P}$-a.s.

$$\mathbb{P}(\tau \in du | \mathcal{F}_t) \sim \eta(du),$$

i.e. the $\mathbb{F}$-conditional law of τ is equivalent to the law of τ.

As a result, there exists a strictly positive $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^+)$-measurable function $(t, \omega, u) \mapsto q_t(\omega, u)$, such that for every $u \ge 0$, $(q_t(u))_{t\ge0}$ is $(\mathbb{P}, \mathbb{F})$-martingale and

$$\mathbb{P}(\tau > \theta | \mathcal{F}_t) = \int_\theta^\infty q_t(u)\eta(du) \qquad \forall t \ge 0, \quad \mathbb{P} - a.s.$$

or equivalently

$$\mathbb{E}_\mathbb{P}(Z_t | \mathcal{F}_t) = \mathbb{E}_\mathbb{P}(z_t(\tau) | \mathcal{F}_t) = \int_0^\infty z_t(u) q_t(u) \eta(du),$$

for any $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^+)$-measurable random variable $Z_t = z_t(\tau)$. The family of the processes $q(u)$ is called the $(\mathbb{P}, \mathbb{F})$-conditional density of τ with respect to η. In particular,

$$\mathbb{P}(\tau > \theta) = \mathbb{P}(\tau > \theta | \mathcal{F}_0) = \int_\theta^\infty q_0(u)\eta(du) \text{ and } q_0(u) = 1, \forall u \ge 0.$$

Remark 5.1. One can consider a particular case when $\forall u \geq 0$

$$q_t(u) = q_u(u), \quad \forall t \geq u \quad d\mathbb{P} - a.s.$$

It means that

$$\mathbb{P}(\tau > s|\mathcal{F}_t) = \mathbb{P}(\tau > s|\mathcal{F}_s), \quad 0 \leq s \leq t$$

and new information does not change the conditional distribution of τ.

In the structural approach introduced by Merton τ is an $\mathbb{F}$-stopping time. In the reduced-form approach which we work with this property is no longer fulfilled. Let us now present a proposition which shows that under the special assumption concerning the measure η, τ avoids $\mathbb{F}$-stopping times.

Assumption 5.1. We assume that the law of τ is non-atomic.

Proposition 5.4. *[3] The Assumption 5.1 and the Hypothesis 5.1 are satisfied. Then, we have for every $\mathbb{F}$-stopping time ξ bounded by T that*

$$\mathbb{P}(\tau = \xi) = 0.$$

Proof. From the tower property we have

$$\mathbb{P}(\tau = \xi) = \mathbb{E}_{\mathbb{P}}(\mathbb{I}_{\{\tau=\xi\}}) = \mathbb{E}_{\mathbb{P}}(\mathbb{E}_{\mathbb{P}}(\mathbb{I}_{\{\tau=\xi\}})|\mathcal{F}_t) = \mathbb{E}_{\mathbb{P}}(\mathbb{E}_{\mathbb{P}}(\mathbb{I}_{\{\tau=\xi\}}|\mathcal{F}_t)).$$

Let us prove firstly that $\mathbb{E}_{\mathbb{P}}(\mathbb{I}_{\{\tau=\xi\}}|\mathcal{F}_t) = 0$. Again, using the tower property

$$\mathbb{E}_{\mathbb{P}}(\mathbb{I}_{\{\tau=\xi\}}|\mathcal{F}_t) = \mathbb{E}_{\mathbb{P}}(\mathbb{E}_{\mathbb{P}}(\mathbb{I}_{\{\tau=\xi\}}|\mathcal{F}_t)|\mathcal{F}_T) = \mathbb{E}_{\mathbb{P}}(\mathbb{E}_{\mathbb{P}}(\mathbb{I}_{\{\tau=\xi\}}|\mathcal{F}_T)|\mathcal{F}_t).$$

Since τ admits the conditional density we can write that

$$\mathbb{E}_{\mathbb{P}}(\mathbb{E}_{\mathbb{P}}(\mathbb{I}_{\{\tau=\xi\}}|\mathcal{F}_T)|\mathcal{F}_t) = \mathbb{E}_{\mathbb{P}}\bigg(\int_0^\infty \mathbb{I}_{\{u=\xi\}} q_t(u)\eta(du)|\mathcal{F}_t\bigg).$$

The integral $\int_0^\infty \mathbb{I}_{\{u=\xi\}} q_t(u)\eta(du)$ is a Lebesgue integral with respect to the measure η for each fixed ω. Since the measure η is non-atomic, $\eta(\{\xi(\omega)\}) = 0$, the mentioned integral is also equal to 0, as well as its conditional expectation. Thus,

$$\mathbb{E}_{\mathbb{P}}\Big(\mathbb{I}_{\{\tau=\xi\}}|\mathcal{F}_t\Big) = 0$$

and

$$\mathbb{P}(\tau = \xi) = 0.$$

□

5.5 The change of measure on $\mathbb{G}^\tau$

Due to the fact that working with τ independent of the prices filtration $\mathbb{F}$ is easier we have to introduce a decoupling measure which provides this property.

Proposition 5.5. *[2] Let us suppose that $\mathcal{E}$-hypothesis holds. There exists a process $L = (L_t)_{t\geq 0}$ with $L_t = \frac{1}{q_t(\tau)}$ and $\mathbb{E}_{\mathbb{P}}(L_t) = L_0 = 1$ which is a strictly positive $(\mathbb{P}, \mathbb{G}^\tau)$-martingale and thus defines a probability measure $\mathbb{P}^*$ - locally equivalent to $\mathbb{P}$ such that*

$$d\mathbb{P}^*{}_{|\mathcal{G}_t^\tau} = L_t d\mathbb{P}_{|\mathcal{G}_t^\tau}, \quad i.e. \qquad \forall A \in \mathcal{G}_t^\tau \quad \mathbb{P}^*(A) = \int_A L_t d\mathbb{P}.$$

The martingale L is called the Radon-Nikodým density of $\mathbb{P}^*$ with respect to $\mathbb{P}$.

The measure $\mathbb{P}^*$ has the following properties

i) Under $\mathbb{P}^*$, the random time τ is independent of $\mathcal{F}_t$, $\forall\ t \geq 0$;

ii) $\forall\ t \geq 0\ \mathbb{P}^*{}_{|\mathcal{F}_t} = \mathbb{P}_{|\mathcal{F}_t}$;

iii) $\mathbb{P}^*{}_{|\sigma(\tau)} = \mathbb{P}_{|\sigma(\tau)}$;

iv) $\mathbb{P}^*(\tau \in du|\mathcal{F}_t) = \mathbb{P}^*(\tau \in du)$;

v) $(\mathbb{P}^*, \mathbb{F})$-martingales remain $(\mathbb{P}^*, \mathbb{G}^\tau)$-martingales.

For the proof of the proposition and the properties, see [2] and [4].

The following lemma presents the Bayes formula which plays a crucial role in the proof of the next proposition.

Lemma 5.1. *[4] We assume that $\mathcal{E}$-hypothesis holds, the measures $\mathbb{P}$ and $\mathbb{P}^*$ are equivalent on $\mathcal{G}_t^\tau$ and Y_t - an $\mathcal{F}_t$-measurable, $\mathbb{P}^*$-integrable random variable. Then, for any $s < t$*

$$\mathbb{E}_{\mathbb{P}^*}(Y_t|\mathcal{G}_s^\tau) = \frac{\mathbb{E}_{\mathbb{P}}(L_t Y_t|\mathcal{G}_s^\tau)}{L_s},$$

where L is a Radon-Nikodým density of $\mathbb{P}^*$ with respect to $\mathbb{P}$.

Proof. Let us denote ζ as $\frac{\mathbb{E}_{\mathbb{P}}(L_t Y_t|\mathcal{G}_s^{\tau})}{L_s}$. We will show that ζ is a $\mathcal{G}_s^{\tau}$-conditional expectation of Y_t under the measure $\mathbb{P}^*$. We have that

$$\mathbb{E}_{\mathbb{P}^*}(Y_t|\mathcal{G}_s^{\tau}) = \zeta.$$

Let us modify firstly this condition. If we multiply both sides by a $\mathcal{G}_s^{\tau}$-measurable random variable $\tilde{Y}_s$, as a result we get

$$\mathbb{E}_{\mathbb{P}^*}(\tilde{Y}_s Y_t|\mathcal{G}_s^{\tau}) = \tilde{Y}_s \zeta.$$

We take the expectation with respect to $\mathbb{P}^*$, again on both sides, and apply the tower property on the left-hand side to obtain

$$\mathbb{E}_{\mathbb{P}^*}(\tilde{Y}_s Y_t) = \mathbb{E}_{\mathbb{P}^*}(\tilde{Y}_s \zeta). \tag{5.1}$$

We transformed (5.1) to the equality above. Therefore, to prove (5.1) we can show that (5.1) is fulfilled. Starting from the left-hand side and changing the measure, we obtain

$$\mathbb{E}_{\mathbb{P}^*}(\tilde{Y}_s Y_t) = \mathbb{E}_{\mathbb{P}}(L_t \tilde{Y}_s Y_t),$$

since $\tilde{Y}_s Y_t$ is $\mathcal{G}_t^{\tau}$-measurable. Then, we condition on $\mathcal{G}_s^{\tau}$ and we use the tower property. Therefore, we have that

$$\mathbb{E}_{\mathbb{P}}(L_t \tilde{Y}_s Y_t) = \mathbb{E}_{\mathbb{P}}(\mathbb{E}_{\mathbb{P}}(L_t \tilde{Y}_s Y_t|\mathcal{G}_s^{\tau})).$$

Since $\tilde{Y}_s$ is $\mathcal{G}_s^{\tau}$-measurable, we can take it outside the conditional expectation. $\tilde{Y}_s \mathbb{E}_{\mathbb{P}}(L_t Y_t|\mathcal{G}_s^{\tau})$ is a $\mathcal{G}_s^{\tau}$-measurable random variable so we can, again, change the measure to obtain

$$\mathbb{E}_{\mathbb{P}}(\tilde{Y}_s \mathbb{E}_{\mathbb{P}}(L_t Y_t|\mathcal{G}_s^{\tau})) = \mathbb{E}_{\mathbb{P}^*}(L_s^{-1} \tilde{Y}_s \mathbb{E}_{\mathbb{P}}(L_t Y_t|\mathcal{G}_s^{\tau})).$$

Replacing

$$L_s^{-1} \mathbb{E}_{\mathbb{P}}(L_t Y_t|\mathcal{G}_s^{\tau})$$

with ζ, we get that

$$\mathbb{E}_{\mathbb{P}^*}(\tilde{Y}_s Y_t) = \mathbb{E}_{\mathbb{P}^*}(\tilde{Y}_s \zeta)$$

and we proved (5.1) which is equivalent to (5.1) being satisfied. □

Let us now analyse the proposition which allows to transform a $\mathcal{G}_t^{\tau}$-expected value to an $\mathcal{F}_t$-expected value under the decoupling measure.

Proposition 5.6. *[2] Let $Z_t = z_t(\tau)$ be $\mathcal{G}_t^{\tau}$-measurable. For $s \leq t$, if $z_t(\tau)$ is $\mathbb{P}^*$-integrable and if $z_t(u)$ is $\mathbb{P}$ (or $\mathbb{P}^*$)-integrable for any $u \geq 0$ then,*

$$\mathbb{E}_{\mathbb{P}^*}(z_t(\tau)|\mathcal{G}_s^{\tau}) = \mathbb{E}_{\mathbb{P}^*}(z_t(u)|\mathcal{F}_s)_{|u=\tau} = \mathbb{E}_{\mathbb{P}}(z_t(u)|\mathcal{F}_s)_{|u=\tau} \quad \mathbb{P} \textit{ (or } \mathbb{P}^*\textit{)-a.s.}$$

See [2] for the proof.

Finally, using the proposition above, we prove in the following proposition that the filtration $\mathbb{G}^\tau$ inherits the right-continuity from the filtration $\mathbb{F}$.

Proposition 5.7. *[4] Let us assume that the Hypothesis 5.1 is satisfied. Then,*

$$\forall t \in [0,T) \quad \mathcal{G}_t^\tau = \mathcal{G}_{t+}^\tau \tag{5.2}$$

Proof. To prove that (5.2) is satisfied we have to show that any $\mathcal{G}_{t+}^\tau$-measurable random variable is $\mathcal{G}_t^\tau$-measurable.

At the beginning, let us fix $t \in [0,T)$ and $\delta \in (0, T-t)$ which preserves $\delta + t$ being in the interval (t,T). The proof will be done according to the following plan.

i) Firstly, we prove that $\mathcal{G}_{t+}^\tau$-conditional expectation of the random variable $Z_{t+\delta}^0 = y_{t+\delta}h(\tau)$ (where $\forall t \geq 0$ y_t is $\mathcal{F}_t$-measurable and h is a bounded Borel function on $\mathbb{R}^+$) is the same as a $\mathcal{G}_t^\tau$-conditional expectation.

ii) Then, we extend the obtained result to any $\mathcal{G}_{t+\delta}^\tau$-measurable random variable $Z_{t+\delta}$.

iii) Finally, we use ii) to show that any $\mathcal{G}_{t+}^\tau$-measurable random variable is also $\mathcal{G}_t^\tau$-measurable.

i) Let us assume that we are working at the beginning under the decoupling measure $\mathbb{P}^*$, i.e. τ is independent of the filtration $\mathbb{F}$. $\forall \varepsilon \in (0,\delta)$ we get

$$\mathbb{E}_{\mathbb{P}^*}(Z_{t+\delta}^0 | \mathcal{G}_{t+}^\tau) = \mathbb{E}_{\mathbb{P}^*}(y_{t+\delta}h(\tau)|\mathcal{G}_{t+}^\tau).$$

Since $\mathcal{G}_{t+}^\tau = \bigcap_{\varepsilon>0} \mathcal{G}_{t+\varepsilon}^\tau = \bigcap_{\varepsilon>0} \mathcal{F}_{t+\varepsilon}^\tau \vee \sigma(\tau)$ and $h(\tau)$ is $\sigma(\tau)$-measurable we have

$$\mathbb{E}_{\mathbb{P}^*}(y_{t+\delta}h(\tau)|\mathcal{G}_{t+}^\tau) = h(\tau)\mathbb{E}_{\mathbb{P}^*}(y_{t+\delta}|\mathcal{G}_{t+}^\tau).$$

Using the tower property and the fact that $\forall \varepsilon > 0$, $\bigcap_{\varepsilon>0} \mathcal{G}_{t+\varepsilon}^\tau \subset \mathcal{G}_{t+\varepsilon}^\tau$ we obtain that

$$h(\tau)\mathbb{E}_{\mathbb{P}^*}(y_{t+\delta}|\mathcal{G}_{t+}^\tau) = h(\tau)\mathbb{E}_{\mathbb{P}^*}(\mathbb{E}_{\mathbb{P}^*}(y_{t+\delta}|\mathcal{G}_{t+\varepsilon}^\tau)|\mathcal{G}_{t+}^\tau).$$

From the Proposition 5.6 and the definition of $\mathcal{G}_{t+\varepsilon}^\tau$ as $\mathcal{F}_{t+\varepsilon} \vee \sigma(\tau)$ it follows that

$$\mathbb{E}_{\mathbb{P}^*}(y_{t+\delta}|\mathcal{G}_{t+\varepsilon}^\tau) = \mathbb{E}_{\mathbb{P}^*}(y_{t+\delta}|\mathcal{F}_{t+\varepsilon})_{|u=\tau} = (y_{t+\delta})_{|u=\tau} = y_{t+\delta} = \mathbb{E}_{\mathbb{P}^*}(y_{t+\delta}|\mathcal{F}_{t+\varepsilon}).$$

From the right-continuity of $\mathbb{F}$ we get that

$$\lim_{\varepsilon \to 0} \mathbb{E}_{\mathbb{P}^*}(y_{t+\delta}|\mathcal{F}_{t+\varepsilon}) = \mathbb{E}_{\mathbb{P}^*}(y_{t+\delta}|\mathcal{F}_t)$$

and

$$\lim_{\varepsilon \to 0}[h(\tau)\mathbb{E}_{\mathbb{P}^*}(\mathbb{E}_{\mathbb{P}^*}(y_{t+\delta}|\mathcal{F}_{t+\varepsilon})|\mathcal{G}^{\tau}_{t+})] = h(\tau)\mathbb{E}_{\mathbb{P}^*}(\mathbb{E}_{\mathbb{P}^*}(y_{t+\delta}|\mathcal{F}_t)|\mathcal{G}^{\tau}_{t+}).$$

Since $\forall\ t \geq 0$, $\mathcal{G}^{\tau}_{t+} \supset \mathcal{G}^{\tau}_t \supset \mathcal{F}_t$, we can omit the conditional expectation with respect to σ-algebra $\mathcal{G}^{\tau}_{t+}$ and in the result we obtain that

$$h(\tau)\mathbb{E}_{\mathbb{P}^*}(y_{t+\delta}|\mathcal{F}_t).$$

Now from the independence of τ and $\mathbb{F}$ we can replace $\mathcal{F}_t$ by $\mathcal{G}^{\tau}_t$ and put $h(\tau)$ inside the conditional expectation what follows that

$$h(\tau)\mathbb{E}_{\mathbb{P}^*}(y_{t+\delta}|\mathcal{F}_t) = \mathbb{E}_{\mathbb{P}^*}(h(\tau)y_{t+\delta}|\mathcal{G}^{\tau}_t) = \mathbb{E}_{\mathbb{P}^*}(Z^0_{t+\delta}|\mathcal{G}^{\tau}_t).$$

As a result, we obtained that

$$\mathbb{E}_{\mathbb{P}^*}(Z^0_{t+\delta}|\mathcal{G}^{\tau}_{t+}) = \mathbb{E}_{\mathbb{P}^*}(Z^0_{t+\delta}|\mathcal{G}^{\tau}_t).$$

ii) Since the property is fulfilled for the random variables $Z^0_{t+\delta}$ of the form $y_{t+\delta}h(\tau)$, which are $\mathcal{F}_{t+\delta} \otimes \mathcal{B}(\mathbb{R}^+)$, using the property of the mathematical expectation, we can state that (5.7) is satisfied for the sum of such variables. From the Proposition 5.2 which establishes form of $\mathcal{G}^{\tau}_{t+\delta}$-measurable random variable and by passing to the limit, we obtain that (5.7) is satisfied for any $\mathcal{G}^{\tau}_{t+\delta}$-measurable random variable $Z_{t+\delta}$.

iii) Since $\mathcal{G}^{\tau}_{t+\delta} \supset \mathcal{G}^{\tau}_{t+\varepsilon} \supset \bigcap_{\varepsilon>0} \mathcal{G}^{\tau}_{t+\varepsilon} = \mathcal{G}^{\tau}_{t+}$ we can apply the result from ii) to any $\mathcal{G}^{\tau}_{t+}$-measurable random variable Z_{t+}, hence

$$Z_{t+} = \mathbb{E}_{\mathbb{P}^*}(Z_{t+}|\mathcal{G}^{\tau}_{t+}) = \mathbb{E}_{\mathbb{P}^*}(Z_{t+}|\mathcal{G}^{\tau}_t).$$

Since $\mathbb{P}^* \sim \mathbb{P}$ and $\mathcal{G}_0$ contains all $\mathbb{P}$-negligible events, Z_{t+} is also $\mathcal{G}_t$-measurable.

□

5.5.1 The survival process under measure $\mathbb{P}$ and $\mathbb{P}^*$

Let us finally introduce the conditional survival process R applying the density approach under the measure $\mathbb{P}$ and $\mathbb{P}^*$. More precisely,

$$R_t := \mathbb{P}(\tau > t|\mathcal{F}_t) = \int_t^\infty q_t(u)\eta(du),$$

$$R_t^* := \mathbb{P}^*(\tau > t|\mathcal{F}_t) = \int_t^\infty \eta(du).$$

The form of R_t is straightforward while the form of R_t^* requires more detailed explanation. Due to the properties of the measure $\mathbb{P}^*$ in relation with the measure $\mathbb{P}$ (see section 5.5) we have

$$\mathbb{P}^*(\tau > t|\mathcal{F}_t) = \mathbb{P}^*(\tau > t)$$

and

$$\mathbb{P}^*(\tau > t) = \mathbb{P}^*(\tau > t|\mathcal{F}_0) = \mathbb{P}(\tau > t|\mathcal{F}_0) = \mathbb{P}(\tau > t) = \int_t^\infty \eta(du).$$

As a result we obtained that

$$\mathbb{P}^*(\tau > t|\mathcal{F}_t) = \int_t^\infty \eta(du).$$

Remark 5.2. Properties of the process R

i) $(R^*)_{t\geq 0}$ is a deterministic, continuous and decreasing function;

i) $(R_t)_{t\geq 0}$ is an $(\mathbb{P}, \mathbb{F})$-supermartingale (called Azéma supermartingale).

Chapter 6

The initial enlargement framework

In this chapter we explore some propositions concerning the expectation tools and the martingales characterization in the initially enlarged filtrations. We assume that the Hypothesis 5.1 is satisfied throughout the entire chapter and we show finally that it is a sufficient condition for the defaultable market to be arbitrage-free for the agent with initially enlarged filtration as an information flow. Let us introduce firstly an auxiliary lemma which will be used in the proofs below.

Lemma 6.1. *[2] Let $Z_t = z_t(\tau)$ be a $\mathcal{G}_t^\tau$-measurable, $\mathbb{P}$-integrable random variable and*

$$z_t(\tau) = 0 \quad \mathbb{P} - a.s.$$

Then, for η-a.e. $u \geq 0$,

$$z_t(u) = 0 \quad \mathbb{P} - a.s.$$

Proof. Since $z_t(\tau)$ is integrable, $\mathbb{E}_\mathbb{P}(|z_t(\tau)|) < \infty$. On the other hand, $z_t(\tau) = 0$ $\mathbb{P}$-a.s. Therefore, if we put the conditional expectation on both sides and apply the tower property thereafter, we will obtain that

$$\mathbb{E}_\mathbb{P}(z_t(\tau)) = \mathbb{E}_\mathbb{P}(0) = 0$$

and

$$0 = \mathbb{E}_\mathbb{P}(z_t(\tau)) = \mathbb{E}_\mathbb{P}(\mathbb{E}_\mathbb{P}(z_t(\tau))|\mathcal{F}_t) = \mathbb{E}_\mathbb{P}(\mathbb{E}_\mathbb{P}(z_t(\tau)|\mathcal{F}_t)).$$

From the Hypothesis 5.1 we obtain that

$$\mathbb{E}_\mathbb{P}(\mathbb{E}_\mathbb{P}(z_t(\tau)|\mathcal{F}_t)) = \mathbb{E}_\mathbb{P}\left(\int_0^\infty |z_t(u)| q_t(u)\eta(du)\right)$$

and from the previous results

$$\mathbb{E}_{\mathbb{P}}\left(\int_0^\infty |z_t(u)|q_t(u)\eta(du)\right) = 0.$$

Due to the fact that $\forall t \geq 0$ $z_t(u) \geq 0$, $\forall u \geq 0$ $(q_t(u))_{t\geq 0}$ is a strictly positive martingale $\mathbb{P}$-a.s. and η is a positive measure, we get that

$$\int_0^\infty |z_t(u)|q_t(u)\eta(du) \geq 0.$$

Given that the expected value from this integral is equal 0, we conclude that

$$\int_0^\infty |z_t(u)|q_t(u)\eta(du) = 0 \quad \mathbb{P} - a.s.$$

Again, from the fact that $(q_t(u))_{t\geq 0}$ is a strictly positive process $\mathbb{P}$-a.s. and η is a positive measure, we obtain that for $\eta - a.e.$ $u \geq 0$ $z_t(u) = 0 - \mathbb{P}$-a.s. □

6.1 Expectation tools

In the following lemma we make precise how to express the $\mathcal{G}_s^\tau$-conditional expectation in terms of the $\mathcal{F}_s$-conditional expectation under the same measure $\mathbb{P}$.

Lemma 6.2. *[2] Let $Z_t = z_t(\tau)$ be $\mathcal{G}_t^\tau$-measurable. For $s \leq t$, if $z_t(\tau)$ is $\mathbb{P}$-integrable then,*

$$\mathbb{E}_{\mathbb{P}}(z_t(\tau)|\mathcal{G}_s^\tau) = \frac{1}{q_s(\tau)}\mathbb{E}_{\mathbb{P}}(z_t(u)q_t(u)|\mathcal{F}_s)_{|u=\tau}.$$

Proof. Since $\mathbb{P}$ and $\mathbb{P}^*$ are equivalent on $\mathcal{G}_s^\tau$ and $L = (\frac{1}{q_t(\tau)})_{t\geq 0}$ is a Radon-Nikodým density of $\mathbb{P}^*$ with respect to $\mathbb{P}$, we can apply the Bayes formula (see Lemma 5.1) to obtain

$$\mathbb{E}_{\mathbb{P}}(z_t(\tau)|\mathcal{G}_s^\tau) = \frac{\mathbb{E}_{\mathbb{P}^*}(L_t^{-1}z_t(\tau)|\mathcal{G}_s^\tau)}{L_s^{-1}}.$$

Using the explicit form for L_t, we get

$$\frac{\mathbb{E}_{\mathbb{P}^*}(L_t^{-1}z_t(\tau)|\mathcal{G}_s^\tau)}{L_s^{-1}} = \frac{\mathbb{E}_{\mathbb{P}^*}(q_t(\tau)z_t(\tau)|\mathcal{G}_s^\tau)}{q_s(\tau)}.$$

Eventually, from the Proposition 5.6, we have

$$\frac{\mathbb{E}_{\mathbb{P}^*}(q_t(\tau)z_t(\tau)|\mathcal{G}_s^\tau)}{q_s(\tau)} = \frac{\mathbb{E}_{\mathbb{P}^*}(q_t(u)z_t(u)|\mathcal{F}_s)_{|u=\tau}}{q_s(\tau)}.$$

Since $\mathbb{P}$ and $\mathbb{P}^*$ coincide on $\mathcal{F}_s$

$$\mathbb{E}_{\mathbb{P}}(z_t(\tau)|\mathcal{G}_s^\tau) = \frac{\mathbb{E}_{\mathbb{P}}(q_t(u)z_t(u)|\mathcal{F}_s)_{|u=\tau}}{q_s(\tau)}.$$

□

6.2 The martingales characterization

Our task now is to find a characterization of $(\mathbb{P}, \mathbb{G}^\tau)$-martingales in terms of $(\mathbb{P}, \mathbb{F})$-martingales. Let us consider the following proposition.

Proposition 6.1. *[2] A process $Z = z(\tau)$ is a $(\mathbb{P}, \mathbb{G}^\tau)$-martingale if and only if the process $(z_t(u)q_t(u))_{t\geq 0}$ is a $(\mathbb{P}, \mathbb{F})$-martingale, for η-a.e. $u \geq 0$.*

Proof. Let us prove firstly the necessity condition by assuming that Z is a $(\mathbb{P}, \mathbb{G}^\tau)$-martingale. As a result, we have

$$z_s(\tau) = \mathbb{E}_{\mathbb{P}}(z_t(\tau)|\mathcal{G}_s^\tau).$$

Using the Lemma 6.2, we get that

$$\mathbb{E}_{\mathbb{P}}(z_t(\tau)|\mathcal{G}_s^\tau) = \frac{1}{q_s(\tau)}\mathbb{E}_{\mathbb{P}}(z_t(u)q_t(u)|\mathcal{F}_s)_{|u=\tau}$$

and hence,

$$z_s(\tau)q_s(\tau) = \mathbb{E}_{\mathbb{P}}(z_t(u)q_t(u)|\mathcal{F}_s)_{|u=\tau}.$$

$z_s(\tau)q_s(\tau) - \mathbb{E}_{\mathbb{P}}(z_t(u)q_t(u)|\mathcal{F}_s)_{|u=\tau}$ is a $\mathcal{G}_s^\tau$-measurable random variable and it is equal to 0. Therefore, we can use the Lemma 6.1 and write that η-a.s. for all $u > 0$

$$z_s(u)q_s(u) - \mathbb{E}_{\mathbb{P}}(z_t(u)q_t(u)|\mathcal{F}_s) = 0.$$

Finally we have that η-a.s. for all $u > 0$

$$\mathbb{E}_{\mathbb{P}}(z_t(u)q_t(u)|\mathcal{F}_s) = z_s(u)q_s(u),$$

which proves that $(z_t(u)q_t(u))_{t\geq 0}$ is a $(\mathbb{P}, \mathbb{F})$-martingale.

To prove the sufficiency part, let us assume that the process $(z_t(u)q_t(u))_{t\geq 0}$ is a $(\mathbb{P}, \mathbb{F})$-martingale, for η-a.e. $u \geq 0$. We have to show that

$$\mathbb{E}_{\mathbb{P}}(Z_t|\mathcal{G}_s^\tau) = Z_s.$$

If we apply the Lemma 6.2 for the left-hand side we obtain that

$$\mathbb{E}_{\mathbb{P}}(Z_t|\mathcal{G}_s^\tau) = \mathbb{E}_{\mathbb{P}}(z_t(\tau)|\mathcal{G}_s^\tau) = \frac{1}{q_s(\tau)}\mathbb{E}_{\mathbb{P}}(z_t(u)q_t(u)|\mathcal{F}_s)_{|u=\tau}.$$

From the martingale property stated above, we get

$$\frac{1}{q_s(\tau)}\mathbb{E}_{\mathbb{P}}(z_t(u)q_t(u)|\mathcal{F}_s)_{|u=\tau} = \frac{1}{q_s(\tau)}(z_s(u)q_s(u))_{|u=\tau} = z_s(\tau).$$

□

6.3 The $\mathcal{E}$-hypothesis and the absence of arbitrage in the filtration $\mathbb{G}^\tau$

We shall remind in the beginning the general condition for the absence of arbitrage. It is a well-known fact that if there exists at least one martingale measure (a measure equivalent to the physical measure such that a stock price process is a martingale with respect to the given filtration), i.e. the set of all martingales measures is not empty, then the market is arbitrage-free.

Let us now consider a default-free and arbitrage-free market with assets remaining assets of the full filtration $\mathbb{G}^\tau$. We set $\mathbb{Q}$ as one of the martingale measures equivalent to $\mathbb{P}$ on $\mathbb{F}$ and assume that the set of measures equivalent to $\mathbb{P}$ on $\mathbb{G}^\tau$ is non-empty. We showed before that if $\mathcal{E}$-hypothesis holds, then there exists a decoupling measure $\mathbb{P}^*$ making τ independent of the reference filtration and coinciding with $\mathbb{Q}$ on $\mathbb{F}$ (see section 5.5). As a result, $\mathbb{P}^*$ preserves martingale property in the initially enlarged filtration and a set of martingale measures equivalent to $\mathbb{P}$ on $\mathbb{G}^\tau$ is non-empty. Therefore, $\mathcal{E}$-hypothesis is a suitable condition to make the defaultable market arbitrage-free for the agent with the initially enlarged filtration as an information flow.

Chapter 7

The progressive enlargement framework

7.1 The intensity approach

In this section we study the progressive enlargement of filtration which we introduced before in the preceding chapter. Hereafter, we assume that the price process follows the log-normal distribution. Thus, the filtration $\mathbb{F}$ is considered as a Brownian filtration (i.e. $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ and $\mathcal{F}_t = \sigma(B_s, s \leq t)$, where B_s is a standard Brownian motion). It is not necessary to require the Hypothesis 5.1 to hold.

We can easily describe $\mathcal{G}_t$-measurable events on the set $\{\tau > t\}$. Any $\mathcal{G}_t$-measurable random variable X_t satisfies $X_t\mathbb{I}_{\{\tau>t\}} = Y_t\mathbb{I}_{\{\tau>t\}}$, where Y_t is an $\mathcal{F}_t$-measurable random variable.

7.1.1 Expectation tools

Proposition 7.1. *Let ζ be an integrable random variable. Let T be a fixed time horizon. Then, for any $t < T$,*

$$\mathbb{E}_{\mathbb{P}}(\zeta|\mathcal{G}_t)\mathbb{I}_{\{\tau>t\}} = \mathbb{I}_{\{\tau>t\}}\frac{\mathbb{E}_{\mathbb{P}}(\zeta\mathbb{I}_{\{\tau>t\}}|\mathcal{F}_t)}{\mathbb{E}_{\mathbb{P}}(\mathbb{I}_{\{\tau>t\}}|\mathcal{F}_t)}.$$

Proof. Since ζ is an integrable random variable we can write the conditional expectation

$$X_t = \mathbb{E}_{\mathbb{P}}(\zeta|\mathcal{G}_t)$$

which is a $\mathcal{G}_t$-measurable random variable. We make the assumption that $\mathcal{G}_t \subset \mathcal{G}_t^*$, where

$$\mathcal{G}_t^* = \{A \in \mathcal{G}_t : \exists B \in \mathcal{F}_t \quad A \cap \{\tau > t\} = B \cap \{\tau > t\}\}.$$

Thus, there exists an $\mathcal{F}_t$-measurable random variable Y_t such that

$$X_t \mathbb{I}_{\{\tau>t\}} = Y_t \mathbb{I}_{\{\tau>t\}}.$$

Taking the conditional expectation with respect to $\mathcal{F}_t$ from both sides we obtain

$$\mathbb{E}_{\mathbb{P}}(X_t \mathbb{I}_{\{\tau>t\}} | \mathcal{F}_t) = \mathbb{E}_{\mathbb{P}}(Y_t \mathbb{I}_{\{\tau>t\}} | \mathcal{F}_t).$$

Knowing that Y_t is an $\mathcal{F}_t$-measurable random variable we take it outside the conditional expectation and get

$$\mathbb{E}_{\mathbb{P}}(X_t \mathbb{I}_{\{\tau>T\}} | \mathcal{F}_t) = Y_t \mathbb{E}_{\mathbb{P}}(\mathbb{I}_{\{\tau>t\}} | \mathcal{F}_t).$$

Thus,

$$Y_t = \frac{\mathbb{E}_{\mathbb{P}}(X_t \mathbb{I}_{\{\tau>t\}} | \mathcal{F}_t)}{\mathbb{E}_{\mathbb{P}}(\mathbb{I}_{\{\tau>t\}} | \mathcal{F}_t)}.$$

We have

$$\mathbb{E}_{\mathbb{P}}(\zeta | \mathcal{G}_t) \mathbb{I}_{\{\tau>t\}} = Y_t \mathbb{I}_{\{\tau>t\}} = \mathbb{I}_{\{\tau>t\}} \frac{\mathbb{E}_{\mathbb{P}}(X_t \mathbb{I}_{\{\tau>t\}} | \mathcal{F}_t)}{\mathbb{E}_{\mathbb{P}}(\mathbb{I}_{\{\tau>t\}} | \mathcal{F}_t)}.$$

We get that

$$X_t = \mathbb{E}_{\mathbb{P}}(\zeta | \mathcal{G}_t)$$

and $\mathbb{I}_{\{\tau>t\}}$ is $\mathcal{F}_t$-measurable. Hence we have

$$\mathbb{I}_{\{\tau>t\}} \frac{\mathbb{E}_{\mathbb{P}}(X_t \mathbb{I}_{\{\tau>t\}} | \mathcal{F}_t)}{\mathbb{E}_{\mathbb{P}}(\mathbb{I}_{\{\tau>t\}} | \mathcal{F}_t)} = \mathbb{I}_{\{\tau>t\}} \frac{\mathbb{I}_{\{\tau>t\}} \mathbb{E}_{\mathbb{P}}(\mathbb{E}_{\mathbb{P}}(\zeta | \mathcal{G}_t) | \mathcal{F}_t)}{\mathbb{E}_{\mathbb{P}}(\mathbb{I}_{\{\tau>t\}} | \mathcal{F}_t)}.$$

From the fact that $\mathbb{F} \subset \mathbb{G}$ we deduce

$$\mathbb{I}_{\{\tau>t\}} \frac{\mathbb{I}_{\{\tau>t\}} \mathbb{E}_{\mathbb{P}}(\mathbb{E}_{\mathbb{P}}(\zeta | \mathcal{G}_t) | \mathcal{F}_t)}{\mathbb{E}_{\mathbb{P}}(\mathbb{I}_{\{\tau>t\}} | \mathcal{F}_t)} = \mathbb{I}_{\{\tau>t\}} \frac{\mathbb{I}_{\{\tau>t\}} \mathbb{E}_{\mathbb{P}}(\zeta | \mathcal{F}_t)}{\mathbb{E}_{\mathbb{P}}(\mathbb{I}_{\{\tau>t\}} | \mathcal{F}_t)}.$$

Consequently, using again the fact that $\mathbb{I}_{\{\tau>t\}}$ is $\mathcal{F}_t$-measurable, we obtain

$$\mathbb{I}_{\{\tau>t\}} \frac{\mathbb{I}_{\{\tau>t\}} \mathbb{E}_{\mathbb{P}}(\zeta | \mathcal{F}_t)}{\mathbb{E}_{\mathbb{P}}(\mathbb{I}_{\{\tau>t\}} | \mathcal{F}_t)} = \mathbb{I}_{\{\tau>t\}} \frac{\mathbb{E}_{\mathbb{P}}(\zeta \mathbb{I}_{\{\tau>t\}} | \mathcal{F}_t)}{\mathbb{E}_{\mathbb{P}}(\mathbb{I}_{\{\tau>t\}} | \mathcal{F}_t)}.$$

□

Corollary 7.1. *Let X be a $\mathcal{G}_t$-measurable integrable random variable. Let T be a fixed time horizon. Then, for any $t \in [0,T)$,*

$$\mathbb{E}_{\mathbb{P}}(X|\mathcal{G}_t)\mathbb{I}_{\{\tau>t\}} = \mathbb{I}_{\{\tau>t\}}\frac{\mathbb{E}_{\mathbb{P}}(X\mathbb{I}_{\{\tau>t\}}|\mathcal{F}_t)}{\mathbb{E}_{\mathbb{P}}(\mathbb{I}_{\{\tau>t\}}|\mathcal{F}_t)}.$$

Proof. When we put $X = \mathbb{E}_{\mathbb{P}}(X|\mathcal{G}_t)$ we can use the proof of the Proposition 7.1 to obtain the result. □

Proposition 7.2. *Let X be a $\mathcal{G}_T$-measurable random variable where T is a fixed time horizon. Then*

$$\mathbb{E}_{\mathbb{P}}(X|\mathcal{G}_t)\mathbb{I}_{\{\tau>t\}} = \mathbb{I}_{\{\tau>t\}}\frac{\mathbb{E}_{\mathbb{P}}(X\mathbb{I}_{\{\tau>t\}}|\mathcal{F}_t)}{\mathbb{E}_{\mathbb{P}}(\mathbb{I}_{\{\tau>t\}}|\mathcal{F}_t)}.$$

Proof. We have that $\mathbb{E}_{\mathbb{P}}(X|\mathcal{G}_t)$ is $\mathcal{G}_t$-measurable. Hence, there exists an $\mathcal{F}_t$-measurable random variable Y_t such that

$$\mathbb{E}_{\mathbb{P}}(X|\mathcal{G}_t)\mathbb{I}_{\{\tau>t\}} = Y_t\mathbb{I}_{\{\tau>t\}}.$$

Taking the conditional expectation with respect to $\mathcal{F}_t$ from both sides we obtain

$$\mathbb{E}_{\mathbb{P}}(\mathbb{E}_{\mathbb{P}}(X|\mathcal{G}_t)\mathbb{I}_{\{\tau>t\}}|\mathcal{F}_t) = \mathbb{E}_{\mathbb{P}}(Y_t\mathbb{I}_{\{\tau>t\}}|\mathcal{F}_t).$$

Knowing that Y_t is $\mathcal{F}_t$-measurable and $\mathbb{I}_{\{\tau>t\}}$ is $\mathcal{H}_t$-measurable, we can write

$$\mathbb{I}_{\{\tau>t\}}\mathbb{E}_{\mathbb{P}}(\mathbb{E}_{\mathbb{P}}(X|\mathcal{G}_t)|\mathcal{F}_t) = \mathbb{I}_{\{\tau>t\}}\mathbb{E}_{\mathbb{P}}(X|\mathcal{F}_t) = \mathbb{E}_{\mathbb{P}}(X\mathbb{I}_{\{\tau>t\}}|\mathcal{F}_t) = Y_t\mathbb{E}_{\mathbb{P}}(\mathbb{I}_{\{\tau>t\}}|\mathcal{F}_t).$$

We obtain

$$Y_t = \frac{\mathbb{E}_{\mathbb{P}}(X\mathbb{I}_{\{\tau>t\}}|\mathcal{F}_t)}{\mathbb{E}_{\mathbb{P}}(\mathbb{I}_{\{\tau>t\}}|\mathcal{F}_t)}.$$

Then

$$\mathbb{E}_{\mathbb{P}}(X|\mathcal{G}_t)\mathbb{I}_{\{\tau>t\}} = \mathbb{I}_{\{\tau>t\}}\frac{\mathbb{E}_{\mathbb{P}}(X\mathbb{I}_{\{\tau>t\}}|\mathcal{F}_t)}{\mathbb{E}_{\mathbb{P}}(\mathbb{I}_{\{\tau>t\}}|\mathcal{F}_t)}.$$

□

Proposition 7.3. *Let X be a $\mathcal{G}_T$-measurable integrable random variable and T a fixed time horizon, then*

$$\mathbb{E}_{\mathbb{P}}(X\mathbb{I}_{\{\tau>T\}}|\mathcal{G}_t) = \mathbb{I}_{\{\tau>t\}}\frac{\mathbb{E}_{\mathbb{P}}(X\mathbb{I}_{\{\tau>T\}}|\mathcal{F}_t)}{\mathbb{E}_{\mathbb{P}}(\mathbb{I}_{\{\tau>t\}}|\mathcal{F}_t)}.$$

Proof. We can write the expectation on two sets

$$\mathbb{E}_{\mathbb{P}}(X\mathbb{I}_{\{\tau>T\}}|\mathcal{G}_t) = \mathbb{I}_{\{\tau>t\}}\mathbb{E}_{\mathbb{P}}(X\mathbb{I}_{\{\tau>T\}}|\mathcal{G}_t) + \mathbb{I}_{\{\tau\leq t\}}\mathbb{E}_{\mathbb{P}}(X\mathbb{I}_{\{\tau>T\}}|\mathcal{G}_t)$$

Then we can take the indicators inside the expectations because they are $\mathcal{G}_t$-measurable. We get

$$\mathbb{E}_{\mathbb{P}}(X\mathbb{I}_{\{\tau>t\}}\mathbb{I}_{\{\tau>T\}}|\mathcal{G}_t) + \mathbb{E}_{\mathbb{P}}(X\mathbb{I}_{\{\tau\leq t\}}\mathbb{I}_{\{\tau>T\}}|\mathcal{G}_t).$$

The second term on the right-hand side vanishes and we get

$$\mathbb{I}_{\{\tau>t\}}\mathbb{E}_{\mathbb{P}}(X\mathbb{I}_{\{\tau>T\}}|\mathcal{G}_t).$$

Finally, from the Proposition 7.2 we have

$$\mathbb{I}_{\{\tau>t\}}\frac{\mathbb{E}_{\mathbb{P}}(X\mathbb{I}_{\{\tau>T\}}\mathbb{I}_{\{\tau>t\}}|\mathcal{F}_t)}{\mathbb{E}_{\mathbb{P}}(\mathbb{I}_{\{\tau>t\}}|\mathcal{F}_t)} = \mathbb{I}_{\{\tau>t\}}\frac{\mathbb{E}_{\mathbb{P}}(X\mathbb{I}_{\{\tau>T\}}|\mathcal{F}_t)}{\mathbb{E}_{\mathbb{P}}(\mathbb{I}_{\{\tau>t\}}|\mathcal{F}_t)}.$$

□

7.1.2 The $\mathbb{F}$-hazard process $(\Gamma_t)_{t\geq 0}$

In the previous chapter we introduced a hazard function in a framework of the filtration $\mathbb{H}$. We had a supposition that the agent pricing the defaultable contingent claims knows only the distribution function $F(t) = \mathbb{P}(\tau \leq t)$ of default time τ. Accordingly, the hazard function was purely deterministic. Nevertheless, in this chapter we assume that the agent also observes the price process. Thus, we add to our study this information and the hazard function is no more deterministic. Moreover, while calculating the probability of τ we take under consideration the flow of information about the prices process $\mathbb{F}$. We denote

$$\tilde{F}(t) = \mathbb{P}(\tau \leq t|\mathcal{F}_t)$$

and make the following assumption.

Assumption 7.1. We assume that $\forall t \geq 0$ $\tilde{F}(t) < 1$.

Consequently, we define an $\mathbb{F}$-hazard process as follows.

Definition 7.1. We call a process $\Gamma = (\Gamma_t)_{t\geq 0}$ an $\mathbb{F}$-hazard process where

$$\Gamma_t = -\ln(1 - \tilde{F}(t)).$$

We can easily check that the process is a submartingale.

Proposition 7.4. *$\mathbb{P}(\tau \le t|\mathcal{F}_t)$ is an $\mathbb{F}$-submartingale.*

Proof. We have

$$\mathbb{E}_{\mathbb{P}}(\mathbb{P}(\tau \le t|\mathcal{F}_t)|\mathcal{F}_s) = \mathbb{E}_{\mathbb{P}}(\mathbb{E}_{\mathbb{P}}(\mathbb{I}_{\{\tau\le t\}}|\mathcal{F}_t)|\mathcal{F}_s) = \mathbb{E}_{\mathbb{P}}(\mathbb{I}_{\{\tau\le t\}}|\mathcal{F}_s) = \mathbb{P}(\tau \le t|\mathcal{F}_s)$$

and clearly

$$\mathbb{P}(\tau \le t|\mathcal{F}_s) \ge \mathbb{P}(\tau \le s|\mathcal{F}_s)$$

Thus, $\tilde{F}(t)$ is an $\mathbb{F}$-submartingale. □

Proposition 7.5. *Let T be a fixed time horizon and Y be an $\mathcal{F}_T$-measurable integrable random variable. Then*

$$\mathbb{E}_{\mathbb{P}}(Y\mathbb{I}_{\{\tau>t\}}|\mathcal{G}_t) = \mathbb{I}_{\{\tau>t\}}\mathbb{E}_{\mathbb{P}}(Y\exp(\Gamma_t - \Gamma_T)|\mathcal{F}_t).$$

Proof. If Y is $\mathcal{F}_T$-measurable, then Y is $\mathcal{G}_T$-measurable because $\mathcal{F}_T \subset \mathcal{F}_T \vee \mathcal{H}_T = \mathcal{G}_T$.
According to the Proposition 7.3 we have

$$\mathbb{E}_{\mathbb{P}}(Y\mathbb{I}_{\{\tau>T\}}|\mathcal{G}_t) = \mathbb{I}_{\{\tau>t\}}\frac{\mathbb{E}_{\mathbb{P}}(Y\mathbb{I}_{\{\tau>T\}}|\mathcal{F}_t)}{\mathbb{E}_{\mathbb{P}}(\mathbb{I}_{\{\tau>t\}}|\mathcal{F}_t)}.$$

From the definition of a hazard process we have

$$\mathbb{E}_{\mathbb{P}}(\mathbb{I}_{\{\tau>t\}}|\mathcal{F}_t) = \mathbb{P}(\tau > t|\mathcal{F}_t) = \exp(-\Gamma_t),$$

which is $\mathcal{F}_t$-measurable.
Function $f(x) = \frac{1}{x}$, where $x \in \mathbb{R}^+\backslash\{0\}$, is a Borel function. Hence, if $\exp(-\Gamma_t)$ is $\mathcal{F}_t$-measurable then

$$\frac{1}{\exp(-\Gamma_t)}$$

is $\mathcal{F}_t$-measurable. Thus, we can take $\frac{1}{\exp(-\Gamma_t)}$ inside the expectation in the nominator and obtain

$$\mathbb{I}_{\{\tau>t\}}\mathbb{E}_{\mathbb{P}}(Y\exp(\Gamma_t)\mathbb{I}_{\{\tau>T\}}|\mathcal{F}_t).$$

From the tower property we can condition the expectation in the nominator with a bigger σ-algebra $\mathcal{F}_T$. $\exp(\Gamma_t)$ is $\mathcal{F}_t$-measurable so it is also $\mathcal{F}_T$-measurable because $\mathcal{F}_t \subset \mathcal{F}_T$. We can take $Y\exp(\Gamma_t)$ outside this expected value because the function $f(x, y) = xy$ is a Borel function so $Y\exp(\Gamma_t)$ is $\mathcal{F}_T$-measurable. We get

$$\mathbb{I}_{\{\tau>t\}}\mathbb{E}_{\mathbb{P}}(Y\exp(\Gamma_t)\mathbb{E}_{\mathbb{P}}(\mathbb{I}_{\{\tau>T\}}|\mathcal{F}_T)|\mathcal{F}_t).$$

Then again from the definition of a hazard process we have

$$\mathbb{E}_{\mathbb{P}}(\mathbb{I}_{\{\tau>T\}}|\mathcal{F}_T) = \mathbb{P}(\tau > T|\mathcal{F}_T) = \exp(-\Gamma_T).$$

We obtain the result

$$\mathbb{I}_{\{\tau>t\}}\mathbb{E}_{\mathbb{P}}(Y \exp(\Gamma_t) \exp(-\Gamma_T)|\mathcal{F}_t).$$

□

Corollary 7.2. *Let T be a fixed time horizon. Then*

$$\mathbb{E}_{\mathbb{P}}(\mathbb{I}_{\{\tau>T\}}|\mathcal{G}_t) = \mathbb{I}_{\{\tau>t\}}\mathbb{E}_{\mathbb{P}}(\exp(\Gamma_t - \Gamma_T)|\mathcal{F}_t).$$

Proof. The proof is straightforward from the Proposition 7.5. □

7.1.3 The $\mathbb{G}$-intensity of τ

In the enlarged filtration we can also define a $\mathbb{G}$-intensity of default time τ. From the Definition 4.2, $\tilde{\lambda}_s$ is a $\mathbb{G}$-intensity of τ if

i) $\tilde{\lambda}_s$ is a $\mathbb{G}$-adapted non-negative predictable process,

ii) $(\mathbb{I}_{\{\tau\leq t\}} - \int_0^t \tilde{\lambda}_u \mathbb{I}_{\{\tau\geq u\}} du)_{t\geq 0}$ is a $(\mathbb{P}, \mathbb{G})$-martingale.

Proposition 7.6. *If $(\tilde{k}_t)_{t\geq 0}$ is a $\mathbb{G}$-predictable bounded process, then there exists an $\mathbb{F}$-predictable bounded process $(k_t)_{t\geq 0}$ such that*

$$\tilde{k}_t\mathbb{I}_{\{\tau\geq t\}} = k_t\mathbb{I}_{\{\tau\geq t\}}.$$

Proof. On the set $\mathbb{I}_{\{\tau\geq t\}}$, i.e. before the default appears, we do not have any information about the distribution of default time. We observe the default only when it occurs. From this it follows that on $\mathbb{I}_{\{\tau\geq t\}}$ any $\mathbb{G}$-predictable process is an $\mathbb{F}$-predictable process $(k_t)_{t\geq 0}$, i.e.

$$\tilde{k}_t\mathbb{I}_{\{\tau\geq t\}} = k_t\mathbb{I}_{\{\tau\geq t\}}.$$

□

Corollary 7.3. *There exists an $\mathbb{F}$-predictable process $\lambda = (\lambda_t)_{t\geq 0}$, such that*

$$\tilde{\lambda}_t\mathbb{I}_{\{\tau\geq t\}} = \lambda_t\mathbb{I}_{\{\tau\geq t\}}$$

and

$$(N_t - \int_0^t \lambda(u)\mathbb{I}_{\{\tau\geq u\}} du)_{t\geq 0}$$

is a $\mathbb{G}$-martingale.

Proof. The existence follows from the Proposition 7.6. □

In the future calculations we change the measure so that τ is independent of $\mathbb{F}$. Thus, we consider the following proposition.

Proposition 7.7. *If τ is independent of $\mathbb{F}$ then the $\mathbb{G}$-intensity of τ on the set $\mathbb{I}_{\{\tau>t\}}$ is*

$$\lambda_s = \frac{f(s)}{1-F(s)}.$$

Equivalently,

$$\tilde{\lambda}_s \mathbb{I}_{\{\tau>t\}} = \frac{f(s)}{1-F(s)}\mathbb{I}_{\{\tau>t\}}.$$

7.1.4 $\mathcal{H}$-hypothesis and the absence of arbitrage in the filtration $\mathbb{G}$

The $\mathcal{H}$-hypothesis

Let us consider the $\mathcal{H}$-hypothesis (or the immersion property) which is strongly related to the absence of arbitrage in the progressively enlarged filtration.

Hypothesis 7.1. (**$\mathcal{H}$-hypothesis**) Every square-integrable $\mathbb{F}$-martingale remains square-integrable $\mathbb{G}$-martingale.

It is also pivotal to give the conditions equivalent to the $\mathcal{H}$-hypothesis.

Proposition 7.8. *[1] The following statements are equivalent:*

($\mathcal{H}$) *Every $\mathbb{F}$-square integrable martingale is a $\mathbb{G}$-square integrable martingale,*

($\mathcal{H}1$) $\forall t \geq 0$, $\forall Y \in \mathcal{F}_\infty$, $\forall X \in \mathcal{G}_t$ $\mathbb{E}(YX|\mathcal{F}_t) = \mathbb{E}(Y|\mathcal{F}_t)\mathbb{E}(X|\mathcal{F}_t)$,

($\mathcal{H}2$) $\forall t \geq 0$, $\forall X \in \mathcal{G}_t$ $\mathbb{E}(X|\mathcal{F}_\infty) = \mathbb{E}(X|\mathcal{F}_t)$,

($\mathcal{H}3$) $\forall t \geq 0$, $\forall Y \in \mathcal{F}_\infty$, $\mathbb{E}(Y|\mathcal{G}_t) = \mathbb{E}(Y|\mathcal{F}_t)$,

($\mathcal{H}4$) $\forall s \leq t$, $\mathbb{P}(\tau \leq s|\mathcal{F}_\infty) = \mathbb{P}(\tau \leq s|\mathcal{F}_t)$.

For the proof see [1].

The absence of arbitrage

Let us consider a default-free market with the property of the absence of arbitrage and assume that assets of the reference filtration $\mathbb{F}$ remain assets of the full filtration $\mathbb{G}$. Moreover, $\mathbb{Q}$ is a martingale measure equivalent to $\mathbb{P}$ on $\mathbb{F}$. Consequently, if the immersion property holds under a risk-neutral measure $\mathbb{Q}$, i.e. $\mathbb{F}$-martingales are $\mathbb{G}$-martingales, then the set of all martingale measures equivalent to $\mathbb{P}$ on $\mathbb{G}$ contains already the measure $\mathbb{Q}$. Therefore, it is not empty and the market is arbitrage-free.

7.1.5 The value of information

To price the contingent claims we have to use the martingale measure. Let us consider a complete market with the risk-free interest rate $r(t) \equiv 0$ and $B_0 = 0$. Equivalently, every $\mathcal{F}_T$-measurable claim Y is hedgeable and the price of Y is

$$\mathbb{E}_{\mathbb{Q}}(Y|\mathcal{F}_t),$$

where $\mathbb{Q}$ is a martingale measure equivalent to $\mathbb{P}$. We consider default time such that $\{\tau > T\}$ is $\mathcal{G}_T$-measurable. Let us denote by C_t^{un} the price of the defaultable contingent claim Y calculated by an agent who knows only the price process (i.e. the agent does not know the distribution of default time τ) and by C_t^{in} - the price of Y calculated by the agent who knows the price process as well as observes the default when it happens (i.e. the agent knows the distribution of default time τ). Let us study the following proposition.

Proposition 7.9. *The difference between the prices calculated by these two agents is*

$$C_t^{in} - C_t^{un} = \mathbb{E}_{\mathbb{Q}}(Y\mathbb{I}_{\{\tau>T\}}|\mathcal{F}_t)\Big(\frac{1}{\mathbb{E}_{\mathbb{Q}}(\mathbb{I}_{\{\tau>T\}}|\mathcal{F}_t)} - 1\Big).$$

Proof. The informed agent knows the price process and the default distribution. It means that at time t he has the information $\mathcal{G}_t$. We can write

$$C_t^{in} = \mathbb{E}_{\mathbb{Q}}(Y\mathbb{I}_{\{\tau>T\}}|\mathcal{G}_t.$$

Further, from the Proposition 7.3 we have

$$C_t^{in} = \mathbb{I}_{\{\tau>t\}}\frac{\mathbb{E}_{\mathbb{Q}}(Y\mathbb{I}_{\{\tau>T\}}|\mathcal{F}_t)}{\mathbb{E}_{\mathbb{Q}}(\mathbb{I}_{\{\tau>t\}}|\mathcal{F}_t)}.$$

At time t the uninformed agent has only the information $\mathcal{F}_t$. Hence,

$$C_t^{un} = \mathbb{E}_{\mathbb{Q}}(Y\mathbb{I}_{\{\tau>T\}}|\mathcal{F}_t).$$

Finally,

$$C_t^{in} - C_t^{un} = \mathbb{E}_{\mathbb{Q}}(Y\mathbb{I}_{\{\tau>T\}}|\mathcal{F}_t)\Big(\frac{1}{\mathbb{E}_{\mathbb{Q}}(\mathbb{I}_{\{\tau>T\}}|\mathcal{F}_t)} - 1\Big).$$

□

7.2 The density approach

Let us now present alternative approach for the default modelling, namely the density approach. It was proved in [3] that the $\mathbb{G}$-intensity (see 7.1.3) can be completely derived from the conditional density process $q(u)$. However, given the $\mathbb{G}$-intensity, we can only obtain the knowledge of $q_t(u)$ for $u \geq t$. As a result, the intensity-based approach is not suitable for after the default case.

In the following subsection we study the projections tools. The lemmas below allow us to express $\sigma(\tau)$- and a $\mathcal{G}_t^\tau$-measurable random variable in terms of a $\mathcal{G}_t$-measurable random variable, i.e. project the results obtained in $\mathbb{G}^\tau$ and the filtration generated by $\sigma(\tau)$ on the filtration $\mathbb{G}$.

7.2.1 Projection tools

We begin with a $\mathbb{G}$-projection of a $\sigma(\tau)$-measurable random variable.

Lemma 7.1. *[2] Let $V = h(\tau)$ be $\sigma(\tau)$-measurable and $\mathbb{P}$-integrable random variable. Then, for $s \leq t$,*

$$\mathbb{E}_{\mathbb{P}}(V|\mathcal{G}_s) = \mathbb{E}_{\mathbb{P}}(h(\tau)|\mathcal{G}_s) = \tilde{y}_s \mathbb{I}_{\{\tau > s\}} + h(\tau)\mathbb{I}_{\{\tau \leq s\}},$$

with

$$\tilde{y}_s = \frac{1}{R_s} \int_s^\infty v(u) q_t(u) \eta(du),$$

where $\tilde{y}_s$ is an $\mathcal{F}_s$-measurable random variable and h - a Borel function on $\mathbb{R}^+$.

See [2] for the proof.

In the lemma below we establish analogous results for a $\mathcal{G}_t^\tau$-measurable random variable.

Lemma 7.2. *[2] Let $Z_t = z_t(\tau)$ be a $\mathcal{G}_t^\tau$-measurable and $\mathbb{P}$-integrable random variable. Then, for $s \leq t$,*

$$\mathbb{E}_{\mathbb{P}}(Z_t^\tau|\mathcal{G}_s) = \mathbb{E}_{\mathbb{P}}(z_t(\tau)|\mathcal{G}_s) = \tilde{y}_s \mathbb{I}_{\{\tau > s\}} + \hat{z}_s(\tau)\mathbb{I}_{\{\tau \leq s\}},$$

with

$$\tilde{y}_s = \frac{1}{R_s}\mathbb{E}\bigg(\int_s^\infty z_t(u) q_t(u) \eta(du)|\mathcal{F}_s\bigg),$$

$$\hat{z}_s(u) = \frac{1}{q_s(u)} \mathbb{E}_{\mathbb{P}}(z_t(u) q_t(u)|\mathcal{F}_s).$$

For the proof see [2].

As an application, let us consider the following lemma which by projecting the martingale L defined earlier in section 5.5 gives us a Radon-Nikodým density on $\mathbb{G}$.

Lemma 7.3. *[3], [2] Let us assume that $\mathbb{P}^*$ is equivalent to $\mathbb{P}$ on $\mathbb{G}$. Then, there exists a process $l = (l_t)_{t\geq 0}$ such that*

$$d\mathbb{P}_{|G_t} = l_t d\mathbb{P}^*{}_{|G_t},$$

i.e. $\forall t \geq 0$ l_t defines the corresponding Radon-Nikodým density on $\mathcal{G}_t$. Moreover,

$$l_t = \mathbb{E}_{\mathbb{P}}(L_t|\mathcal{G}_t) = \mathbb{I}_{\{\tau>t\}}\frac{R_t}{R_t^*} + \mathbb{I}_{\{\tau\leq t\}}\frac{1}{q_t(\tau)},$$

where L was defined earlier as $(L_t)_{t\geq 0}$ with

$$L_t = \frac{1}{q_t(\tau)}.$$

For the proof, see [3].

7.2.2 The $\mathcal{H}$-hypothesis and special property of the conditional density process

Let us now study the relation between the $\mathcal{H}$-hypothesis introduced in the Proposition 7.8 and the conditional density process $q(u)$ with the special property shown in the Remark 5.1, namely

$$q_t(u) = q_u(u), \quad \forall t \geq u \geq 0 \quad d\mathbb{P} - a.s.$$

One can consider the following proposition.

Proposition 7.10. *[3] We recall the $\mathcal{H}$-hypothesis, in the form of $(\mathcal{H}2)$, which can be stated as: for any fixed t and any bounded $\mathcal{G}_t$-measurable random variable X_t,*

$$\mathbb{E}_{\mathbb{P}}(X_t|\mathcal{F}_\infty) = \mathbb{E}_{\mathbb{P}}(X_t|\mathcal{F}_t) \quad \mathbb{P} - a.s.$$

Then, the $\mathcal{H}$-hypothesis is fulfilled if and only if

$$q_t(u) = q_u(u), \quad \forall t \geq u \geq 0 \quad d\mathbb{P} - a.s.$$

One can find the proof in [3].

Remark 7.1. In the subsection 7.1.4 we established that the $\mathcal{H}$-hypothesis satisfied under a risk-neutral measure is a suitable condition for the absence of arbitrage. If we combine this result with the proposition above, we can state that to provide the arbitrage-free market, it is sufficient to introduce the Hypothesis 5.1 and assume that the new information does not change the conditional distribution of τ.

7.2.3 The martingales characterization

In this subsection we give some results concerning the characterization of $(\mathbb{P}, \mathbb{G})$-martingales in terms of $(\mathbb{P}, \mathbb{F})$-martingales.

Proposition 7.11. *[2] A $\mathbb{G}$-adapted process $X = (X_t)_{t\geq 0}$, given by $X_t := \tilde{x}_t \mathbb{I}_{\{\tau > t\}} + \hat{x}_t(\tau)\mathbb{I}_{\{\tau \leq t\}}$, is a $(\mathbb{P}, \mathbb{G})$-martingale if and only if the following conditions are satisfied*

i) the process $(\hat{x}(u)q_t(u))_{t\geq u}$ is a $(\mathbb{P}, \mathbb{G})$-martingale for η-a.e. $u \geq 0$;

ii) the process $m = (m_t)_{t\geq 0}$ with

$$m_t := \mathbb{E}_{\mathbb{P}}(X_t|\mathcal{F}_t) = \tilde{x}_t Z_t + \int_0^t \hat{z}_t(u) q_t(u) \eta(du),$$

is a $(\mathbb{P}, \mathbb{F})$-martingale.

For the proof, see [2].

Chapter 8

Pricing and hedging of Black-Scholes type models with default

8.1 The model evaluation and the description of the task

We consider two companies which may be related to each other. The default event is triggered by the second company while the first company is default-free with respect to that default. However, it does not necessarily mean that the first company is default-free in general. We assume that the regular investor observes only the stock prices of the default-free company (1) but he wants to price a European call option written on the investment consisting of

- a stock of the company (1),
- a defaultable corporate bond issued by the company (2) (see Figure 8.1).

One may interpret this situation in the following way. The issuer of the option knows that the companies may be correlated. Thus, he adds to the stock of the default-free company, a defaultable corporate bond issued by the defaultable company as an additional gain opportunity.

Basic assumptions

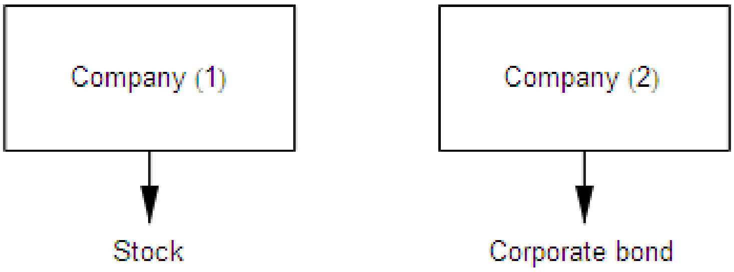

Figure 8.1: The task model.

We fix T as a maturity time for the option and from now on we consider all the price processes and filtrations up to moment T. Let $(\Omega, \mathcal{G}, \mathbb{P})$ be a probability space on which we define two-dimensional standard Brownian motion $W = (W_t^{(1)}, W_t^{(2)})_{t\in[0,T]}$. We endow $(\Omega, \mathcal{G}, \mathbb{P})$ with a filtration $\mathbb{F}^{(1)} = (\mathcal{F}_t^{(1)})_{0\le t\le T}$ generated by $W^{(1)}$, i.e. $\forall\, t \in [0,T] \quad \mathcal{F}_t^{(1)} = \sigma(W_s^{(1)}, 0 \le s \le t)$.

The default-free market

We consider a default-free Black-Scholes market $(B, S^{(1)})$ consisting of one riskless asset $B = (B_t)_{t\in[0,T]}$ and one risky asset $S^{(1)} = (S_t^{(1)})_{t\in[0,T]}$. Their prices follow the random walk with the dynamics

$$dB_t = rB_t dt, \quad t \in [0,T], \quad B_0 = 1,$$
$$dS_t^{(1)} = S_t^{(1)}(\mu_{(1)} dt + \sigma_{(1)} dW_t^{(1)}), \quad t \in [0,T], \tag{8.1}$$

where r, $\mu_{(1)}$, $\sigma_{(1)}$ are real constants, $\sigma_{(1)} > 0$. For the simplicity we put $r = 0$.

The defaultable market

Furthermore, we can establish a defaultable market $(B,S^{(1)},S^{(2)})$ by adding to the default-free market one defaultable asset $S^{(2)} = (S_t^{(2)})_{t\in[0,T]}$ which follows the random walk with the dynamics

$$dS_t^{(2)} = S_t^{(2)}(\mu_{(2)} dt + \sigma_{(2)} dW_t^{(2)}), \quad t \in [0,T],$$

where $\mu_{(2)}$, $\sigma_{(2)}$ are real constants, $\sigma_{(2)} > 0$. The processes $S^{(1)}$ and $S^{(2)}$ represent the stock price processes of respectively company (1) and (2). Additionally, we assume that there is a defaultable bond traded in the market. The bond consists of a payment of one monetary unit at time T if and only if default has not occurred before time T, i.e. the payment is $\mathbb{I}_{\{\tau>T\}}$.

The default time

Finally, we define default time τ as the first time when the stock price process $S^{(2)}$ hits a barrier a, i.e.

$$\tau = \inf\{t \in [0, T] : S_t^{(2)} \leq a\}, \tag{8.2}$$

where $0 < a < 1$. Since we determined all the price processes up to the

Table 8.1: Definition of default time τ

$\{t \in [0,T] : S_t^{(2)} \leq a\} \neq \emptyset$	$\{t \in [0,T] : S_t^{(2)} \leq a\} = \emptyset$
$\downarrow$	$\downarrow$
$\tau = \inf\{t \in [0,T] : S_t^{(2)} \leq a\}$	$\tau = T$

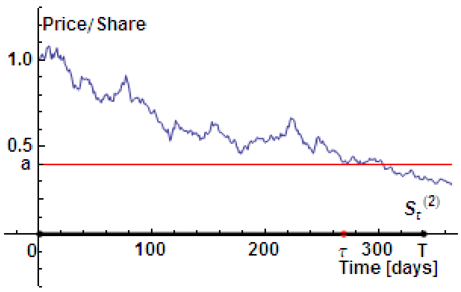

Figure 8.2: Default time occurs before time T

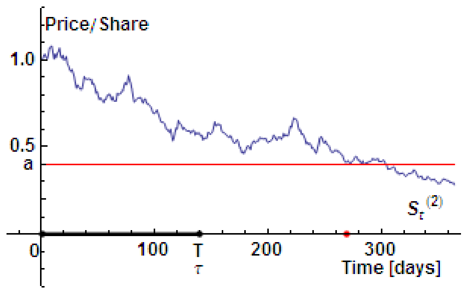

Figure 8.3: Default time occurs after time T

maturity time T we have to take under consideration the fact that the default may not occur before time T. If this is the case, then the set in (8.2) is empty and inf of such a set is equal to ∞. To avoid this situation, we put τ equal to T instead. Nevertheless, if the barrier a was crossed at least once by the process $S^{(2)}$ in the time interval $[0, T]$, then τ is min of all time moments for which it occurred. As a result, we have to consider a random variable of the form $\tau \wedge T$.

The insider

Let us now present a special type of an investor trading in such a defaultable market, we call this investor an insider of the company (2). The insider observes the prices of the stock (1) and has access to some additional information concerning default time τ of the company (2). In our case this additional information consists of the distribution of τ. Moreover, the insider has it from the beginning, i.e. the filtration $\mathbb{F}^{(1)}$ is enlarged with the default time τ in an initial manner (see chapter 5). Consequently, we make precise that the information available to the insider at time t is represented by σ-algebra $\mathcal{G}_t^\tau = \mathcal{F}_t \vee \sigma(\tau)$. We assume that the regular investor who wants to price the option knows that there exists an insider in the market.

The wealth process

Let us define X^ϕ, where $X^\phi = (X_t^\phi)_{t\in[0,T]}$, as a wealth process obtained by the regular investor using a self-financing strategy ϕ, where $\phi = (\phi_t)_{t\in[0,T]}$ with $\phi_t = (\phi_t^S, \phi_t^B)$ - an $\mathbb{F}^{(1)}$-predictable strategy. We remind that the self-financing property means that no money is withdrawn or added to the portfolio. More precisely, $(\phi_t^S)_{t\in[0,T]}$ indicates the financial position of the investor in the stocks of the company (1) and $(\phi_t^B)_{t\in[0,T]}$ describes the position in riskless bonds. Specifically, if we denote by

$$\pi = (\pi_t)_{t\in[0,T]}$$

ratio of wealth from shares of the company (1) and the whole wealth X^ϕ then

$$1 - \pi = (1 - \pi_t)_{t\in[0,T]}$$

is the ratio of wealth from bonds and the whole wealth X^ϕ. We can write

$$\phi_u^S S_u = X_u^\phi \pi_u$$

and

$$\phi_u^B B_u = X_u^\phi (1 - \pi_u).$$

Therefore, the wealth at time t is defined as

$$X_t^\phi = x + \int_0^t \phi_u^S dS_u^{(1)} + \int_0^t \phi_u^B dB_u,$$

where x is the initial capital. Furthermore, by (8.1) we obtain

$$X_t^\phi = x + \int_0^t \Big(\mu_{(1)}\pi_u + r(1-\pi_u)\Big) X_u du + \int_0^t \pi_u X_u dW_u^{(1)}.$$

Let us make precise in the end of this chapter that our goal is to price a European call option written on the investment consisting of a default-free and defaultable assets. However, we want to find this price for the regular investor whose information flow is only the filtration generated by the stock price process of the default-free company. Finally, we assume that the regular investor knows that there exists an insider in the market.

8.2 Methods of pricing in arbitrage-free and incomplete market

In this section we begin with explaining why the defaultable market is arbitrage-free and incomplete for the regular investor. Then, we continue with finding a pricing measure via minimizing f-divergence method which is strongly related to the utility approach.

8.2.1 The arbitrage-free market

From the fact that default time τ and the reference filtration are independent under the physical measure $\mathbb{P}$, $\mathbb{P}$ admits the properties of the decoupling measure. Thus, it preserves the martingale properties in the initially enlarged filtration and according to the section 6.3 the market is arbitrage-free. As a result, there exists at least one martingale measure.

8.2.2 The incomplete market

The incompleteness of the market is caused by the influence on the stock prices by an informed investor. The additional information is considered as a strong initial information which we model by the initial enlargement. On the one hand, for an informed investor the influenced market is complete. On the other hand, for a common investor it is incomplete which means that there exists more than one martingale measure. Consequently, one of the most challenging tasks is to choose a martingale measure for pricing financial derivatives.

8.2.3 The f-divergence minimization approach

A common method of pricing derivatives in incomplete market is to base the prices on a martingale measure which minimizes certain distance, namely f-divergence which measures the difference between the probability measures $\mathbb{P}$ and $\mathbb{Q}$.

Definition 8.1. If f is a convex function on $[0,\infty)$, $\mathbb{Q}$ and $\mathbb{P}$ are the probability measures such that $\mathbb{Q} << \mathbb{P}$ and $\frac{d\mathbb{Q}}{d\mathbb{P}}$ is a Radon-Nikodým density of $\mathbb{Q}$ with respect to $\mathbb{P}$, then we call

$$f(\mathbb{Q}|\mathbb{P}) = \mathbb{E}_{\mathbb{P}}\Big(f\Big(\frac{d\mathbb{Q}}{d\mathbb{P}}\Big)\Big)$$

an f-divergence of $\mathbb{Q}$ with respect to $\mathbb{P}$.

As a function f we can take for example $f(x) = (\sqrt{x}-1)^2$, $f(x) = 2(1-\sqrt{x})$ (Hellinger distances) or $f(x) = |x-1|$ (total variation distance). The standard approach is to choose as a pricing measure f-divergence minimal equivalent martingale measure $\mathbb{Q}^*$ such that

$$\mathbb{E}_{\mathbb{P}}\Big(f\Big(\frac{d\mathbb{Q}^*}{d\mathbb{P}}\Big)\Big) = \inf_{\mathbb{Q}\in\mathcal{M}(\mathbb{P})} \mathbb{E}_{\mathbb{P}}\Big(f\Big(\frac{d\mathbb{Q}}{d\mathbb{P}}\Big)\Big),$$

where $\mathcal{M}(\mathbb{P})$ is the set of all martingale measures equivalent to $\mathbb{P}$.

It is crucial in our case that the f-divergence minimization is closely related to the utility maximization via the Legendre transform. Let us introduce now the utility approach in more details.

8.2.4 The utility approach

The utility approach is based on the fact that one has to estimate the value of some (defaultable) contingent claim seen from the perspective of an agent who optimizes his behavior relative to some utility function. The utility function measures the investor's satisfaction. Therefore, in this section we have to introduce briefly some known results concerning the maximizing expected utility theory. Let us begin with the following definition.

Definition 8.2. We define the utility function u as a strictly increasing, strictly concave, twice continuously differentiable function on $dom(u) = \{x \in \mathbb{R}, u(x) > -\infty\}$ which satisfies

$$u'(\infty) = \lim_{x\to\infty} u'(x) = 0,$$

$$u'(\underline{x}) = \lim_{x\to\underline{x}} u'(x) = \infty,$$

where $\underline{x} = \inf\{u \in dom(u)\}$.

It is important to comment the definition above. We require that the utility function is an increasing function of wealth because with the growth of wealth the usefulness which the investor has also grows. The concavity of the function stands for an investor who is risk-averse. The utility function's slope gets flatter as the wealth increases. It means that the first unit of wealth yields more utility (satisfaction) than the second and subsequent units.

The standard utility functions

Let us consider three standard utility functions:

i)
$$u(x) = 1 - e^{-x}, \tag{8.3}$$

ii)
$$u(x) = lnx, \tag{8.4}$$

iii)
$$u(x) = \frac{x^p}{p}, p \in (-\infty, 0) \cup (0, 1). \tag{8.5}$$

The maximization of the expected value of the power utility $u(x) = \frac{x^p}{p}, p \in (-\infty, 0) \cup (0, 1)$ is equivalent to the maximization of the expected rate of return compounded $\frac{1}{pT}$ times per year:

$$\frac{1}{pT}\mathbb{E}\Big(\Big(\frac{X_T}{x}\Big)^p - 1\Big).$$

The values $p < 0$ correspond to the discount rate. With the increase in p the investor's risk tolerance also increases.

The case of the logarithmic utility function $u(x) = lnx$ we consider as a limiting case of a power utility function as $p \to 0$. The application is in the maximization of the expected continuously compounded growth rate:

$$\frac{1}{T}\mathbb{E}\Big(\ln\Big(\frac{X_T}{x}\Big)\Big).$$

The exponential utility function $u(x) = 1 - e^{-x}$ corresponds to the entropic measure.

Reformulating the problem

Our task now is to formulate the problem in terms of the expected utility theory. We begin with reminding that the wealth at time t obtained using strategy ϕ is defined as

$$X_t^\phi = X_0^\phi + \int_0^t \phi_u^S dS_u^{(1)} + \int_0^t \phi_u^B dB_u,$$

where X_0^ϕ is the initial capital and $\phi = (\phi_t)_{0\le t\le T}$ with $\phi_t = (\phi_t^S, \phi_t^B)$ is the self-financing strategy (see section 8.1). Let us assume for the simplicity that the risk-free interest rate $r = 0$. It means that

$$X_t^\phi = X_0^\phi + \int_0^t \phi_u^S dS_u^{(1)}.$$

According to [5], to avoid phenomena like doubling strategies (doubling the position), we make an assumption that during the trading the losses do not exceed a finite credit line, i.e.

$$\exists b > 0 \quad \text{such that} \quad \forall t \in [0, T] \quad \int_0^t \phi_u^S dS_u^{(1)} \ge -b.$$

We say that such a strategy ϕ is admissible. The preferences of the investor are represented by the utility functions described above. The resulting optimization problem is of the form

$$\sup_{\phi\in\mathcal{A}} \mathbb{E}_{\mathbb{P}}(u(X_T^\phi)) = \sup_{\phi\in\mathcal{A}} \mathbb{E}_{\mathbb{P}}(u(X_0^\phi + \int_0^T \phi_u^S dS_u^{(1)})) = \mathbb{E}_{\mathbb{P}}(u(X_0^\phi + \int_0^T \phi_u^{*S} dS_u^{(1)})),$$

where $\mathcal{A}$ is a set of admissible strategies.

The Legendre transformation and the dual approach

Let us now explain briefly the relationship between the utility maximization and the f-divergence minimization. We start with the definition of the Legendre transformation.

Definition 8.3. If $u : \mathbb{R} \to \mathbb{R}$ is twice continuously differentiable and $\forall x \in \mathbb{R}$ $u''(x) < 0$ (u is concave), then we call

$$\hat{u}(x) = u(I(y)) - yI(y)$$

a Legendre transform of u, where $I = (u')^{-1}$.

The function $\hat{u}$ which we obtain in this case is convex. However, u is also a Legendre transform of $\hat{u}$ and

$$u(x) = \inf_{y\in\mathbb{R}}\{\hat{u}(y) + xy\} = \hat{u}(I(x)) + xI(x).$$

This property is called duality. Here we give the form of the convex duals of the standard utility functions given by (8.3), (8.4) and (8.5).

i)

$$u(x) = 1 - e^{-x} \quad \rightarrow \quad \hat{u}(x) = 1 - x + xlnx \tag{8.6}$$

ii)

$$u(x) = lnx \quad \rightarrow \quad \hat{u}(x) = -lnx - 1 \tag{8.7}$$

iii)

$$u(x) = \frac{x^p}{p}, p \in (-\infty, 0) \cup (0, 1) \quad \rightarrow \quad \hat{u}(x) = -\frac{p-1}{p}x^{\frac{p}{p-1}}. \tag{8.8}$$

Via the Legendre transformation one can obtain the equivalent problem in the following form

$$\sup_{\phi\in\mathcal{A}} \mathbb{E}_{\mathbb{P}}(u(X_T^\phi)) = \inf_{y>0}\{X_0^\phi y + \mathbb{E}_{\mathbb{P}}(\hat{u}(y\frac{d\mathbb{Q}_T^*}{d\mathbb{P}_T}))\}, \tag{8.9}$$

where $\mathbb{Q}^*$ is $\hat{u}$-minimal equivalent martingale measure. As a result we can base the price of the option in our task on the $\hat{u}$-minimal equivalent martingale measure. The problem now is to find $\mathbb{Q}^*$ such that

$$\mathbb{E}_{\mathbb{P}}\Big(\hat{u}\Big(\frac{d\mathbb{Q}^*}{d\mathbb{P}}\Big)\Big) = \inf_{\mathbb{Q}\in\mathcal{M}(\mathbb{P})} \mathbb{E}_{\mathbb{P}}\Big(\hat{u}\Big(\frac{d\mathbb{Q}}{d\mathbb{P}}\Big)\Big).$$

8.3 Martingale measures on $\mathbb{G}^\tau$

Let us denote the set of martingale measures equivalent to $\mathbb{P}$ on $\mathbb{F}^{(1)}$ as $\mathcal{M}_{\mathbb{F}^{(1)}}(\mathbb{P})$ and the set of martingale measures equivalent to $\mathbb{P}$ on $\mathbb{G}^\tau$ as $\mathcal{M}_{\mathbb{G}^\tau}(\mathbb{P})$. Our goal now is to choose one measure from the set $\mathcal{M}_{\mathbb{G}^\tau}(\mathbb{P})$ as a pricing measure. We remind firstly that $\mathbb{Q}$ equivalent to $\mathbb{P}$ is a martingale measure on $\mathbb{G}^\tau$ when the discounted price process $\left(S_t^{(1)}\right)_{t\in[0,T]}$ is a $(\mathbb{Q}, \mathbb{G}^\tau)$-martingale. We have to consider prices as $(\mathbb{Q}, \mathbb{G}^\tau)$-martingales since the regular investor knows that in the market there is also an insider who influences the prices. For the simplicity we assumed that the risk-free interest rate r is equal to 0 and then $\forall t \in [0, T]$ $B_t = 1$.

The ordinary investor with the public information flow $\mathbb{F}^{(1)}$ does not have the arbitrage opportunities since the default-free market in our case is arbitrage-free. In addition, it is complete. This means that there exists a uniquely defined martingale measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ such that the discounted price process $\left(S_t^{(1)}\right)_{t\in[0,T]}$ is a $(\mathbb{Q},\mathbb{F}^{(1)})$-martingale, i.e. $\mathcal{M}_{\mathbb{F}^{(1)}}(\mathbb{P})=\{\mathbb{Q}\}$.

Let us denote $p=(p_t)_{t\in[0,T]}$ as a corresponding density process, i.e.

$$d\mathbb{Q}_{|\mathcal{F}_t^{(1)}}=p_t d\mathbb{P}_{|\mathcal{F}_t^{(1)}},\text{ i.e. }\forall A\in\mathcal{F}_t^{(1)}\quad \mathbb{Q}(A)=\int_A p_t d\mathbb{P}. \tag{8.10}$$

It is well known that for a Black-Scholes market $(B,S^{(1)})$ and the filtration $\mathbb{F}^{(1)}$ (see section 8.1 for the definition) the density process p is defined as such that

$$p_t=\exp\Big\{-\frac{\theta^2 t}{2}-\theta W_t^{(1)}\Big\},\text{ where }\theta=\frac{\mu_{(1)}-r}{\sigma_{(1)}}. \tag{8.11}$$

In the end of the previous chapter we established that to find a pricing measure it is necessary to find a density process such that

$$\mathbb{E}_{\mathbb{P}}\Big(\hat{u}\Big(\frac{d\mathbb{Q}^*}{d\mathbb{P}}\Big)\Big)=\inf_{\mathbb{Q}\in\mathcal{M}(\mathbb{P})}\mathbb{E}_{\mathbb{P}}\Big(\hat{u}\Big(\frac{d\mathbb{Q}}{d\mathbb{P}}\Big)\Big),$$

where $\frac{d\mathbb{Q}}{d\mathbb{P}}$ has simply the following form

$$d\mathbb{Q}_{|\mathcal{G}_T^\tau}=P_T(\tau)d\mathbb{P}_{|\mathcal{G}_T^\tau}.$$

However, let us remind that $P_T(\tau)$ has to be a positive random variable with $\mathbb{E}_{\mathbb{P}}(P_T(\tau))=1$.

We start with bounding $\mathbb{E}_{\mathbb{P}}\Big(\hat{u}\Big(\frac{d\mathbb{Q}}{d\mathbb{P}}\Big)\Big)$ from below. Firstly we need to condition $\mathbb{E}_{\mathbb{P}}\Big(\hat{u}\Big(P_T(\tau)\Big)\Big)$ on $\mathcal{F}_T^{(1)}$ to obtain

$$\mathbb{E}_{\mathbb{P}}\Big(\mathbb{E}_{\mathbb{P}}\Big(\hat{u}\Big(P_T(\tau)\Big)\Big)|\mathcal{F}_T^{(1)}\Big)$$

which from the tower property is equal to

$$\mathbb{E}_{\mathbb{P}}\Big(\mathbb{E}_{\mathbb{P}}\Big(\hat{u}\Big(P_T(\tau)\Big)|\mathcal{F}_T^{(1)}\Big)\Big).$$

Using Jensen inequality we get that

$$\mathbb{E}_{\mathbb{P}}\Big(\mathbb{E}_{\mathbb{P}}\Big(\hat{u}\Big(P_T(\tau)|\mathcal{F}_T^{(1)}\Big)\Big)\Big)\geq\mathbb{E}_{\mathbb{P}}\Big(\hat{u}\Big(\mathbb{E}_{\mathbb{P}}\Big(P_T(\tau)|\mathcal{F}_T^{(1)}\Big)\Big)\Big).$$

Let us now specify the form of $\mathbb{E}_{\mathbb{P}}\Big(P_T(\tau)|\mathcal{F}_T^{(1)}\Big)$. We can rewrite it as an integral

$$\mathbb{E}_{\mathbb{P}}\Big(P_T(\tau)|\mathcal{F}_T^{(1)}\Big)(\omega) = \int_0^{\infty} P_T(\omega, u)q_T(u)\eta(du) = \tilde{P}_T(\omega).$$

We omit ω for the simplicity.

It is necessary to check the properties of $\tilde{P}_T$. From the fact that $P_T(\tau)$ is a positive random variable, the conditional density process $q(u)$ and the measure η are strictly positive, we obtain that $\tilde{P}_T \geq 0$. We need to calculate also the expected value of $\tilde{P}_T$. We begin with rewriting it using the definition of $\tilde{P}_T$ and the tower property. We obtain that

$$\mathbb{E}_{\mathbb{P}}\tilde{P}_T = \mathbb{E}_{\mathbb{P}}(\mathbb{E}_{\mathbb{P}}(P_T(\tau)|\mathcal{F}_T^{(1)})) = \mathbb{E}_{\mathbb{P}}(\mathbb{E}_{\mathbb{P}}(P_T(\tau))|\mathcal{F}_T^{(1)}).$$

Since $P_T(\tau)$ is a density process and $\mathbb{E}_{\mathbb{P}}(P_T(\tau)) = 1$, we have that

$$\mathbb{E}_{\mathbb{P}}(\mathbb{E}_{\mathbb{P}}(P_T(\tau))|\mathcal{F}_T^{(1)}) = \mathbb{E}_{\mathbb{P}}(1|\mathcal{F}_T^{(1)}) = 1$$

and hence

$$\mathbb{E}_{\mathbb{P}}\tilde{P}_T = 1.$$

What is more this process is lying on the filtration $\mathbb{F}^{(1)}$. As a result, we get a candidate for a density process, which is $\tilde{P}_T$ such that

$$\mathbb{E}_{\mathbb{P}}\Big(\hat{u}\Big(P_T(\tau)\Big)\Big) \geq \mathbb{E}_{\mathbb{P}}\Big(\hat{u}\Big(\tilde{P}_T\Big)\Big).$$

However, to use a measure defined by this density process, we need to prove that $(S_t^{(1)}\tilde{P}_t)_{t\geq 0}$ is a $(\mathbb{P}, \mathbb{G}^{\tau})$-martingale.

To prove that $(S_t^{(1)}\tilde{P}_t)_{t\geq 0}$ is a $(\mathbb{P}, \mathbb{G}^{\tau})$-martingale, we can use the fact that $\forall u \geq 0$ $(P_t(u))_{t\geq 0}$ is a density process lying on the filtration $\mathbb{G}^{\tau}$ and hence $(S_t^{(1)}P_t(u))_{t\geq 0}$ is a $(\mathbb{P}, \mathbb{G}^{\tau})$-martingale.

Proposition 6.1 defines martingales in $\mathbb{G}^{\tau}$. It states that $\mathbb{G}^{\tau}$-adapted process $(z_t(u))_{t\geq 0}, u \geq 0$ is a $(\mathbb{P}, \mathbb{G}^{\tau})$-martingale if and only if the process $(z_t(u)q_t(u))_{t\geq 0}$ is a $(\mathbb{P}, \mathbb{F}^{(1)})$-martingale, where $(q_t(u))_{t\geq 0}$ is the conditional density process (see Proposition 5.1). In our case random variable $z_t(u)$ is of the form $S_t^{(1)}P_t(u)$ and it clearly is measurable with respect to σ-algebra $\mathcal{G}_t^{\tau}$.

As a result we obtain that since $(S_t^{(1)}P_t(u))_{t\geq 0}$ is a $(\mathbb{P},\mathbb{G}^\tau)$-martingale then, $(S_t^{(1)}P_t(u)q_t(u))_{t\geq 0}$ is a $(\mathbb{P},\mathbb{F}^{(1)})$-martingale. It means that

$$\mathbb{E}_{\mathbb{P}}(S_t^{(1)}P_t(u)q_t(u)) = S_0^{(1)}.$$

We put conditional expectation on both sides

$$\mathbb{E}_{\mathbb{P}}(\mathbb{E}_{\mathbb{P}}(S_t^{(1)}P_t(u)q_t(u))) = \mathbb{E}_{\mathbb{P}}(S_0^{(1)})$$

and get that

$$\mathbb{E}_{\mathbb{P}}\Big(\int_0^\infty S_t^{(1)}P_t(u)q_t(u)\eta(du)\Big) = S_0^{(1)}.$$

We can take $S_t^{(1)}$ outside the integral and obtain

$$\mathbb{E}_{\mathbb{P}}\Big(S_t^{(1)}\int_0^\infty P_t(u)q_t(u)\eta(du)\Big) = S_0^{(1)}.$$

Since $\int_0^\infty P_t(u)q_t(u)\eta(du)$ is the definition of $\tilde{P}_t$ we obtain that

$$\mathbb{E}_{\mathbb{P}}(S_t^{(1)}\tilde{P}_t) = S_0^{(1)} = S_0^{(1)}\tilde{P}_0.$$

As a result, we proved that $(S_t^{(1)}\tilde{P}_t)_{t\geq 0}$ is a $(\mathbb{P},\mathbb{G}^\tau)$-martingale and we can take $(\tilde{P}_t)_{t\geq 0}$ as a density process. However, it is crucial to notice that the process $(\tilde{P}_t)_{t\geq 0}$ is lying on the filtration $\mathbb{F}^{(1)}$ and since the density process is defined uniquely on the filtration $\mathbb{F}^{(1)}$, the process $(\tilde{P}_t)_{t\geq 0}$ has to coincide with the process $(p_t)_{t\geq 0}$. Finally we showed that minimum is attained at $(p_t)_{t\geq 0}$ and we take as a pricing measure, the measure $\mathbb{Q}^*$ such that

$$d\mathbb{Q}^*_{|\mathcal{G}_T^\tau} = p_T d\mathbb{P}_{|\mathcal{G}_T^\tau}.$$

8.4 The distribution of τ with respect to $\mathbb{P}$

In this section we study the distribution of default time. Firstly, for simplification we consider the well known results for a hitting time defined as the first time when the barrier value is crossed by a Brownian motion with drift or the first time when a linear function barrier is crossed by a standard Brownian motion. Then, we apply them to our stopping time τ.

The density for a hitting time

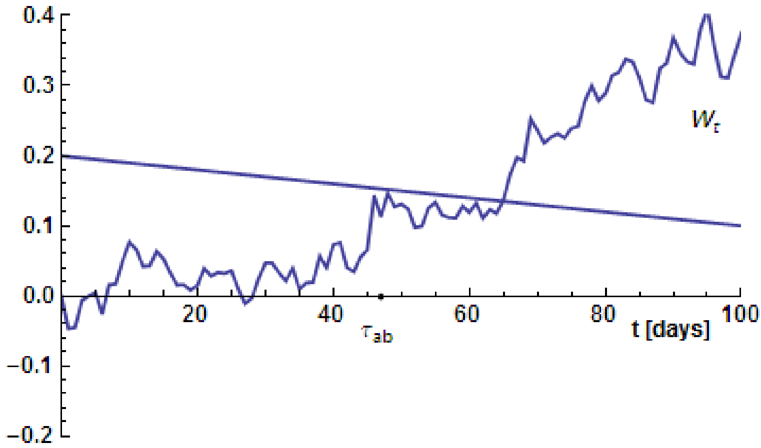

Figure 8.4: A hitting time of a linear function barrier by a standard Brownian motion.

From [7] we know that for a hitting time $\tau_{ab} = \inf\{t \geq 0 : W_t^{(2)} \geq a + bt\}$, where $a > 0$ and $b < 0$ (see Figure 8.4), the density function is of the form

$$f_{\tau_{ab}}(t) = \frac{a}{t}\Psi_t(a + bt), \tag{8.12}$$

where $\Psi_t(u) = \frac{1}{\sqrt{2\pi t}} \exp\{-\frac{u^2}{2t}\}$. To get the distribution function we integrate (8.12) and obtain

$$F_{\tau_{ab}}(t) = \mathbb{P}(\tau_{ab} \leq t) = 1 - \Phi\Big(\frac{a + bt}{\sqrt{t}}\Big) + \exp\{-2ab\}\Phi\Big(\frac{bt - a}{\sqrt{t}}\Big).$$

To check (8.12) we verify that

$$\mathbb{E}_{\mathbb{P}}(\exp\{-\lambda\tau_{ab}\}) = \exp\{-a(b + \sqrt{b^2 + 2\lambda})\}. \tag{8.13}$$

We have

$$\int_{[0,\infty]} \frac{1}{\sqrt{2\pi t}} \frac{a}{t} \exp\{-\frac{(a + bt)^2}{2t}\} \exp\{-\lambda\tau_{ab}\}dt$$

which we can write as

$$\exp\{-ab\} \int_{[0,\infty]} \frac{1}{\sqrt{2\pi t}} \frac{a}{t} \exp\{\frac{a^2}{2t}\} \exp\{-(\lambda + \frac{b^2}{2})t\}dt$$

where

$$\int_{[0,\infty]} \frac{1}{\sqrt{2\pi t}} \frac{a}{t} \exp\{\frac{a^2}{2t}\} \exp\{-(\lambda + \frac{b^2}{2})t\}dt = \mathbb{E}_{\mathbb{P}}(\exp\{-(\lambda + \frac{b^2}{2})\tau_a\})$$

with

$$\tau_a = \inf\{t \geq 0 : W_t \leq a\}.$$

From the fact that

$$\mathbb{E}_{\mathbb{P}}(\exp\{-\Lambda\tau_a\}) = \exp\{-a\sqrt{2\Lambda}\}$$

we have

$$\mathbb{E}_{\mathbb{P}}(\exp\{-(\lambda + \frac{b^2}{2})\tau_a\}) = \exp\{-a\sqrt{2(\lambda + \frac{b^2}{2})}\}. \tag{8.14}$$

After multiplying the right-hand side of (8.14) by $\exp\{-ab\}$ we get (8.13) which proves that our formula for the density (8.12) is correct.

The density of default time τ

In our case

$$\tau = \inf\{t \geq 0 : S_t^{(2)} \leq a\}$$

which is equivalent to

$$\tau = \inf\left\{t \geq 0 : W_t^{(2)} \leq \frac{lna}{\sigma_{(2)}} + \Big(\frac{\sigma_{(2)}}{2} - \frac{\mu_{(2)}}{\sigma_{(2)}}\Big)t\right\}.$$

Let us denote

$$a_1 = \frac{lna}{\sigma_{(2)}}$$

and

$$b_1 = \frac{\sigma_{(2)}}{2} - \frac{\mu_{(2)}}{\sigma_{(2)}}.$$

We see that $a_1 < 0$ and we assume that $b_1 > 0$. Therefore, we can not apply the result established in [7] directly. Nevertheless, we can use a Brownian motion's reflection property and calculate the density function for a hitting time of a reflected barrier by a reflected standard Brownian motion.

We have that

$$\tau = \inf\{t \geq 0 : -W_t^{(2)} \geq -a_1 + (-b_1)t\}.$$

We denote $a_2 = -a_1 > 0$ and $b_2 = -b_1 < 0$ and get (see Figure 8.5)

$$\tau = \inf\{t \geq 0 : -W_t^{(2)} \geq a_2 + b_2 t\}.$$

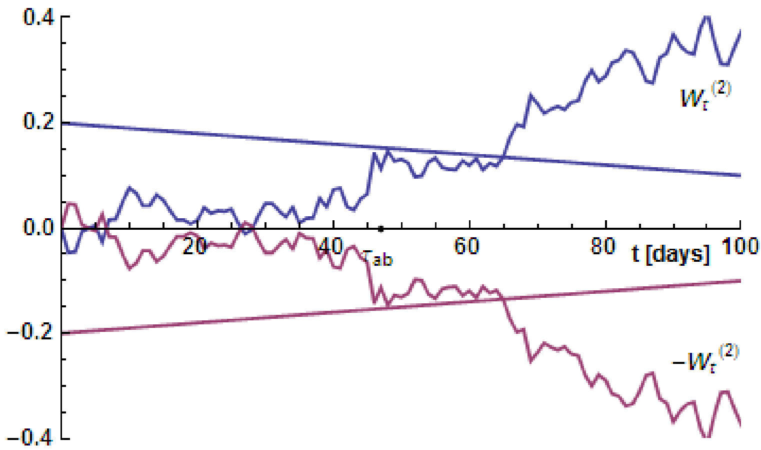

Figure 8.5: A hitting time of a linear function barrier by a standard Brownian motion - dual problems.

Finally, we obtain the form of the density

$$f_\tau(t) = -\frac{\ln a}{\sigma_{(2)}}\Psi_t(-\frac{\ln a}{\sigma_{(2)}} - (\frac{\sigma_{(2)}}{2} - \frac{\mu_{(2)}}{\sigma_{(2)}})t). \tag{8.15}$$

We get the distribution function $F_\tau(t) = \mathbb{P}(\tau \leq t)$ as

$$\mathbb{P}(\tau \leq t) = \Phi\Big(\frac{\ln a - (\mu_{(2)} - \frac{\sigma^2_{(2)}}{2})t}{\sigma_{(2)}\sqrt{t}}\Big) \tag{8.16}$$

$$+a^{2\frac{\mu_{(2)}}{\sigma^2_{(2)}}-1}\Phi\Big(\frac{\ln a + (\mu_{(2)} - \frac{\sigma_{(2)}}{2})t}{\sigma_{(2)}\sqrt{t}}\Big).$$

From the fact that we have to consider the random variable $\tau \wedge T$ it is crucial to determine its distribution. In general, it has the form as follows

$$\mathbb{P}(\tau \wedge T \leq t) = \begin{cases} F_\tau(t), & \text{if } t < T, \\ 1, & \text{if } t = T, \end{cases}$$

where $F_\tau(t)$ is the distribution of τ which has the density with respect to the Lebesgue measure given by (8.15).

Thus in our case,

$$\mathbb{P}(\tau \wedge T \leq t) = \begin{cases} \Phi\Big(\frac{\ln a-(\mu_{(2)}-\frac{\sigma^2_{(2)}}{2})t}{\sigma_{(2)}\sqrt{t}}\Big) & \\ +a^{2\frac{\mu_{(2)}}{\sigma^2_{(2)}}-1}\Phi\Big(\frac{\ln a+(\mu_{(2)}-\frac{\sigma_{(2)}}{2})t}{\sigma_{(2)}\sqrt{t}}\Big), & \text{if } t < T, \\ 1, & \text{if } t = T, \end{cases}$$

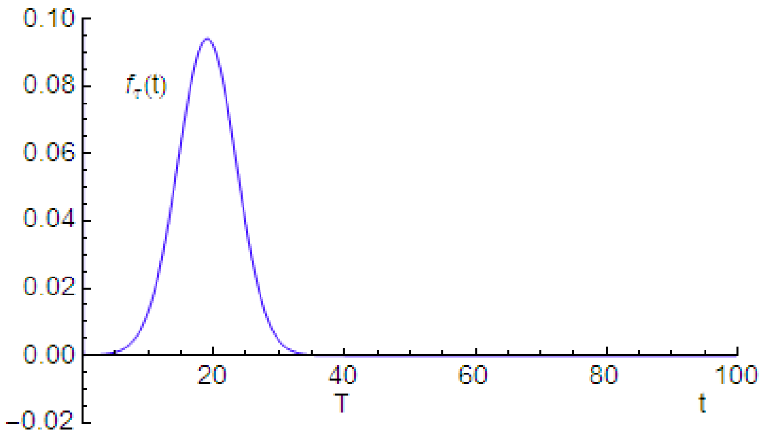

Figure 8.6: The density of τ.

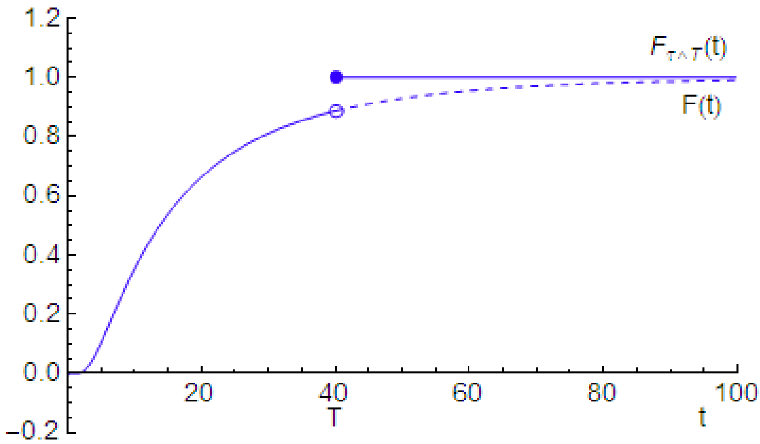

Figure 8.7: The distribution of $\tau \wedge T$.

(see Figure 8.7). We can also calculate

$$\mathbb{P}(\tau > t) = 1 - \Phi\Big(\frac{\ln a - (\mu_{(2)}\frac{\sigma^2_{(2)}}{2})t}{\sigma_{(2)}\sqrt{t}}\Big) \tag{8.17}$$

$$- a^{2\frac{\mu_{(2)}}{\sigma^2_{(2)}}-1}\Phi\Big(\frac{\ln a + (\mu_{(2)} - \frac{\sigma_{(2)}}{2})t}{\sigma_{(2)}\sqrt{t}}\Big). \tag{8.18}$$

8.5 European call option pricing

The goal of this section is to calculate a price of a European call option written on a stock of the company (1) and on a bond issued by a defaultable company (2) which stock price process is described by $S^{(2)}$. Specifically, we consider an option with the following payoff

$$g\Big(S_T^{(1)} + \mathbb{I}_{\{\tau>T\}}\Big) = \Big(S_T^{(1)} + \mathbb{I}_{\{\tau>T\}} - K\Big)^+, \tag{8.19}$$

where T is the maturity time and K is the strike price of the option.

Clearly, the payoff function can be written in the form

$$g_T = \mathbb{I}_{\{S_T^{(1)} + \mathbb{I}_{\{\tau>T\}} - K > 0\}} \Big(S_T^{(1)} + \mathbb{I}_{\{\tau>T\}} - K\Big), \tag{8.20}$$

where

$$\tau = \inf\{t \in [0,T] : S_t^{(2)} \leq a\}.$$

Under the assumption that the $(B, S^{(1)}, S^{(2)})$-market is arbitrage-free we use $\mathbb{Q}^* \in \mathcal{M}_{\mathbb{G}^\tau}(\mathbb{P})$ (see section 8.3) which has the following form

$$d\mathbb{Q}^*_{|\mathcal{G}_T^\tau} = p_T d\mathbb{P}_{|\mathcal{G}_T^\tau}, \tag{8.21}$$

where p_T is a density process such that

$$d\mathbb{Q}_{|\mathcal{F}_T} = p_T d\mathbb{P}_{|\mathcal{F}_T}, \tag{8.22}$$

and $\mathbb{Q} \in \mathcal{M}_{\mathbb{F}^{(1)}}(\mathbb{P})$. See section 8.3 for definitions of $\mathcal{M}_{\mathbb{F}^{(1)}}(\mathbb{P})$ and $\mathcal{M}_{\mathbb{G}^\tau}(\mathbb{P})$. The density process $p = (p_t)_{t\in[0,T]}$ is defined as in (8.11).

8.5.1 Pricing in the Black-Scholes market with default

Under the assumptions of the Black-Scholes model we have the following formula for pricing the financial derivatives.

$$\mathbb{C}_0(T) = B_0 \mathbb{E}_{\mathbb{Q}}\Big(\frac{g_T}{B_T}\Big),$$

where g_T is an $\mathcal{F}_T^{(1)}$-measurable payoff function and $B = (B_t)_{t\in[0,T]}$ is a riskless asset. $\mathbb{C}_0(T)$ denotes the price at time $t = 0$ of a financial derivative with the maturity time T.

In our case the payoff $g_T = g\Big(S_T^{(1)} + \mathbb{I}_{\{\tau>T\}}\Big)$ is $\mathcal{G}_T^\tau$-measurable, where $\forall t \in [0,T]$ $\mathcal{G}_t^\tau = \mathcal{F}_t^{(1)} \vee \sigma(\tau)$. To price a $\mathcal{G}_T^\tau$-measurable payoff g_T we need to use $\mathbb{Q}^* \in \mathcal{M}_{\mathbb{G}^\tau}(\mathbb{P})$ defined in section 8.3. We have

$$\mathbb{C}_0(T) = B_0 \mathbb{E}_{\mathbb{Q}^*}\Big(\frac{g_T}{B_T}\Big).$$

Let us remind that the risk-free interest rate $r = 0$. Hence, $dB_t = 0$ and $\forall t \in [0,T]$ $B_t = 1$. Thus,

$$\mathbb{C}_0(T) = \mathbb{E}_{\mathbb{Q}^*}(g_T).$$

According to (8.19) we get

$$\mathbb{C}_0(T) = \mathbb{E}_{\mathbb{Q}^*}\Big(\Big(S_T^{(1)} + \mathbb{I}_{\{\tau>T\}} - K\Big)^+\Big).$$

By (8.21) we change the measure and get

$$\mathbb{C}_0(T) = \mathbb{E}_{\mathbb{P}}\Big(\Big(S_T^{(1)} + \mathbb{I}_{\{\tau>T\}} - K\Big)^+ p_T\Big).$$

8.5.2 The case when $W^{(1)}$ and $W^{(2)}$ are uncorrelated

In this section we formulate and prove a theorem which gives us an exact formula for pricing the European option with the payoff given by (8.19).

Theorem 8.1. *In the case of uncorrelated Brownian motions the pricing formula for the European option with the payoff given by (8.19) has the following form*

$$\mathbb{C}_0(T) = ((\Phi(\tilde{d}_1) - \tilde{K}\Phi(\tilde{d}_2))\mathbb{P}(\{\tau > T\}) \tag{8.23}$$

$$+ (\Phi(d_1) - K\Phi(d_2))\mathbb{P}(\{\tau \le T\}), \tag{8.24}$$

where $\mathbb{P}(\{\tau \le T\})$, $\mathbb{P}(\{\tau > T\})$ *are defined by (8.16), (8.17) respectively,*

$$\tilde{d}_{1,2} = \frac{\ln\frac{1}{\tilde{K}} \pm \frac{\sigma_{(1)}^2}{2}T}{\sigma_{(1)}\sqrt{T}} \text{ and } d_{1,2} = \frac{\ln\frac{1}{K} \pm \frac{\sigma_{(1)}^2}{2}T}{\sigma_{(1)}\sqrt{T}}.$$

We shall give the proof.

Proof. With accordance to (2.1) we can represent Ω by two disjoint sets as follows.

$$\Omega = \{\omega \in \Omega : \mathbb{I}_{\{\tau(\omega)>T\}} = 1\} \cup \{\omega \in \Omega : \mathbb{I}_{\{\tau(\omega)>T\}} = 0\}. \tag{8.25}$$

We see that

$$\{\omega \in \Omega : \tau(\omega) > T\}^C = \{\omega \in \Omega : \tau(\omega) \le T\}.$$

Consequently, we can write (8.25) in the simplified notations as follows.

$$\Omega = \{\tau > T\} \cup \{\tau \leq T\}$$

and by the Lemma 2.1 we obtain

$$\mathbb{C}_0(T) = \mathbb{E}_{\mathbb{P}}\Big(\Big(S_T^{(1)} + \mathbb{I}_{\{\tau>T\}} - K\Big)^+ p_T \Big| \{\tau > T\}\Big)\mathbb{P}(\{\tau > T\}) +$$

$$\mathbb{E}_{\mathbb{P}}\Big(\Big(S_T^{(1)} + \mathbb{I}_{\{\tau>T\}} - K\Big)^+ p_T \Big| \{\tau \leq T\}\Big)\mathbb{P}(\{\tau \leq T\}).$$

On the set $\{\tau > T\}$ the random variable $\mathbb{I}_{\{\tau>T\}}$ has a value 1. However, on the set $\{\tau \leq T\}$ it vanishes. Thus, we can write the conditional expectations as follows.

$$\mathbb{E}_{\mathbb{P}}\Big(\Big(S_T^{(1)} + \mathbb{I}_{\{\tau>T\}} - K\Big)^+ p_T \Big| \{\tau > T\}\Big) = \mathbb{E}_{\mathbb{P}}\Big(\Big(S_T^{(1)} + 1 - K\Big)^+ p_T\Big) \quad (8.26)$$

$$\mathbb{E}_{\mathbb{P}}\Big(\Big(S_T^{(1)} + \mathbb{I}_{\{\tau>T\}} - K\Big)^+ p_T \Big| \{\tau \leq T\}\Big) = \mathbb{E}_{\mathbb{P}}\Big(\Big(S_T^{(1)} - K\Big)^+ p_T\Big) \quad (8.27)$$

By inserting (8.26) and (8.27) into the pricing formula we get

$$\mathbb{C}_0(T) = \mathbb{E}_{\mathbb{P}}\Big(\Big(S_T^{(1)} + 1 - K\Big)^+ p_T\Big)\mathbb{P}(\{\tau > T\}) +$$

$$+\mathbb{E}_{\mathbb{P}}\Big(\Big(S_T^{(1)} - K\Big)^+ p_T\Big)\mathbb{P}(\{\tau \leq T\}).$$

Let us consider for simplification the expectation values separately. We have that

$$\mathbb{E}_{\mathbb{P}}\Big(\Big(S_T^{(1)} + 1 - K\Big)^+ p_T\Big) = \mathbb{E}_{\mathbb{P}}\Big(\Big(S_T^{(1)} - \tilde{K}\Big)\mathbb{I}_{\{S_T^{(1)} - \tilde{K} > 0\}} p_T\Big),$$

where $\tilde{K} = K - 1$.

We can calculate the expectation value by integration

$$\int_\Omega \Big(S_T^{(1)} - \tilde{K}\Big)\mathbb{I}_{\{S_T^{(1)} - \tilde{K} > 0\}} p_T d\mathbb{P}. \quad (8.28)$$

Firstly, we write the inequality $S_T^{(1)} - \tilde{K} > 0$ in the form

$$W_T^{(1)} \geq \frac{\ln \tilde{K} - \Big(\mu_{(1)} - \frac{\sigma_{(1)}^2}{2}\Big)T}{\sigma_{(1)}}.$$

Let us denote

$$\tilde{d}_1' = \frac{\ln \frac{1}{\tilde{K}} + \Big(\mu_{(1)} - \frac{\sigma_{(1)}^2}{2}\Big)T}{\sigma_{(1)}}. \tag{8.29}$$

Then, we can extend (8.28) to the following form

$$\int_{-\tilde{d}_1'}^{\infty} \frac{1}{\sqrt{2\pi T}} \Big(\exp\Big\{ \Big(\mu_{(1)} - \frac{\sigma_{(1)}^2}{2}\Big)T + \sigma_{(1)} x \Big\} - \tilde{K} \Big)$$

$$\cdot \exp\Big\{ - \frac{\mu_{(1)}}{\sigma_{(1)}} x - \frac{1}{2}\Big(\frac{\mu_{(1)}}{\sigma_{(1)}}\Big)^2 \Big\} \exp\Big\{ - \frac{x^2}{2T} \Big\} dx. \tag{8.30}$$

We can calculate the integral analogically as in the derivation of the Black-Scholes formula by completing the square. We obtain the result

$$\mathbb{E}_{\mathbb{P}}\Big(\Big(S_T^{(1)} - \tilde{K}\Big) \mathbb{I}_{\{S_T^{(1)} - \tilde{K} > 0\}} p_T \Big) = \Phi(\tilde{d}_1) - \tilde{K}\Phi(\tilde{d}_2), \tag{8.31}$$

where

$$\tilde{d}_{1,2} = \frac{\ln \frac{1}{\tilde{K}} \pm \frac{\sigma_{(1)}^2}{2} T}{\sigma_{(1)}\sqrt{T}}. \tag{8.32}$$

We obtain

$$\mathbb{E}_{\mathbb{P}}\Big(\Big(S_T^{(1)} - K\Big)^{+} p_T \Big) = \Phi(d_1) - K\Phi(d_2), \tag{8.33}$$

where

$$d_{1,2} = \frac{\ln \frac{1}{K} \pm \frac{\sigma_{(1)}^2}{2} T}{\sigma_{(1)}\sqrt{T}}. \tag{8.34}$$

$\Phi(.)$ is the cumulative distribution function of a normally distributed random variable with mean 0 and variance 1 such that $\Phi(u) = \int_{-\infty}^{u} \frac{1}{\sqrt{2\pi}} \exp\{-\frac{y^2}{2}\} dy$. Finally, we get

$$\mathbb{C}_0(T) = ((\Phi(\tilde{d}_1) - \tilde{K}\Phi(\tilde{d}_2))\mathbb{P}(\{\tau > T\}) \tag{8.35}$$

$$+ (\Phi(d_1) - K\Phi(d_2))\mathbb{P}(\{\tau \leq T\}), \tag{8.36}$$

where $\Phi(.)$ is a cumulative distribution function of a normally distributed random variable with mean 0, variance 1 and $\mathbb{P}(\{\tau \leq T\})$, $\mathbb{P}(\{\tau > T\})$, $\tilde{d}_{1,2}$ and $d_{1,2}$ are defined by (8.16), (8.17), (8.32) and (8.34), respectively.

□

8.5.3 The case when $W^{(1)}$ and $W^{(2)}$ are correlated with the correlation coefficient ρ

In this section we consider the case where $W_T^{(1)}$ and $W_T^{(2)}$ are the correlated standard Brownian motions with a correlation parameter $\rho \in (-1, 1)$. Namely, we formulate a theorem and give a proof for a pricing formula in this case.

For the later purpose we calculate the conditional density function of $W_T^{(1)}$ condition on $W_T^{(2)} = y$.

Firstly, let us consider the fact that if two integrable random variables X and Y defined on $(\Omega, \mathcal{F}, \mathbb{P})$ are dependent then

$$\mathbb{E}_{\mathbb{P}}(g(X)|Y) = \Big(\int_{\mathbb{R}} g(x) f_{X|Y=y}(x|y) dx \Big)_{|y=Y}, \tag{8.37}$$

where g is a Borel function and $f_{X|Y=y}(x|y)$ is a conditional density function such that

$$f_{X|Y=y}(x|y) = \frac{f_{(X,Y)}(x, y)}{f_Y(y)},$$

where $f_{(X,Y)}(x, y)$ is the joint density function of a random vector (X, Y) and $f_Y(y)$ is a density function of random variable Y.

For our further purposes we need a conditional density function $f_{W_T^{(1)}|W_T^{(2)}=y}(x|y)$.

Proposition 8.1. *The law of* $W_T^{(1)}|W_T^{(2)}$ *is equivalent to a law of a normally distributed random variable with mean* ρy *and variance* $T(1-\rho^2)$. *Specifically,*

$$f_{W_T^{(1)}|W_T^{(2)}=y}(x|y) = f_{Z_T}(z),$$

where Z_T *is a normally distributed random variable with mean* ρy *and variance* $T(1 - \rho^2)$.

Proof. We begin with the formula for the conditional density, namely

$$f_{W_T^{(1)}|W_T^{(2)}=y}(x|y) = \frac{f_{(W_T^{(1)}, W_T^{(2)})}(x, y)}{f_{W_T^{(2)}}(y)}.$$

We use also the miltivariate normal distribution to establish the following form of the joint density of the vector $(W_T^{(1)}, W_T^{(2)})$

$$f_{(W_T^{(1)}, W_T^{(2)})}(x, y) = \frac{1}{2\pi T\sqrt{1-\rho^2}} \exp\{-\frac{x^2 + y^2 - 2\rho xy}{2(1-\rho^2)T}\}.$$

Since

$$f_{W_T^{(2)}}(y) = \frac{1}{\sqrt{2\pi T}} \exp\{-\frac{y^2}{2T}\},$$

we obtain

$$f_{W_T^{(1)}|W_T^{(2)}=y}(x|y) = \frac{1}{\sqrt{2\pi T(1-\rho^2)}} \exp\{-\frac{(x-\rho y)^2}{2T(1-\rho^2)}\}. \tag{8.38}$$

Thus,

$$f_{W_T^{(1)}|W_T^{(2)}=y}(x|y) = f_{Z_T}(z),$$

where Z_T is a normally distributed random variable with mean ρy and variance $T(1-\rho^2)$. □

Let us now formulate the theorem concerning the pricing formula for the case of two correlated Brownian motions.

Theorem 8.2. *In the case of two correlated Brownian motions with the correlation coefficient $\rho \in (0,1)$ the pricing formula for the European option with the payoff given by (8.19) has the following form*

$$\mathbb{C}_0 = \int_{-\infty}^{D} C_3(x)\frac{1}{\sqrt{2\pi T}} \exp\Big\{-\frac{x^2}{2T}\Big\}dx + \int_{D}^{\infty} C_5(x)\frac{1}{\sqrt{2\pi T}} \exp\Big\{-\frac{x^2}{2T}\Big\}dx, \tag{8.39}$$

where

$$D = \frac{\ln a - (\mu_{(2)} - \frac{\sigma_{(2)}^2}{2})T}{\sigma_{(2)}}$$

and

$$C_5(x) = \gamma(x)C_3(x) + (1-\gamma(x))C_4(x),$$

where

$$\gamma(x) = a^{2\left(\frac{x}{\sigma_{(2)}T} + \frac{\mu_{(2)}}{\sigma_{(2)}^2} - \frac{1}{2}\right)} \exp\Big\{-\frac{2}{T}\Big(\frac{\ln a}{\sigma_{(2)}}\Big)^2\Big\},$$

$$C_3(x) = \exp\Big\{-\frac{((\theta-\sigma_{(1)})\rho)^2}{2}T - (\theta-\sigma_{(1)})\rho x\Big\}\Phi(D_1),$$

$$-K\exp\Big\{-\frac{(\theta\rho)^2}{2}T - \theta\rho x\Big\}\Phi(D_2),$$

$$C_4(x) = \exp\Big\{-\frac{((\theta-\sigma_{(1)})\rho)^2}{2}T - (\theta-\sigma_{(1)})\rho x\Big\}\Phi(\tilde{D}_1)$$

$$-\tilde{K}\exp\Big\{-\frac{(\theta\rho)^2}{2}T - \theta\rho x\Big\}\Phi(\tilde{D}_2),$$

$$D_1 = \frac{\ln \frac{1}{K} + \frac{\sigma_{(1)}^2}{2} T + \rho x \sigma_{(1)}}{\sigma_{(1)} \sqrt{T(1-\rho^2)}} + (\theta - \sigma_{(1)}) \sqrt{T} \frac{\rho^2}{\sqrt{1-\rho^2}}, \quad D_2 = D_1 - \sigma_{(1)} \sqrt{T(1-\rho^2)},$$

$$\tilde{D}_1 = \frac{\ln \frac{1}{\tilde{K}} + \frac{\sigma_{(1)}^2}{2} T + \rho x \sigma_{(1)}}{\sigma_{(1)} \sqrt{T(1-\rho^2)}} + (\theta - \sigma_{(1)}) \sqrt{T} \frac{\rho^2}{\sqrt{1-\rho^2}}, \quad \tilde{D}_2 = \tilde{D}_1 - \sigma_{(1)} \sqrt{T(1-\rho^2)}.$$

Proof. To price we need to calculate the following mathematical expectation

$$\mathbb{C}_0 = \mathbb{E}_{\mathbb{P}}(g(S_T^{(1)}) p_T),$$

where

$$g(S_T^{(1)}) = (S_T^{(1)} + \mathbb{I}_{\{\tau > T\}} - K)^+$$

and

$$p_T = \exp\Big\{ -\frac{\theta^2}{2} T - \theta W_T^{(1)} \Big\}. \tag{8.40}$$

From the fact that we can express $g(S_T^{(1)})$ as

$$g(S_T^{(1)}) = g(S_T^{(1)}) \mathbb{I}_\Omega = g(S_T^{(1)}) \mathbb{I}_{\{\tau \le T\}} + g(S_T^{(1)}) \mathbb{I}_{\{\tau > T\}}$$

so that

$$\mathbb{C}_0 = \mathbb{E}_{\mathbb{P}}((S_T^{(1)} + \mathbb{I}_{\{\tau > T\}} - K)^+ p_T \mathbb{I}_{\{\tau \le T\}} + (S_T^{(1)} + \mathbb{I}_{\{\tau > T\}} - K)^+ p_T \mathbb{I}_{\{\tau > T\}}),$$

we obtain

$$\mathbb{C}_0 = \mathbb{E}_{\mathbb{P}}((S_T^{(1)} + 1 - K)^+ p_T \mathbb{I}_{\{\tau > T\}} + (S_T^{(1)} - K)^+ p_T \mathbb{I}_{\{\tau \le T\}}).$$

We condition this mathematical expectation on $\mathcal{F}_T^{(2)} = \sigma(W_s^{(2)}, 0 \le s \le t)$ and take the random variables $\mathbb{I}_{\{\tau > T\}}$ and $\mathbb{I}_{\{\tau \le T\}}$ outside the conditional expectation because they are $\mathcal{F}_T^{(2)}$-measurable. We get that

$$\mathbb{C}_0 = \mathbb{E}_{\mathbb{P}}(\mathbb{E}_{\mathbb{P}}((S_T^{(1)} - K)^+ p_T | \mathcal{F}_T^{(2)}) \mathbb{I}_{\{\tau \le T\}}) + \mathbb{E}_{\mathbb{P}}(\mathbb{E}_{\mathbb{P}}((S_T^{(1)} + 1 - K)^+ p_T | \mathcal{F}_T^{(2)}) \mathbb{I}_{\{\tau > T\}}).$$

To simplify, we denote

$$C_3(W_T^{(2)}) = \mathbb{E}_{\mathbb{P}}((S_T^{(1)} - K)^+ p_T | \mathcal{F}_T^{(2)}),$$

$$\tilde{K} = K - 1$$

and

$$C_4(W_T^{(2)}) = \mathbb{E}_{\mathbb{P}}((S_T^{(1)} - \tilde{K})^+ p_T | \mathcal{F}_T^{(2)}).$$

Let us now show that

$$\mathbb{E}_{\mathbb{P}}((S_T^{(1)}-K)^+ p_T|\mathcal{F}_T^{(2)}) = \mathbb{E}_{\mathbb{P}}((S_T^{(1)}-K)^+ p_T|W_T^{(2)})$$

so we can replace $\mathcal{F}_T^{(2)}$ with $W_T^{(2)}$. It is enough to prove that $\mathbb{E}_{\mathbb{P}}((S_T^{(1)}-K)^+ p_T|\mathcal{F}_T^{(2)})$ is $W_T^{(2)}$-measurable. We start with the fact that $\mathbb{E}_{\mathbb{P}}((S_T^{(1)}-K)^+ p_T|\mathcal{F}_T^{(2)})$ is $\mathcal{F}_T^{(2)}$-measurable and we denote $W^{(3)} = (W_t)_{t\in[0,T]}$ as the third Brownian motion which is independent of $W^{(2)}$. Let us also remind that

$$S_T^{(1)} = \exp\Big\{\Big(\mu_{(1)} - \frac{\sigma_{(1)}^2}{2}\Big)T + \sigma_{(1)}W_T^{(1)}\Big\}$$

and p_T is defined as in (8.40). Using Normal Correlation Theorem which says that we can represent $W_T^{(1)}$ as

$$W_T^{(1)} = \rho W_T^{(2)} + \sqrt{1-\rho^2}W_T^{(3)},$$

we obtain

$$\mathbb{E}_{\mathbb{P}}\Big(\Big(\exp\Big\{\Big(\mu_{(1)} - \frac{\sigma_{(1)}^2}{2}\Big)T + \sigma_{(1)}\Big(\rho W_T^{(2)} + \sqrt{1-\rho^2}W_T^{(3)}\Big)\Big\} - K\Big)^+$$
$$\cdot \exp\Big\{-\frac{\theta^2}{2}T - \theta\Big(\rho W_T^{(2)} + \sqrt{1-\rho^2}W_T^{(3)}\Big)\Big\}|\mathcal{F}_T^{(2)}\Big).$$

Then, applying the formula for the calculation of the conditional expectation with one measurable and one independent variable, we can conclude that $\mathbb{E}_{\mathbb{P}}((S_T^{(1)}-K)^+ p_T|\mathcal{F}_T^{(2)}) = \mathbb{E}_{\mathbb{P}}((S_T^{(1)}-K)^+ p_T|W_T^{(2)})$. Thus, we can examine the latter.

To calculate this expectation we use the fact that

$$\mathbb{E}_{\mathbb{P}}((S_T^{(1)}-K)^+ p_T|W_T^{(2)}) = \Big(\mathbb{E}_{\mathbb{P}}((S_T^{(1)}-K)^+ p_T|W_T^{(2)}=w)\Big)_{|w=W_T^{(2)}}.$$

Let us calculate

$$C_3(w) = \mathbb{E}_{\mathbb{P}}((S_T^{(1)}-K)^+ p_T|W_T^{(2)}=w)$$

which equals to

$$C_3(w) = \mathbb{E}_{\mathbb{P}}((S_T^{(1)}-K)\mathbb{I}_{\{S_T^{(1)}-K\geq 0\}} p_T|W_T^{(2)}=w).$$

Using the conditional density function (8.38) we calculate that

$$\mathbb{E}_{\mathbb{P}}(\mathbb{I}_{\{S_T^{(1)}-K\geq 0\}} p_T|W_T^{(2)}=w)$$

$$= \int_{-d_1'}^{\infty} \exp\Big\{-\frac{\theta^2}{2}T - \theta x\Big\}\frac{1}{\sqrt{2\pi T(1-\rho^2)}}\exp\{-\frac{(x-\rho w)^2}{2T(1-\rho^2)}\}dx,$$

where

$$d_1' = \frac{\ln\frac{1}{K} + (\mu_{(1)} - \frac{\sigma_{(1)}^2}{2})T}{\sigma_{(1)}}.$$

We get

$$\mathbb{E}_{\mathbb{P}}(\mathbb{I}_{\{S_T^{(1)}-K\geq 0\}}p_T|W_T^{(2)}) = \exp\Big\{-\frac{(\theta\rho)^2}{2}T - \theta\rho W_T^{(2)}\Big\}\Phi(D_2),$$

where $\Phi(.)$ is a cumulative distribution function of a normally distributed random variable with mean 0 and variance 1 and

$$D_2 = \frac{\ln\frac{1}{K} - \frac{\sigma_{(1)}^2}{2}T + \rho\sigma_{(1)}W_T^{(2)}}{\sigma_{(1)}\sqrt{T(1-\rho^2)}} + \theta\sqrt{T}\frac{\rho^2}{\sqrt{1-\rho^2}}. \tag{8.41}$$

Analogically, we calculate that

$$\mathbb{E}_{\mathbb{P}}(S_T^{(1)}\mathbb{I}_{\{S_T^{(1)}-K\geq 0\}}p_T|W_T^{(2)}) = \exp\Big\{-\frac{((\theta-\sigma_{(1)})\rho)^2}{2}T - (\theta-\sigma_{(1)})\rho W_T^{(2)}\Big\}\Phi(D_1),$$

where

$$D_1 = \frac{\ln\frac{1}{K} + \frac{\sigma_{(1)}^2}{2}T + \rho W_T^{(2)}\sigma_{(1)}}{\sigma_{(1)}\sqrt{T(1-\rho^2)}} + (\theta-\sigma_{(1)})\sqrt{T}\frac{\rho^2}{\sqrt{1-\rho^2}}. \tag{8.42}$$

We got the formula

$$C_3(W_T^{(2)}) = \exp\Big\{-\frac{((\theta-\sigma_{(1)})\rho)^2}{2}T - (\theta-\sigma_{(1)})\rho W_T^{(2)}\Big\}\Phi(D_1) \tag{8.43}$$

$$-K\exp\Big\{-\frac{(\theta\rho)^2}{2}T - \theta\rho W_T^{(2)}\Big\}\Phi(D_2),$$

where D_1 and D_2 are defined by (8.42) and (8.41), respectively.
The formula for $C_4(W_T^{(2)})$ is analogical but instead of K we have $\tilde{K}$. Specifically,

$$C_4(W_T^{(2)}) = \exp\Big\{-\frac{((\theta-\sigma_{(1)})\rho)^2}{2}T - (\theta-\sigma_{(1)})\rho W_T^{(2)}\Big\}\Phi(\tilde{D}_1) \tag{8.44}$$

$$-\tilde{K}\exp\Big\{-\frac{(\theta\rho)^2}{2}T - \theta\rho W_T^{(2)}\Big\}\Phi(\tilde{D}_2),$$

where

$$\tilde{D}_1 = \frac{\ln \frac{1}{\tilde{K}} + \frac{\sigma_{(1)}^2}{2}T + \rho W_T^{(2)}\sigma_{(1)}}{\sigma_{(1)}\sqrt{T(1-\rho^2)}} + (\theta - \sigma_{(1)})\sqrt{T}\frac{\rho^2}{\sqrt{1-\rho^2}} \tag{8.45}$$

and

$$\tilde{D}_2 = \frac{\ln \frac{1}{\tilde{K}} - \frac{\sigma_{(1)}^2}{2}T + \rho\sigma_{(1)} W_T^{(2)}}{\sigma_{(1)}\sqrt{T(1-\rho^2)}} + \theta\sqrt{T}\frac{\rho^2}{\sqrt{1-\rho^2}}. \tag{8.46}$$

Consequently, we have the pricing formula as following

$$\mathbb{C}_0 = \mathbb{E}_{\mathbb{P}}(C_3(W_T^{(2)})\mathbb{I}_{\{\tau \le T\}}) + \mathbb{E}_{\mathbb{P}}(C_4(W_T^{(2)})\mathbb{I}_{\{\tau > T\}}).$$

We condition on $W_T^{(2)}$ and can take $C_3(W_T^{(2)})$ and $C_4(W_T^{(2)})$ outside the expectation value. We obtain

$$\mathbb{C}_0 = \mathbb{E}_{\mathbb{P}}(C_3(W_T^{(2)})\mathbb{E}_{\mathbb{P}}(\mathbb{I}_{\{\tau \le T\}}|W_T^{(2)})) + \mathbb{E}_{\mathbb{P}}(C_4(W_T^{(2)})\mathbb{E}_{\mathbb{P}}(\mathbb{I}_{\{\tau > T\}}|W_T^{(2)})).$$

Thus, we need to calculate

$$\mathbb{P}(\tau > T|W_T^{(2)}) \text{ and } \mathbb{P}(\tau \le T|W_T^{(2)}).$$

Let us calculate the latter. We use the fact that

$$\mathbb{P}(\tau \le T|W_T^{(2)} = w) = \mathbb{P}(\sup_{t\in[0,T]}(-W_t^{(2)} + a_1 + b_1 t) \ge 0|W_T^{(2)} = w),$$

where

$$a_1 = \frac{\ln a}{\sigma_{(2)}} < 0 \tag{8.47}$$

and

$$b_1 = \frac{\sigma_{(2)}}{2} - \frac{\mu_{(2)}}{\sigma_{(2)}} > 0. \tag{8.48}$$

We use the formula from [8] such that

$$\mathbb{P}(\sup_{t\in[0,T]} W_\alpha(t) > z|W_\alpha(T) = y) = \begin{cases} \exp\left\{-\frac{2z(z-y)}{T}\right\}, & \text{if } z > \max(0,y), \\ 1, & \text{otherwise,} \end{cases}$$

where W_α is a Brownian motion with drift α. We get that

$$\mathbb{P}(\tau \le T|W_T^{(2)}) = \begin{cases} \gamma(W_T^{(2)}), & \text{if } -a_1 > \max(0, b_1T - W_T^{(2)}), \\ 1, & \text{otherwise} \end{cases}$$

where

$$\gamma(x) = \exp\left\{\frac{2a_1(x - a_1 - b_1 T)}{T}\right\} \tag{8.49}$$

and

$$\mathbb{P}(\tau > T | W_T^{(2)}) = \begin{cases} 1 - \gamma(W_T^{(2)}), & \text{if } -a_1 > \max(0, b_1 T - W_T^{(2)}), \\ 0, & \text{otherwise.} \end{cases}$$

Let us firstly simplify the formula above. Since $-a_1 > 0$ we have three possibilities:

i) $W_T^{(2)} > b_1 T$,

ii) $b_1 T + a_1 < W_T^{(2)} < b_1 T$,

iii) $W_T^{(2)} < b_1 T + a_1$.

Thus we can write

$$\mathbb{P}(\tau \leq T | W_T^{(2)})$$

$$= \gamma(W_T^{(2)}) \mathbb{I}_{\left\{W_T^{(2)} > b_1 T + a_1\right\}} + \mathbb{I}_{\left\{W_T^{(2)} < b_1 T + a_1\right\}}$$

and

$$\mathbb{P}(\tau > T | W_T^{(2)}) = \Big(1 - \gamma(W_T^{(2)})\Big) \mathbb{I}_{\left\{W_T^{(2)} > b_1 T + a_1\right\}},$$

where a_1, b_1, $\gamma(.)$ are defined by (8.47), (8.48), (8.49) respectively.
Finally, we denote $a_1 + b_1 T$ as D and we extend it to the form

$$D = a_1 + b_1 T = \frac{\ln a - (\mu_{(2)} - \frac{\sigma_{(2)}^2}{2})T}{\sigma_{(2)}}. \tag{8.50}$$

We get the pricing formula

$$\mathbb{C}_0 = \int_{-\infty}^{D} C_3(x) \frac{1}{\sqrt{2\pi T}} \exp\Big\{-\frac{x^2}{2T}\Big\} dx + \int_{D}^{\infty} C_5(x) \frac{1}{\sqrt{2\pi T}} \exp\Big\{-\frac{x^2}{2T}\Big\} dx,$$

where

$$C_5(x) = \gamma(x) C_3(x) + (1 - \gamma(x)) C_4(x),$$

where $\gamma(x)$, $C_3(x)$ and $C_4(x)$ are defined by (8.49), (8.43), (8.44) respectively and we transform the function γ to the form

$$\gamma(x) = \exp\left\{\frac{2a_1(x - a_1 - b_1 T)}{T}\right\} = a^{2\left(\frac{x}{\sigma_{(2)} T} + \frac{\mu_{(2)}}{\sigma_{(2)}^2} - \frac{1}{2}\right)} \exp\Big\{-\frac{2}{T}\Big(\frac{\ln a}{\sigma_{(2)}}\Big)^2\Big\}.$$

□

To verify if the final formula is correct let us present a corollary showing that if we put $\rho \equiv 0$ then we obtain the formula for uncorrelated Brownian motions.

Corollary 8.1. *Using the formula for correlated Brownian motions in the case of uncorrelated Brownian motions ($\rho \equiv 0$) we get the same result as in Theorem 8.1.*

Proof. Let us start the proof by putting $\rho = 0$ to the formula (8.39).

We obtain that D_1, D_2, $\tilde{D}_1$, $\tilde{D}_2$ defined by (8.42), (8.41), (8.45), (8.46) respectively, have the form as follows

$$D_{1,2} = \frac{\ln \frac{1}{K} + \frac{\sigma_{(1)}^2}{2} T}{\sigma_{(1)} \sqrt{T}} = d_{1,2},$$

$$\tilde{D}_{1,2} = \frac{\ln \frac{1}{\tilde{K}} + \frac{\sigma_{(1)}^2}{2} T}{\sigma_{(1)} \sqrt{T}} = \tilde{d}_{1,2},$$

for the formula of $d_{1,2}$ and $\tilde{d}_{1,2}$ see (8.34), (8.32).

With $\rho = 0$ the functions C_3, C_4 and C_5 become

$$C_3(x) = \Phi(d_1) - K\Phi(d_2) = C_3,$$

$$C_4(x) = \Phi(\tilde{d}_1) - \tilde{K}\Phi(\tilde{d}_2) = C_4,$$

$$C_5(x) = C_3\gamma(x) + C_4(1 - \gamma(x)).$$

As a result the formula (8.39) has the following form

$$\mathbb{C}_0 = C_3 \int_{-\infty}^{D} \frac{1}{\sqrt{2\pi T}} \exp\Big\{ - \frac{x^2}{2T} \Big\} dx + C_3 \int_{D}^{\infty} \gamma(x) \frac{1}{\sqrt{2\pi T}} \exp\Big\{ - \frac{x^2}{2T} \Big\} dx$$

$$+ C_4 \int_{D}^{\infty} \frac{1}{\sqrt{2\pi T}} \exp\Big\{ - \frac{x^2}{2T} \Big\} dx - C_4 \int_{D}^{\infty} \gamma(x) \frac{1}{\sqrt{2\pi T}} \exp\Big\{ - \frac{x^2}{2T} \Big\} dx, \quad (8.51)$$

where D is defined by (8.50).
We notice that

$$\int_{-\infty}^{D} \frac{1}{\sqrt{2\pi T}} \exp\Big\{ - \frac{x^2}{2T} \Big\} dx = \Phi\Big(\frac{D}{\sqrt{T}} \Big)$$

and
$$\int_D^\infty \frac{1}{\sqrt{2\pi T}} \exp\Big\{-\frac{x^2}{2T}\Big\} = 1 - \Phi\Big(\frac{D}{\sqrt{T}}\Big).$$

Putting these results and the formulas for C_3 and C_4 to the (8.5.3) we get

$$\mathbb{C}_0 = \Big(\Phi(d_1) - K\Phi(d_2)\Big)\Big(\Phi\Big(\frac{D}{\sqrt{T}}\Big) + \int_D^\infty \gamma(x)\frac{1}{\sqrt{2\pi T}} \exp\Big\{-\frac{x^2}{2T}\Big\}dx\Big)$$

$$+\Big(\Phi(\tilde{d}_1) - \tilde{K}\Phi(\tilde{d}_2)\Big)\Big(1 - \Big(\Phi\Big(\frac{D}{\sqrt{T}}\Big) + \int_D^\infty \gamma(x)\frac{1}{\sqrt{2\pi T}} \exp\Big\{-\frac{x^2}{2T}\Big\}dx\Big)\Big).$$

We compare this formula with the final result for the uncorrelated case and we realize that $\Phi\Big(\frac{D}{\sqrt{T}}\Big) + \int_D^\infty \gamma(x)\frac{1}{\sqrt{2\pi T}} \exp\Big\{-\frac{x^2}{2T}\Big\}dx$ is staying in the place of $\mathbb{P}(\tau \le T)$ defined by (8.16). We shall prove that they are equal.

For the simplicity we use firstly the initial form of the function γ, namely the form established in (8.49), and by the method of completing the square we calculate the integral. We obtain that

$$\int_D^\infty \gamma(x)\frac{1}{\sqrt{2\pi T}} \exp\Big\{-\frac{x^2}{2T}\Big\}dx = \exp\Big\{-2a_1b_1\Big\}\Big(1 - \Phi\Big(\frac{D - 2a_1}{\sqrt{T}}\Big)\Big).$$

Putting this result and the final form of D we get that (8.5.3) is equal to

$$\Phi\Big(\frac{\ln a - (\mu_{(2)} - \frac{\sigma_{(2)}^2}{2})t}{\sigma_{(2)}\sqrt{t}}\Big) + a^{2\frac{\mu_{(2)}}{\sigma_{(2)}^2} - 1}\Phi\Big(\frac{\ln a + (\mu_{(2)} - \frac{\sigma_{(2)}}{2})t}{\sigma_{(2)}\sqrt{t}}\Big)$$

which is $\mathbb{P}(\tau \le T)$ defined in (8.16).

Finally we have that

$$\mathbb{C}_0 = \Big(\Phi(d_1) - K\Phi(d_2)\Big)\mathbb{P}(\tau \le T) + \Big(\Phi(\tilde{d}_1) - \tilde{K}\Phi(\tilde{d}_2)\Big)\mathbb{P}(\tau > T)$$

which represents exactly the formula for the uncorrelated case.

□

The price of the option with the payoff given by (8.19) depends on the correlation coefficient ρ (see Figure 8.8, 8.9, 8.10, 8.11).

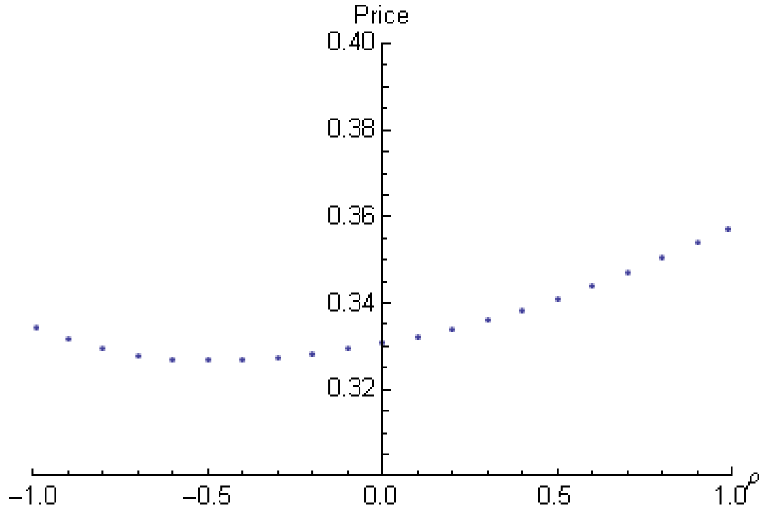

Figure 8.8: A graph illustrating the option's price against ρ for $\mu_{(1)} = 0.1$, $\mu_{(2)} = 0.1$, $\sigma_{(1)} = 0.3$, $\sigma_{(2)} = 0.3$. The strike price $K = 2$, $S_0 = 1$ and the maturity time $T = 10$.

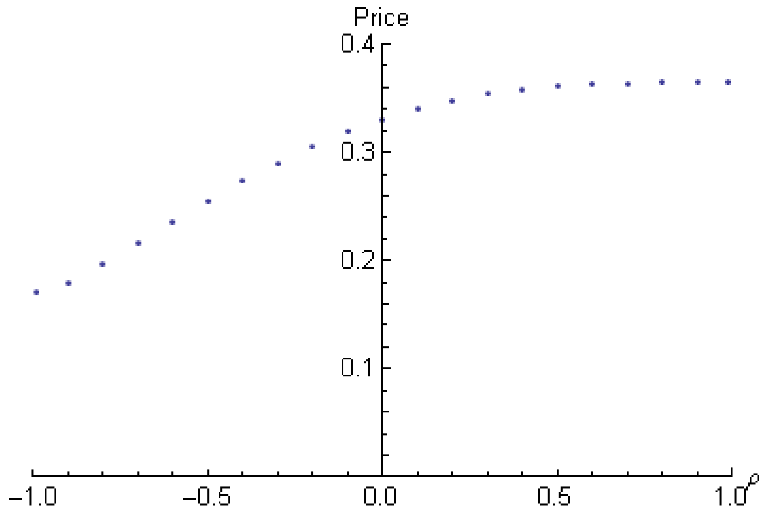

Figure 8.9: A graph illustrating the option's price against ρ for $\mu_{(1)} = -0.1$, $\mu_{(2)} = 0.1$, $\sigma_{(1)} = 0.2$, $\sigma_{(2)} = 0.3$.The strike price $K = 2$, $S_0 = 1$ and the maturity time $T = 10$.

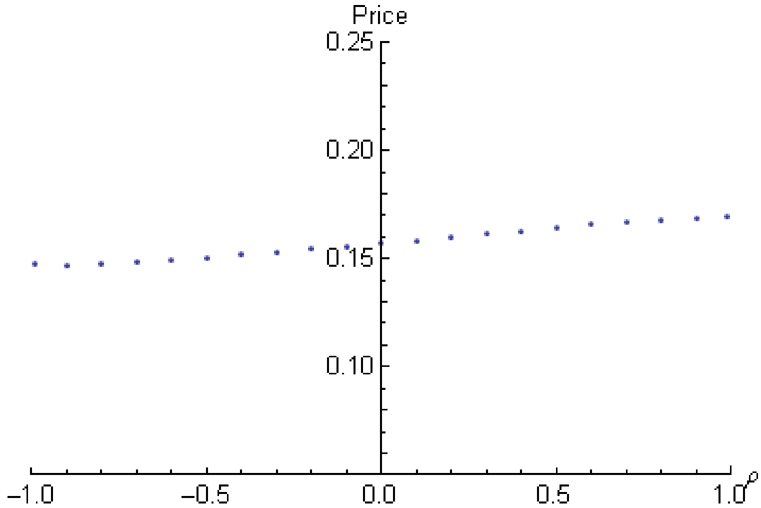

Figure 8.10: A graph illustrating the option's price against ρ for $\mu_{(1)} = 0.1$, $\mu_{(2)} = 0.1$, $\sigma_{(1)} = 0.3$, $\sigma_{(2)} = 0.3$. The strike price $K = 3$, $S_0 = 1$ and the maturity time $T = 10$.

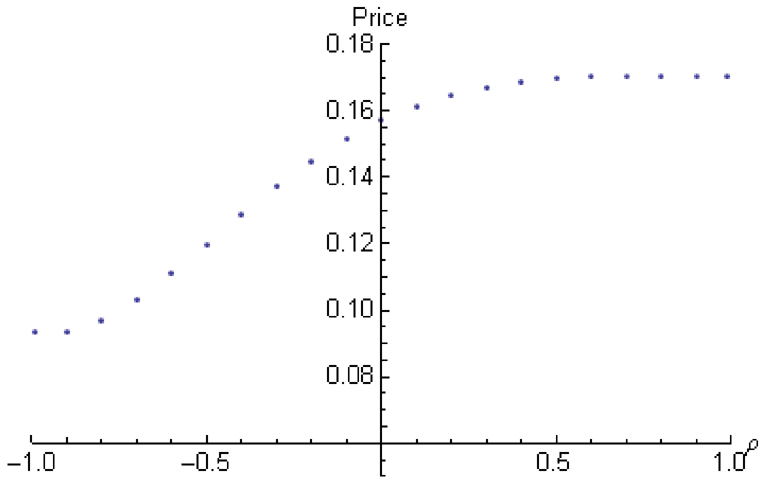

Figure 8.11: A graph illustrating the option's price against ρ for $\mu_{(1)} = -0.1$, $\mu_{(2)} = 0.1$, $\sigma_{(1)} = 0.2$, $\sigma_{(2)} = 0.3$.The strike price $K = 3$, $S_0 = 1$ and the maturity time $T = 10$.

Chapter 9

Conclusions

Due to the different type of information available to the investor in the defaultable market, pricing and hedging of the models with default is not so elementary as of the default-free models. Firstly, since the informed agent influences the stock prices, the market is incomplete for the regular investor. Thus, perfect hedging is not always possible and as a result, we have to choose the minimal martingale measure. What is more, if we work under the measure providing independence of default time and the reference filtration, the market is arbitrage-free since the measure preserves the martingale property in the initially enlarged filtration. Nonetheless, if the measure is not a decoupling one, we have to introduce some hypothesis for the market to be arbitrage-free, namely $\mathcal{H}$-hypothesis or $\mathcal{E}$-hypothesis.

Since for the regular investor market is incomplete and arbitrage-free, there exists more than one martingale measure equivalent to the physical measure in the initially enlarged filtration. However, they are all defined by the Radon-Nikodým density process being the product of the corresponding Radon-Nikodým density process from the reference filtration and a Borel function of default time.

The most common method of dealing with the minimal martingale measure is the utility approach and it consists of estimating the value of some contingent claim seen from the perspective of an agent who optimizes his behavior relative to some utility function. As a result we obtained for three different utility functions that the function of default time in the enlarged filtration is equal to one under the independence of default time and the reference filtration. It means that for these three utility functions the additional information is not valuable in the sense that it does not increase the expected utility. In the case of the correlated Brownian motions, the calculations involve the conditional density function of two Brownian motions. IT appears that the conditional law of the correlated Brownian motions is still a Brownian motion.

Pricing the option written on the investment consisting of a stock of the default-free company and a corporate bond issued by the defaultable company in such a market is analogous to the pricing of Black-Scholes model. The only difference in the case of the independence of default time and the reference filtration is that as a result we have two formulas with different strike prices combined by the corresponding probability whether the default occurred or not. The dependence between default time and the reference filtration results in calculating the conditional expectation of the discounted payoff instead of the regular expected value. In this particular case, we got the pricing formula using mainly some very important properties of the Brownian motion.

Notation

$(\Omega, \mathcal{F}, \mathbb{P})$	Probability space.
$(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$	Filtered probability space.
$X = (X_t)_{t \geq 0}$	Stochastic process.
$\mathbb{F}^X$	Natural filtration.
$\mathcal{B}(\mathbb{R})$	Borel σ-algebra on $\mathbb{R}$.
$B = (B_t)_{t \geq 0}$	Standard Brownian motion.
$\mathcal{F}^B$	σ-algebra generated by B.
$\mathcal{F}_t^B$	σ-algebra generated by B_t.
$S = (S_t)$	Price process.
τ	Default time.
K	Strike price.
T	Maturity time.
$\mathbb{E}_{\mathbb{P}}$	Mathematical expectation under measure $\mathbb{P}$.
η	Law of τ.
$F(t)$	Distribution function of τ.
$f_\tau(t)$	Density function of τ.

Notation

$N = (N_t)_{t \geq 0}$	Default process.
$\mathcal{H}_t$	σ-algebra generated by N_t.
$\mathbb{H}$	Filtration generated by $\mathcal{H}_t$.
U_t	$\mathcal{H}_t$-measurable random variable.
$\lambda = (\lambda_t)_{t \geq 0}$	Intensity process of τ.
$\mathbb{I}_A$	Indicator function of a set A.
$\Gamma(t)$	Hazard function.
$G(t)$	Survival function.
$r(t)$	Deterministic risk-free interest rate.
$\sigma(\tau)$	σ-algebra generated by τ.
$\mathcal{G}_t^\tau$	Enlarged σ-algebra $\mathcal{F}_t \vee \sigma(\tau)$.
$\mathcal{G}_t$	Enlarged σ-algebra $\mathcal{F}_t \vee \mathcal{H}_t$.
$\mathbb{G}^\tau$	Filtration generated by $\mathcal{G}_t^\tau$.
$\mathbb{G}$	Filtration generated by $\mathcal{G}_t$.

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Appendix

```
(*Functions and data*)
a = 0.4; K = 3; Kf = K - 1; T = 10; θ = μ1/σ1; μ1 = 0.08; μ2 = -0.06;
σ1 = 0.2; σ2 = 0.3;
D0 = (Log[a] - (μ2 - σ2^2/2) * T)/σ2;
D1[x_] := (Log[1/K] + σ1^2/2 * T + ρ * x * σ1)/(σ1 * Sqrt[T * (1 - ρ^2)]) + (θ - σ1) * Sqrt[T] * ρ^2/Sqrt[1 - ρ^2];
D2[x_] := (Log[1/K] - σ1^2/2 * T + ρ * x * σ1)/(σ1 * Sqrt[T * (1 - ρ^2)]) + θ * Sqrt[T] * ρ^2/Sqrt[1 - ρ^2];
D1f[x_] := (Log[1/Kf] + σ1^2/2 * T + ρ * x * σ1)/(σ1 * Sqrt[T * (1 - ρ^2)]) + (θ - σ1) * Sqrt[T] * ρ^2/Sqrt[1 - ρ^2];
D2f[x_] := (Log[1/Kf] - σ1^2/2 * T + ρ * x * σ1)/(σ1 * Sqrt[T * (1 - ρ^2)]) + θ * Sqrt[T] * ρ^2/Sqrt[1 - ρ^2];
γ[x_] := a^(2 * (x/(σ2 * T) + μ2/σ2^2 - 1/2)) * e^(-2/T * (Log[a]/σ2)^2);
C3[x_] := e^(-((θ-σ1)*ρ)^2/2 * T - (θ-σ1)*ρ*x) *
  CDF[NormalDistribution[0, 1], D1[x]]
  -K * e^(-(θ*ρ)^2/2 * T - θ*ρ*x) * CDF[NormalDistribution[0, 1], D2[x]];
C4[x_] := e^(-((θ-σ1)*ρ)^2/2 * T - (θ-σ1)*ρ*x) *
  CDF[NormalDistribution[0, 1], D1f[x]]
  -Kf * e^(-(θ*ρ)^2/2 * T - θ*ρ*x) * CDF[NormalDistribution[0, 1], D2f[x]];
C5[x_] := γ[x] * C3[x] + (1 - γ[x]) * C4[x];
calka1[x_] := C3[x] * 1/Sqrt[2 Pi * T] * e^(-x^2/(2*T));
calka2[x_] := C5[x] * 1/Sqrt[2 Pi * T] * e^(-x^2/(2*T));
```

```
rhochoice = {-0.99, -0.9, -0.8, -0.7, -0.6, -0.5, -0.4, -0.3,
   -0.2, -0.1, 0, 0.1, 0.2, 0.3,
   0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 0.99};
(*Calculating the price depending on ρ*)
inf = 100; step = 1;
licznik = 1;
rholista = {};
While[
 licznik ≤ Length[rhochoice],
 ρ = rhochoice[[licznik]];
 suma = 0;
 x = -inf;
 While[
  x ≤ inf, If[x ≤ D0, suma = suma + (step * calka1[x]),
   suma = suma + (step * calka2[x])]; x = x + step
 ];
 AppendTo[rholista, {ρ, suma}];
 licznik++
]
Export["rho.png", Show[ListPlot[rholista,
   PlotRange → {{-1, 1}, {0.015, 0.045}},
   AxesLabel → {"ρ", "Price"}],
   TextStyle -> {FontFamily -> "Arial", FontSize → 13},
   ImageSize → 400, ImageSize → 500]]
```

Figure 9.1: The source code in Mathematica for derivation of option's price in the case of correlated Brownian Motions.