Exponential Fitting, Finite Volume and Box Methods in Option Pricing

Master’s Thesis in Financial Mathematics

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Preface

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Finally we would like to mention our families, without whose support we would not be here.
Abstract
In this thesis we focus mainly on special finite differences and finite volume methods and apply them to the pricing of barrier options. The structure of this work is the following: in Chapter 1 we introduce the definitions of options and illustrate some properties of vanilla European options and exotic options. Chapter 2 describes a classical model used in the financial world, the Black-Scholes model. We derive the Black-Scholes formula and show how stochastic differential equations model financial instruments prices. The aim of this chapter is also to present the initial boundary value problem and the maximum principle. We discuss boundary conditions such as: the first boundary value problem, also called Dirichlet problem that occur in pricing of barrier options and European options. Some kinds of put options lead to the study of a second boundary value problem (Neumann, Robin problem), while the Cauchy problem is associated with one-factor European and American options. Chapter 3 is about finite differences methods such as theta, explicit, implicit and Crank-Nicolson method, which are used for solving partial differential equations. The exponentially fitted scheme is presented in Chapter 4. It is one of the new classes of a robust difference scheme that is stable, has good convergence and does not produce spurious oscillations. The stability is also advantage of the box method that is presented in Chapter 5. In the beginning of the Chapter 6 we illustrate barrier options and then we consider a novel finite volume discretization for a pricing the above options. Chapter 7 describes discretization of the Black-Scholes equation by the fitted finite volume scheme. In Chapter 8 we present and describe numerical results obtained by using the finite difference methods illustrated in the previous chapters.
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Chapter 1

Introduction

In the last 20 years derivatives become very important in the world of Finance. Futures and option contracts are traded on specialized exchanges throughout the world and its significance is evident. Hull [1] defined a derivative as a financial instrument whose value depends on the values of other, more basic underlying variables. For example, it depends on the prices of the traded assets. They are instruments to assist and regulate agreements on transactions of the future.

A derivatives exchange is a market where individuals trade standardized contracts that can be defined by the exchange.

A future and forward contract is an agreement between two parties to buy or sell an asset at a certain time in the future for a certain price. Each of the parties makes a binding engagement; it is impossible to change later this contract. Unlike forwards, futures are traded on exchanges and more formalized.

Contrary, an option gives the holder the right (not the obligation) to buy or sell a risky asset in the future at a previously agreed price. It is a contract between two parties (writer and holder) about trading the asset at a certain future time. The writer fixes the terms of the option contract and sells the option and the holder purchases the option and pays the market price (premium). Depending on the market situation the holder has to decide what to do with the rights of the option.

Options have a limited life time. At the maturity date $T$ the rights of the holder expire and for later times $t > T$ the contract is worthless. The price of the option contract $K$ is called strike (exercise) price.
There are two basic types of options

- The call option gives the holder the right to buy the underlying asset for a fixed price $K$ by the maturity date $T$.
- The put option gives the holder the right to sell the underlying asset for a fixed price $K$ by the maturity date $T$.

Option contracts can be of European or American type

- The European option can be exercised only at expiry.
- American options can be exercised at any time before expiry.

Consequently, American options are more flexible and more valuable than European ones. This is the reason why options on stocks are mostly of American style. For an American option, mostly there exist no explicit formulas and hence numerical solution techniques are required.

Two sides to every option contract exist. One side is the investor who takes the long position (buy the option). On the other side the investor who takes the short position (sell the option). The writer of an option obtain cash up front, has later some potential money obligations \([3]\). The writer’s gain or loss is the opposite of that for the buyer of the option. There exist four types of option positions:

- A long position in a call option,
- A long position in a put option,
- A short position in a call option,
- A short position in a put option.

The value of the option $V = V(S,t)$ depends on the price of the underlying asset $S$ and the time $t$. The final condition for $V(S,T)$ is called payoff function:

- for a call option
  \[
  V_C(S,T) = \max \{S(T) - K, 0\} = (S(T) - K)^+, \tag{1.1a}
  \]
- for a put option
  \[
  V_P(S,T) = \max \{K - S(T), 0\} = (K - S(T))^+. \tag{1.1b}
  \]
The value of the option also depends on other factors. The dependence on the strike price $K$ and the maturity $T$ is obvious. Market parameters affecting the price are the risk-free interest rate $r$, the volatility $\sigma$ of the price $S_t$, and the dividend yield $D$ if the asset pays dividends. The volatility parameter $\sigma$ can be identified as a standard deviation in $S_t$, for scaling divided by the square root of the observed time period. The larger the fluctuations, represented by large values of $\sigma$, the harder it is to predict a future value of the asset. Hence the volatility is a standard measure of risk. The dependence of $V$ on $\sigma$ is highly sensitive [2].

The standard notation for pricing options are following:

<table>
<thead>
<tr>
<th>Current stock price</th>
<th>$S_t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Strike price</td>
<td>$K$</td>
</tr>
<tr>
<td>Time to expiration (maturity)</td>
<td>$T$</td>
</tr>
<tr>
<td>Volatility of the stock price</td>
<td>$\sigma$</td>
</tr>
<tr>
<td>Risk-free interest rate</td>
<td>$r$</td>
</tr>
<tr>
<td>Continuous dividends (dividend paying asset)</td>
<td>$D$</td>
</tr>
</tbody>
</table>

The options are called *plain vanilla options*, if they depend on one underlying, and their payoffs are given by (1.1a) and (1.1b) with $S$ evaluated at the current time moment. All other options are called *exotic*. One of the main difference between exotic and vanilla options lies in the payoff or in the increasing of the dimension, from a single-factor to a multifactor option. The exotic options are harder to price and can be very model dependent. There are six main features to classify exotic options [3]:

- **Time dependence**
  
  It means that, for example, early exercise might only be permitted on specified dates or during fixed periods. Such options are called *Bermudan options*.

- **Cashflows**
  
  The value of the option depends on an amount of the cashflows. If $g$ denotes the amount of money that the contract pays at time $t_0$ and the value of the option $V(t)$ just before $t_0^-$ and just after $t_0^+$ the cashflow, then arbitrage considerations lead to
  
  $$V(t_0^+) = V(t_0^-) + g.$$

  This is a so-called *jump condition*. 
• Path dependence
A lot of options have payoffs that depend on the path of the asset price, and not only on the value of the asset at expiration. There are two types of path dependence

– Weak path dependency
  This means that the option depends only on the asset price in the past and time. The simplest example is a barrier option. Barrier options become worth (knock-in options) or worthless (knock-out options) when the asset price reaches some setting level (barrier).

– Strong path dependency
  In that case we have to keep track of another variable and the payoffs depend on the properties of the asset price path in addition to the value of the underlying at the present moment. For example, Asian options have a payoff that depends on the average value of the underlying asset.

• Dimensionality
It refers to number of the underlying independent variables. For example, a three-dimensional problem can be represented by the strong path-dependence. In other case, the option contract depends on some another underlying asset.

• Order
Higher order options are options whose payoff is a contingent on the value of another option. For example, compound options (a option gives the right to buy another option)

• Embedded decisions
  Holder and writer can make decisions during the life of the contract. This assumes to have the possibility to early exercise like, for example, American style option.

In order to obtain the fair value of the option we need to use a mathematical model of the market, that can serve as an approximation of the financial world.
Chapter 2

The Black-Scholes Model

In 1973 Fischer Black and Myron Scholes published in their paper "The Pricing of Options and Corporate Liabilities" a new model that is known as the Black-Scholes (BS) partial differential equation (PDE) and is used in financial markets to price different kinds of options. As Gutowska describes, the Black-Scholes model assumes that the financial market consists of shares and risk-free financial instruments (bonds, bank account). Buying and selling these instruments is a continuous process, meaning that their prices processes can be modeled by stochastic differential equations (SDE) in the case of a risk instrument or by ordinary differential equations (ODE) in the case of a risk-free instrument. A share price, which does not pay dividends is modeled by a geometric Brownian motion (GBM), which is written in the form of the following SDE:

\[ dS_t = \mu S_t dt + \sigma S_t dW_t, \quad (2.1) \]

where

\( S_t \) represent the stock price in time \( t \),
\( \mu > 0 \) the drift rate of the stock (constant),
\( \sigma > 0 \) is the volatility of the stock (constant),
\( dW_t \) is a stochastic differential, this means a variable with normal distribution.

The process of a risk-free price of an instrument is described as an ODE

\[ dB_t = rB_t dt, \quad (2.2) \]

where
\[ B_t \] represent the risk-free instrument price in time \( t \),
\( r \) is a short-term riskless interest rate.

The main assumptions in the Black-Scholes model exclude the arbitrage opportunity.

### 2.1 The Black-Scholes Formula

This section is based on the work of Gutowska [8]. Let \( V(S_t, t) \) denote a value of an option in time \( t \) when the asset price is \( S_t \). Assume that \( V(S_t, t) \) is a continuous and a differentiable function with respect to \( S_t \) and \( t \), where \( S_t \) is a GBM process.

**Lemma 1 (Itô) \[2\]**

Suppose \( X_t \) follows an Itô process, \( dX_t = a(X_t, t)dt + b(X_t, t)dW_t \), and let \( g(x, t) \) be a \( C^{2,1} \) smooth function (with continuous \( \frac{\partial g}{\partial x}, \frac{\partial^2 g}{\partial x^2}, \frac{\partial g}{\partial t} \)). Then \( Y_t := g(X_t, t) \) follows an Itô process with the same Wiener process \( W_t \)

\[
dY_t = \left( \frac{\partial g}{\partial t} + a\frac{\partial g}{\partial x} + \frac{1}{2}b^2\frac{\partial^2 g}{\partial x^2} \right) dt + b\frac{\partial g}{\partial x} dW_t
\]

(2.3)

where the derivatives of \( g \) as well as the coefficient functions \( a \) and \( b \) in general depend on the arguments \( (X_t, t) \).

Then, according to Itô’s Lemma

\[
dV = (\mu S \frac{\partial V}{\partial S} + \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2})dt + \sigma S \frac{\partial V}{\partial S} dW.
\]

(2.4)

Now let \( \Pi_t \) be a value of the portfolio in time \( t \), i.e.

\[
\Pi_t = V(S_t, t) - \Delta_t S_t,
\]

(2.5)

where \( \Delta := \frac{\partial V}{\partial S} \). The change of the portfolio value between times \( t \) and \( t + dt \) is given by

\[
d\Pi_t = dV(S_t, t) - \Delta_t dS_t.
\]

(2.6)

By the substitution of (2.1) and (2.4) to equation (2.6) we obtain

\[
d\Pi_t = \left( \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt.
\]

(2.7)
Assuming that, there is no arbitrage opportunity, the rate of return on the portfolio must be equal to the rate of return on the risk-free instrument, what is written as
\[ r \Pi_t dt = d \Pi_t = \left( \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt. \] (2.8)

Finally, after substituting (2.5) and \( \Delta = \frac{\partial V}{\partial S} \) in (2.8) we obtain the so-called Black-Scholes equation
\[ \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0. \] (2.9)

Solving the above PDE backward in time with the final pay-off condition and with appropriate boundary conditions we obtain the price \( V(S_t, t) \), [8].

We obtain for an European call option the following formula
\[ V_C(S_t, t) = S_t N(d_1) - e^{-r(T-t)} KN(d_2), \] (2.10)
and for an European put option
\[ V_P(S_t, t) = -S_t N(-d_1) + e^{-r(T-t)} KN(-d_2), \] (2.11)
where
\[ d_1 = \frac{\ln S_t K + (r + \frac{1}{2} \sigma^2)(T-t)}{\sigma \sqrt{T-t}}, \]
\[ d_2 = \frac{\ln S_t K + (r - \frac{1}{2} \sigma^2)(T-t)}{\sigma \sqrt{T-t}} = d_1 - \sigma (T-t), \]
\( N \) denotes the Normal distribution,
\( K \) denotes the strike price.

To obtain an unique solution we should use the suitable boundary conditions. The next section is devoted to this problem.

### 2.2 An Initial Boundary Value Problem

Let us consider the following general PDE of a second order, [5],
\[ \frac{\partial u}{\partial \tau} = \sum_{i,j=1}^{n} a_{ij}(x, \tau) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{j=1}^{n} b_j(x, \tau) \frac{\partial u}{\partial x_j} + c(x, \tau) u - f(x, \tau), \] (2.12)
where \( f, a_{ij}, b_j \) and \( c \) are real functions and they take finite values, 
\( x \) is a point in the \( n \)-dimensional space, \( x \in \mathbb{R}^n \), 
\( \tau \) is a positive variable representing time.

We will focus on the one-dimensional case. Setting \( n = 1 \) the equation (2.12) covers the case of the well-known Black-Scholes equation
\[
\frac{\partial V}{\partial \tau} = \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - D)S \frac{\partial V}{\partial S} - rV, \tag{2.13}
\]
where we have employed in (2.9) a time-reversal \( \tau := T - t \). Comparing (2.13) and (2.9) we can see that in (2.13), there is a new value. This is a value \( D \) that denotes the continuous dividend yield. Now let us consider the following equation
\[
\frac{\partial V}{\partial \tau} = \frac{1}{2} \sum_{i,j=1}^{n} \rho_{ij} \sigma_i \sigma_j S_i \frac{\partial^2 V}{\partial S_i \partial S_j} + \sum_{j=1}^{n} (r - D_j)S_j \frac{\partial V}{\partial S_j} - rV. \tag{2.14}
\]
This equation models a multi-asset environment and is obtained by generalization of (2.13), where \( \sigma_i \) represent the volatility of the \( i \)th asset, \( \rho_{ij} \) represent the correlation between assets \( i \) and \( j \).

Let us focus on the sequel of equation (2.12). We have to supply (2.12) with an initial condition and boundary conditions. The three types of boundary conditions are

1. The first boundary value problem (Dirichlet problem)

The domain of the first boundary value problem is \( D = \Omega \times (0, T) \) where \( \Omega \) is a bounded subset of \( \mathbb{R}^n \) and \( T \) is a positive number. The solution should satisfy the following conditions
   - \( u|_{t=0} = \varphi(x) \) (initial condition),
   - \( u|_{\Gamma} = \psi(x,t) \) (boundary condition), \( x \in \Gamma \),
where \( \Gamma \) denotes the boundary of \( \Omega \), i.e. \( \Gamma = \partial \Omega \). Such boundary conditions are used during modeling single and double barrier options in the one-factor case and during modeling of an European option. They are called Dirichlet boundary conditions.
2. The second boundary value problem (Neumann, Robin problem)

The second boundary value problem is very similar to the previous one and also the initial condition is the same, just the boundary condition is changed

- \( u|_{t=0} = \varphi(x) \) (initial condition),
- \( \left( \frac{\partial u}{\partial \eta} + a(x, t)u \right)|_{\Gamma} = \psi(x, t) \) (boundary condition), \( x \in \Gamma \),

\( \frac{\partial u}{\partial \eta} \) is the derivative of \( u \) with respect to \( \eta \) at \( \Gamma \),
\( \eta \) represent unit outward normal,
\( a(x, t) \) and \( \psi(x, t) \) are given functions of \( x \) and \( t \).

The special case for above boundary condition is when \( a \equiv 0 \) and than such boundary condition is called the Neumann boundary condition. This arise in modeling of certain kinds of put options.

3. The Cauchy problem

In this case the solution for (2.12) is defined by the following condition

\[ u|_{t=0} = \varphi(x) \] (initial condition)

where
\( u(x, t) \) is a function which satisfies (2.12) in \( \mathbb{R}^n \times (0, T) \) and satisfies above initial condition,
\( \varphi(x) \) is a given continuous function.

Furthermore, \( u(x, t) \) satisfies the above initial condition and (2.12). Also, here we have a special case which arise in the modeling of the one-factor European and American options.
Let us consider the European Call. For an option \( V_C = C(S, \tau) \) the initial condition at \( \tau = 0 \) reads

\[ V_C(S, 0) = \max(S - K, 0), \]

where \( S \) denote the underlying asset price and \( K \) is the strike price.
The boundary conditions are the following

\[ S = 0, \quad V_C(0, \tau) = 0, \]
\[ S \to +\infty, \quad V_C(S \to +\infty, \tau) = S. \]
For the *European put* option the initial condition is

\[ V_P(S, 0) = \max(K - S, 0). \]

The boundary conditions are the following

\[
\begin{align*}
S &= 0, & V_P(0, \tau) &= K e^{-r\tau}, \\
S &\to +\infty, & V_P(S \to +\infty, \tau) &= 0,
\end{align*}
\]

where

\( r \) is risk-free interest rate,
\( \tau \) is current time, for the forward-in-time BS equation \((2.13)\)
\( T \) is the maturity time.

### 2.3 The Maximum Principle

In this section we describe the continuous maximum principles for a parabolic type of PDE. The main purpose of this principle is to show conditions that are needed that solutions to \((2.13)\) remain positive if the initial and boundary conditions are positive.

Let \( D = \Omega \times (0, T) \) and \( \overline{D} \) is closure of \( D \). The following theorem states that if the initial and boundary conditions are positive then the solution in the domain \( D \) is also positive.

**Theorem 1** \([5]\). Assume that the function \( u(x,t) \) is continuous in \( D \) and assume that the coefficients in \((2.14)\) are continuous. Suppose that \( Lu \leq 0 \) in \( \overline{D} \setminus \Gamma \) where \( b(x,t) < M \) (\( M \) is some constant) and suppose furthermore that \( u(x,t) \geq 0 \) on \( \Gamma \). Then

\[ u(x,t) \geq 0 \text{ in } \overline{D}. \]

**Theorem 2** \([5]\). Suppose that \( u(x,t) \) is continuous and satisfies \((2.14)\) in \( \overline{D} \setminus \Gamma \) where \( f(x,t) \) is a bounded function (\( |f| \leq N \)) and \( b(x,t) \leq 0 \). If \( |u(x,t)|_{|\Gamma} \leq m \) then

\[ |u(x,t)| \leq Nt + m, \text{ in } \overline{D}. \]

We can use Theorem 2 in the case, where \( b(x,t) \leq b_0 < 0 \)

\[ |u(x,t)| \leq \max \left\{ \frac{-N}{b_0}, m \right\}. \quad (2.15) \]
From the above inequality follows that the initial and boundary values bound the growth of \( u \). We can obtain the minimum and the maximum principles for the heat equation and its variants in a special case when \( b \equiv 0 \) and \( f \equiv 0 \).

**Corollary 1** [5]. Assume that the conditions of Theorem 2 are satisfied and that \( b \equiv 0 \) and \( f \equiv 0 \). Then the solution \( u(x,t) \) takes its the least and the greatest values on \( \Gamma \), that is

\[
m_1 \equiv \min u(x,t) \leq u(x,t) \leq \max u(x,t) \equiv m_2
\]

The above theorems and the corollary state that the solution \( u \) is always positive if it has a positive input. We have shown that if the initial and boundary conditions are positive then the solution is also positive in the bounded domain \( D \), but what about unbounded domains? Is the solution positive in such domains (for example, for the European and American option problem)?

**Theorem 3** (The Maximum principle for the Cauchy problem), [5]. Let \( u(x,t) \) be continuous and bounded from below in \( H = \mathbb{R}^n \times (0,T) \), that is \( u(x,t) \geq -m, m > 0 \). Suppose further that \( u(x,t) \) has the continuous derivatives in \( H \) up to the second order in \( x \) and the first order in \( t \) and that \( Lu \leq 0 \). Let \( \sigma, \mu \) and \( b \) satisfy the conditions

\[
|\sigma(x,t)| \leq M(x^2 + 1), \\
|\mu(x,t)| \leq M\sqrt{x^2 + 1}, \\
b(x,t) \leq M.
\]

Then \( u(x,t) \geq 0 \) everywhere in \( H \) if \( u \geq 0 \) for \( t = 0 \).

To see that the price of an option can not take negative values we can apply Theorem 3 to the Black-Scholes equation (2.13).

How describe Mark S. Joshi in his article [15], if function \( f \) is constant then \( u \) is also constant, where \( u \) denotes the solution. We can assume that \( f \) is a nonlinear function. Summarizing the maximum principle, it states that if \( m \) is the minimum and \( M \) the maximum of the function \( f \) then respectively \( m \) and \( M \) are also the minimum and the maximum of \( u \) and these values are on the boundary. Assume, that \( X_i \) where \( i = 1, \ldots, n \), denote non-dividend paying financial assets following independent Brownian motions and \( \Omega \) is an open bounded subset of \( \mathbb{R}^n \). Let us consider the value \( C(x) \), where \( x \) is the point in the interior. The above value \( C(x) \) is a product with no expiry which pays a continuous function \( f \) at the first time the vector \( (X_1, \ldots, X_n) \) touches...
The most \( C(x) \) will pay is \( M \) and the minimum \( C(x) \) will pay is \( m \). This is, since it will hit the boundary with probability 1 in finite time. By the no-arbitrage assumption, the value should lie somewhere between \( m \) and \( M \). In the case when its value is \( M \), we consider the portfolio with \( M \) stocks minus one unit of \( C \). Such portfolio has got the zero initial value. Nonetheless, if the function \( f \) is non-linear, than the finishing at a point where \( f(x) < M \) has non-zero probability. The conclusion is that either \( f \) is a constant or there occur an arbitrage opportunity. Then if \( f \) is non-constant function, the maximum of \( C \) cannot be obtained in our open bounded subset.
Chapter 3

Finite Difference Methods

3.1 Fundamentals

Now we are interested in how to find a solution of the Black-Scholes equation. Because the corresponding explicit analytical formula cannot always be found for any initial and boundary conditions we must employ numerical methods, i.e. approximate the PDE. The most popular methods are, for example the Monte Carlo method, the binomial and trinomial methods and the finite difference method. In our work we first focus on one of them, that is the finite difference method. The main idea of the finite difference methods is to replace the partial derivatives which appear in the PDE by difference quotients. In other words, it relies on replacing differential equations by finite differences equations.

3.2 The classical finite difference methods

In this section we consider how to construct finite difference methods to solve the Black-Scholes equation

\[ \frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + r S \frac{\partial f}{\partial S} = rf, \]

supplied with the certain boundary conditions and initial conditions. Assuming a reversal of time, (3.1) reads

\[ \frac{\partial f}{\partial \tau} = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + (r - D)S \frac{\partial f}{\partial S} - rf, \]

where \( \tau = T - t \), \( T \) and \( t \) denote maturity time and current time, respectively, i.e. \( \tau \) denotes the remaining life time. In the above equation \( f = f(S, \tau) \)
represent the value of an option, where \( S = S(\tau) \) is the price of the underlying asset at time \( \tau \). We observe that there appear three kinds of derivatives: with respect to \( \tau \), with respect to \( S \) and the second-order derivative also with respect to \( S \). To solve (3.2) we should approximate the partial derivatives by using the difference quotients. To this end, we apply Taylor’s theorem which shows, that a smooth function \( f(x) \) can be written as

\[
f(x \pm h) = f(x) \pm hf'(x) + \frac{1}{2} h^2 f''(x) + \frac{1}{6} h^3 f'''(x) + ... \tag{3.3}
\]

Figure 3.1: A graphical illustration of the forward, backward and central approximations of a spatial derivative \( f(x) \) with respect to \( x \)

If we approximate the first derivative from equation (3.3) we get respectively the forward and backward approximation

\[
f'(x) = \frac{f(x + h) - f(x)}{h} + O(h), \tag{3.4}
\]

\[
f'(x) = \frac{f(x) - f(x - h)}{h} + O(h) \tag{3.5}
\]

where \( O(h) \) states the order of the discretization error. To obtain a more accurate approximation we subtract the backward approximation from the forward approximation and obtain

\[
f'(x) = \frac{f(x + h) - f(x - h)}{2h} + O(h^2). \tag{3.6}
\]
Such difference quotient is called the central or symmetric approximation. As we mentioned earlier, in the Black-Scholes equation there is also a second-order derivative. Adding the forward and the backward approximation leads to

\[ f(x + h) + f(x - h) = 2f(x)h^2 f''(x) + O(h^4), \]

and thus we get immediately

\[ f''(x) = \frac{f(x + h) - 2f(x) + f(x - h)}{h^2} + O(h^2). \]

Now we can apply this knowledge to our issue. First, we need to create a discrete grid. In our case we consider a discretization with respect to time $\tau$ and to the asset price $S$. We create a grid by dividing the $S$ and $\tau$ axis on points that have the same distance, $\Delta \tau$ and $\Delta S$, accordingly we obtain so-called uniform grid. The spatial-temporal domain $(0, S_{\text{max}}) \times (0, T)$ is divided by the grid, with the points $(j\Delta S, n\Delta \tau)$, as follows

\[
S = 0, \Delta S, 2\Delta S, ..., M\Delta S \equiv S_{\text{max}}, \\
\tau = 0, \Delta \tau, 2\Delta \tau, ..., N\Delta \tau \equiv T.
\]

For brevity we write

\[
\Delta S = h, \\
\Delta \tau = k.
\]

Later we are interested in the values of the function $f(S, \tau)$ in the grid points $(j\Delta S, n\Delta \tau)$ and introduce the following grid notation for the pointwise approximation

\[ f^n_j \approx f(j\Delta S, n\Delta \tau). \]

There exist three ways to approximate the partial derivatives in the Black-Scholes equation

- The forward difference

\[
\frac{\partial}{\partial S} f(j\Delta S, n\Delta \tau) \approx \frac{f^n_{j+1} - f^n_j}{h} =: D^+_h f^n_j, \quad (3.9)
\]

and

\[
\frac{\partial}{\partial \tau} f(j\Delta S, n\Delta \tau) \approx \frac{f^n_{j+1} - f^n_j}{k} =: D^+_k f^n_j; \quad (3.10)
\]
• The backward difference
\[\frac{\partial}{\partial S} f(j\Delta S, n\Delta \tau) \approx \frac{f^n_j - f^{n-1}_j}{h} =: D^-_hf^n_j, \quad (3.11)\]
\[\frac{\partial}{\partial \tau} f(j\Delta S, n\Delta \tau) \approx \frac{f^n_j - f^{n-1}_j}{k} =: D^-_kf^n_j; \quad (3.12)\]

• The central (or symmetric) difference
\[\frac{\partial}{\partial S} f(j\Delta S, n\Delta \tau) \approx \frac{f^{n+1}_j - f^n_j}{2h} =: D^0hf^n_j, \quad (3.13)\]
\[\frac{\partial}{\partial \tau} f(j\Delta S, n\Delta \tau) \approx \frac{f^{n+1}_j - f^n_j}{2k} =: D^0kf^n_j. \quad (3.14)\]

We can also approximate the second derivative
\[\frac{\partial^2}{\partial S^2} f(j\Delta S, n\Delta \tau) \approx \frac{f^{n+1}_j - 2f^n_j + f^{n-1}_j}{h^2} =: D^2_hf^n_j = D^+_hf^n_j - D^-_hf^n_j. \quad (3.15)\]

Then according to the type of the option we apply the boundary conditions.

Further we describe some classical finite difference methods.

### 3.2.1 The theta Method

The \(\theta\)-method represents a whole class of the finite difference methods. In this method the spatial derivative is approximated at the time \(\tau_j + \theta \Delta \tau = \tau_j + \theta k\). The start and the end of the time interval are a weighted average of the approximations, see \([\text{II}]\):

\[\frac{f^{n+1}_j - f^n_j}{k} = \frac{1}{\theta} \left( (1 - \theta) \frac{f^{n+1}_{j-1} - 2f^n_j + f^{n-1}_j}{h^2} + \theta \frac{f^{n+1}_{j+1} - 2f^n_{j+1} + f^{n+1}_{j+1}}{h^2} \right), \quad (3.16)\]

where

I - central difference at \(\tau_{n+1/2}\),

II - central difference at \(\tau_n + \theta k\).

For the appropriately chosen value of the implicitness parameter \(\theta \in [0, 1]\), we obtain different methods.
• for $\theta = 0$, the explicit method (see 3.2.2),
• for $\theta = 1$, the implicit method (see 3.2.3),
• for $\theta = \frac{1}{2}$, the Crank-Nicolson method (see 3.2.4).

3.2.2 The Explicit Method

If we approximate in (3.2) the derivative with respect to $S$ by a central difference and the derivative with respect to $t$ by a forward difference, we obtain the following set of equations, [4],

$$
\frac{f_j^{n+1} - f_j^n}{k} = \frac{1}{2} \sigma^2 j^2 h^2 \frac{f_{j+1}^{n+1} - 2f_j^{n+1} + f_{j-1}^n}{h^2} + r j h \frac{f_{j+1}^{n+1} - f_{j-1}^n}{2h} - rf_j^n,
$$

(3.17)
since $S_j = jh$. Let $n = 0$, we have only one unknown quantity, $f_j^1$. Applying the boundary conditions the equations can be solved forward in time. Rewriting the equations, we get an explicit scheme

$$
f_j^{n+1} = a_j^* f_{j-1}^n + b_j^* f_j^n + c_j^* f_{j+1}^n
\quad n = 0, 1, 2, \ldots, N - 1; \quad j = 1, 2, \ldots, M - 1,
$$

(3.18)

where

$$
a_j^* = \frac{1}{2} k (\sigma^2 j^2 - r j),
$$

$$
b_j^* = 1 - k (\sigma^2 j^2 - r),
$$

$$
c_j^* = \frac{1}{2} k (\sigma^2 j^2 + r j).
$$

(3.19)

Figure 3.2: The stencil of the explicit method for the Black-Scholes equation (3.2).
3.2.3 The Implicit Method

The implicit method is obtained by using a backward difference to approximate the partial derivative with respect to \( t \) and a central difference to approximate the partial derivative with respect to \( S \). The difference equations is following

\[
\frac{f_j^{n+1} - f_j^n}{\Delta t} = \frac{1}{2} \sigma^2 j^2 h^2 \frac{f_j^{n+1} - f_j^{n-1}}{h^2} + r j h \frac{f_j^{n+1} - f_j^{n-1}}{2h} - r f_j^{n+1}. \tag{3.20}
\]

We can rewrite (3.20) (for \( j = 1, 2, \ldots, M-1 \) and \( n = 0, 1, \ldots, N-1 \) as

\[
a_j f_{j-1}^{n+1} + b_j f_j^{n+1} + c_j f_{j+1}^{n+1} = f_j^n, \tag{3.21}
\]

where, for each \( j \),

\[
a_j = \frac{1}{2} r j k - \frac{1}{2} \sigma^2 j^2 k, \]

\[
b_j = 1 + \frac{\sigma^2 j^2 k}{r}, \tag{3.22}
\]

\[
c_j = -\frac{1}{2} r j k - \frac{1}{2} \sigma^2 j^2 k.
\]

For each time level we have \( M - 1 \) equations for \( M - 1 \) unknowns. We have to solve a sequence of systems of linear equations for \( n = 0, 1, 2, \ldots, N-1 \) by going forward in time. The tridiagonal system looks like

\[
\begin{bmatrix}
    b_1 & c_1 \\
    a_2 & b_2 & c_2 \\
    & \ddots & \ddots & \ddots \\
    & & a_{M-2} & b_{M-2} & c_{M-2} \\
    & & & a_{M-1} & b_{M-1}
\end{bmatrix}
\begin{bmatrix}
    f_1^{n+1} \\
    f_2^{n+1} \\
    \vdots \\
    f_{M-2}^{n+1} \\
    f_{M-1}^{n+1}
\end{bmatrix}
= 
\begin{bmatrix}
    f_1^n \\
    f_2^n \\
    \vdots \\
    f_{M-2}^n \\
    f_{M-1}^n
\end{bmatrix}
- 
\begin{bmatrix}
    a_1 f_0^{n+1} \\
    0 \\
    \vdots \\
    0 \\
    c_{M-1} f_M^{n+1}
\end{bmatrix}
\]

and can be solved efficiently with the Thomas algorithm [12].
3.2.4 The Crank-Nicolson Method

The Crank-Nicolson method is a combination of the explicit and the implicit methods. We apply this idea to the Black-Scholes equation and obtain the following grid equations

\[
\frac{f_j^{n+1} - f_j^n}{k} = \frac{r j h}{2} \left( \frac{f_{j+1}^{n+1} - f_{j-1}^{n+1}}{2h} \right) + \frac{r j h}{2} \left( \frac{f_{j+1}^{n} - f_{j-1}^{n}}{2h} \right) \\
+ \frac{\sigma^2 j^2 h^2}{4} \left( \frac{f_{j+1}^{n+1} - 2f_j^{n+1} + f_{j-1}^{n+1}}{h^2} \right) \\
+ \frac{\sigma^2 j^2 h^2}{4} \left( \frac{f_{j+1}^{n} - 2f_j^{n} + f_{j-1}^{n}}{h^2} \right) \\
- \frac{r}{2} f_j^{n+1} - \frac{r}{2} f_j^n.
\]

(3.24)

for \( j = 1, 2, \ldots, M - 1 \) and \( n = 0, 1, \ldots, N - 1 \).

These equations may be written as

\[-\alpha_j f_j^{n-1} + (1 - \beta_j) f_j^{n+1} - \gamma_j f_{j+1}^{n+1} = \alpha_j f_{j-1}^{n} + (1 + \beta_j) f_j^{n} - \gamma_j f_{j+1}^{n},
\]

where

\[
\alpha_j = \frac{k}{4}(\sigma^2 j^2 - rj), \\
\beta_j = -\frac{k}{2}(\sigma^2 j^2 + r), \\
\gamma_j = \frac{k}{4}(\sigma^2 j^2 + rj).
\]

(3.25)

Applying the boundary conditions, we have to solve the tridiagonal system of equations

\[M_1 f^{n-1} = M_2 f^n,
\]
where
\[ M_1 = \begin{bmatrix}
1 - \beta_1 & -\gamma_1 \\
-\alpha_2 & 1 - \beta_2 & -\gamma_2 \\
& \ddots & \ddots & \ddots \\
& -\alpha_{M-2} & 1 - \beta_{M-2} & -\gamma_{M-2} \\
& & -\alpha_{M-1} & 1 - \beta_{M-1}
\end{bmatrix}, \]
\[ M_2 = \begin{bmatrix}
1 + \beta_1 & \gamma_1 \\
\alpha_2 & 1 + \beta_2 & \gamma_2 \\
& \ddots & \ddots & \ddots \\
& \alpha_{M-2} & 1 + \beta_{M-2} & \gamma_{M-2} \\
& & \alpha_{M-1} & 1 + \beta_{M-1}
\end{bmatrix}, \] (3.26)
\[ f^n = [f_1^n, f_2^n, \ldots, f_{M-1}^n]^\top. \]

Figure 3.4: The stencil of the Crank-Nicolson method for the Black-Scholes equation (3.2).
Chapter 4

The Exponentially Fitted Scheme

At the beginning let us consider the linear two-point boundary value problem (TPBVP)

\[
\sigma \frac{d^2u}{dx^2} + 2 \frac{du}{dx} = 0 \quad \text{in} \quad (0, 1),
\]
\[
u(0) = 1, \quad u(1) = 0,
\]

with the assumption that \( \sigma \) is a positive constant. Equation (4.1) contains of a first-order derivative with respect to \( x \) and a second-order derivative also with respect to \( x \) so now we replace these derivatives by the corresponding centered finite differences, where \( u_j \approx u(j \Delta x) \)

\[
\sigma D^+ D^- U_j + 2 D^0 U_j = 0,
\]
\[
U_0 = 1, \quad U_J = 0.
\]

Equation (4.2) is equivalent to the following equation

\[
\sigma \left( \frac{U_{j+1} - 2U_j + U_{j-1}}{h^2} \right) + 2 \left( \frac{U_{j+1} - U_{j-1}}{2h} \right) = 0, \quad j = 1, 2, \ldots, J - 1,
\]
\[
U_0 = 1, \quad U_J = 0.
\]

(4.3)

According to such finite difference scheme our aim is to find a mesh function \( U_j, j = 1, \ldots, J - 1 \) that solves (4.3). We rewrite (4.3) in the following form

\[
\left( 1 + \frac{\sigma}{h} \right) U_{j+1} - 2 \frac{\sigma}{h} U_j + \left( \frac{\sigma}{h} - 1 \right) U_{j-1} = 0.
\]

(4.4)

Equation (4.4) is a homogeneous second-order difference equation with the constant coefficients which have solutions of the form

\[
U_j = \lambda^j.
\]

(4.5)
We insert (4.5) into (4.4) yielding
\[
\left(1 + \frac{\sigma}{h}\right) \lambda^2 - 2\frac{\sigma}{h}\lambda + \left(\frac{\sigma}{h} - 1\right) = 0,
\]
i.e.
\[
\lambda^2 - \frac{2\sigma}{(1 + \frac{\sigma}{h})} \lambda + \frac{\left(\frac{\sigma}{h} - 1\right)}{(1 + \frac{\sigma}{h})} = 0. \tag{4.6}
\]
We obtain a quadratic equation (4.6) so now we must solve this
\[
\Delta := \frac{4}{(1 + \frac{\sigma}{h})^2}
\]
Due to, $\Delta > 0$ we have two solutions $\lambda_1$ or $\lambda_2$:
\[
\lambda_1 = 1, \quad \lambda_2 = \frac{1 - \frac{h}{\sigma}}{1 + \frac{\sigma}{h}}.
\]
The general solution to (4.4) is a linear combination
\[
U_j = a\lambda^j + b = a \frac{1 - \frac{h}{\sigma}}{1 + \frac{\sigma}{h}} + b. \tag{4.7}
\]
Variables $a$ and $b$ are obtained from the boundary conditions (4.3)
- $U_0 = 1$ at $j = 0$,
  \[
a + b = 1, \tag{4.8}
\]
- $U_J = 0$ at $j = J$,
  \[
a\lambda^J + b = 0. \tag{4.9}
\]
Solving the system of equations (4.7) and (4.8) we obtain
\[
a = \frac{1}{1 - \lambda^J}, \quad b = \frac{-\lambda^J}{1 - \lambda^J}. \tag{4.10}
\]
Replacing (4.9) in (4.6) we have the solution
\[
U_j = \frac{\lambda^j - \lambda^J}{1 - \lambda^J}, \quad \text{where} \quad \lambda = \frac{1 - \frac{h}{\sigma}}{1 + \frac{\sigma}{h}}. \tag{4.11}
\]
Let us note that the exact solution of (4.1) is given by
\[
\frac{e^{-2x/\sigma} - e^{-2/\sigma}}{1 - e^{-2/\sigma}}. \tag{4.12}
\]
With the assumption that $\sigma < h$ in (4.10) we can predict that $\lambda < 0$, so depending on the value $j$, $\lambda^j$ is positive or negative. In the limit case $\sigma \to 0$ we have
\[
\lim_{\sigma \to 0} U_j = \frac{(-1)^j + 1}{2}.
\]
Therefore for all $\sigma < h$, $U_j$ oscillates in a bounded region. Hence, we conclude that, the centered finite difference scheme is inadequate for the numerical solution of (4.1) with the assumption $\sigma < h$. A very important remark here is that the grid solutions certainly depend on the chosen discretization.

The exponentially fitted schemes is one of the new class of the robust difference schemes. The properties of such schemes are a good convergence and the fact that they are stable. Furthermore, they do not produce spurious oscillations, as the centered finite difference scheme before. We wish to use the following scheme to solve the Black-Scholes equation. Firstly, let us consider the slightly more general TPBVP
\[
\sigma \frac{d^2u}{dx^2} + \mu \frac{du}{dx} = 0 \quad \text{in} \quad (A, B),
\]
\[
u(A) = \beta_0, \quad u(B) = \beta_1,
\]
with the exact solution
\[
u = \frac{(\beta_0 - \beta_1)e^{-\frac{\mu x}{2}} - \beta_0 e^{-\frac{\mu B}{2}} + \beta_1 e^{-\frac{\mu A}{2}}}{e^{-\frac{\mu A}{2}} - e^{-\frac{\mu B}{2}}}
\] (4.14)
where $\sigma$ and $\mu$ are positive constants. We apply the standard centered difference scheme for the approximation (4.13)

$$\sigma \rho D^2 U_j + \mu D^0 U_j = 0, \quad j = 1, \ldots, J - 1$$

$$U_0 = \beta_0, \quad U_J = \beta_1,$$

with the discrete solution

$$u_j = \frac{(\beta_0 - \beta_1) \lambda^j - \beta_0 \lambda^J + \beta_1}{1 - \lambda^J},$$

where $\rho$ denotes the so-called fitting factor and $\lambda = \frac{2\sigma - \mu h}{2\rho \sigma + \mu h}$.

The fitting factor was introduced to make the solution (4.14) and (4.16) identical at the mesh-points. As it follows from the form of solutions (4.14) and (4.16), the sufficient condition for this purpose is

$$e^{-\frac{\sigma}{\mu} x} = \lambda^j.$$

According to that $x = jh$ and $\lambda = \frac{2\sigma - \mu h}{2\rho \sigma + \mu h}$, we rewrite above equation in the following form

$$e^{-\frac{\sigma}{\mu} h} = \frac{2\rho \sigma - \mu h}{2\rho \sigma + \mu h}.$$

We can derive a value $\rho$ from this equation

$$\rho = \frac{\mu h}{2\sigma} \coth \frac{\mu h}{2\sigma},$$

where $\coth x$ represents the hyperbolic cotangent function defined by

$$\coth x = \frac{e^x + e^{-x}}{e^x - e^{-x}} = \frac{e^{2x} + 1}{e^{2x} - 1}.$$

We can use the fitting factor to construct a fitted difference scheme for the more general TPBVPs. To do so, let us consider

$$\sigma(x) \frac{d^2 u}{dx^2} + \mu(x) \frac{du}{dx} + b(x) u = f(x),$$

$$u(A) = \beta_0, \quad u(B) = \beta_1,$$

where $\sigma$, $\mu$ and $b$ are given continuous functions satisfying the conditions

$$\sigma(x) \geq 0, \quad \mu(x) \geq \alpha > 0, \quad b(x) \leq 0 \quad \text{for} \quad x \in (A, B)$$
The approximation (4.15) by a fitted difference scheme is the following one

\[ \gamma_j^h D^2 U_j + \mu_j D^0 U_j + b_j U_j = f_j, \quad j = 1, \cdots, J - 1, \]

\[ U_0 = \beta_0, \quad U_J = \beta_1, \]

(4.18)

where

\[ \gamma_j^h = \frac{\mu_j h}{2} \coth \frac{\mu_j h}{2\sigma_j}, \]

\[ \sigma_j = \sigma(x_j), \quad \mu_j = \mu(x_j), \quad b_j = b(x_j). \]

(4.19)

Now we can write the suitable results.

**Theorem 4 (Uniform Stability).** [2].

The solution of the scheme (4.18) is uniformly stable, that is

\[ |U_j| \leq |\beta_0| + |\beta_1| + \frac{1}{\alpha} \max_{k=1, \cdots, J} |f_k|, \quad j = 1, \cdots, J - 1 \]

Furthermore, the scheme (4.18) is monotone in the sense that the matrix representation of (4.18)

\[ AU = F, \]

where \( U = (U_1, \cdots, U_{J-1})^\top \), \( F = (f_1, \cdots, f_{J-1})^\top \) and

\[ A = \begin{pmatrix}
0 & \cdots & a_{j,j+1} \\
\cdots & \ddots & \cdots \\
a_{j,j} & \cdots & a_{j,j+1} \\
0 & \cdots & a_{j,j-1}
\end{pmatrix} \]
Chapter 4. The Exponentially Fitted Scheme

\[ a_{j,j-1} = \frac{\gamma_j h}{h^2} - \frac{\mu_j}{2h} > 0, \quad \text{always}, \]

\[ a_{j,j} = -\frac{2\gamma_j}{h^2} + b_j < 0, \quad \text{always}, \]

\[ a_{j,j+1} = \frac{\gamma_j}{h^2} + \frac{\mu_j}{2h} > 0, \quad \text{always}, \quad (4.20) \]

produces positive solutions from the positive input.

**Theorem 5 (Uniform Convergence)** \[5\]

Let \( u \) and \( U \) be the solutions of (4.17) and (4.18), respectively. Then

\[ |u(x_j) - U_j| \leq Mh, \]

where \( M \) is a positive constant that is independent of \( h \) and \( \sigma \).

The conclusion is that the fitted scheme (4.18) is stable, convergent and does not produce any oscillations for all parameters under consideration.
Chapter 5

The Box Method

To reduce sensitivities of a portfolio or an option to the movements of the underlying asset we need to approximate the derivatives of the solution of the Black-Scholes equation, which are called "greeks". The most important ones are

- Delta $\Delta := \frac{\partial V}{\partial S}$,
- Gamma $\Gamma := \frac{\partial^2 V}{\partial S^2}$.

Difference quotient approximations to $\Delta$ and $\Gamma$ may give rise to problems when the stock price is close to the exercise price. In fact we know that the Crank-Nicolson scheme yields oscillating solutions for the values $\Delta$ and $\Gamma$ while no such oscillations occur with the fitted schemes, when the option is at the money.

They were obtained by the following formulas

$$\Delta \approx \frac{(V_{j+1}^n - V_{j-1}^n)}{2h}, \quad h = S_{j+1} - S_j,$$
$$\Gamma \approx \frac{(V_{j+1}^n - 2V_j^n + V_{j-1}^n)}{h^2},$$

(5.1)

where $V_j^n$ is the value of the option at the grid point $S_j$ and at the time level $\tau_n$.

The problem to approximate the greeks in the case of the Neumann boundary conditions becomes more challenging than in the case of the Dirichlet boundary conditions, because we have to approximate the one-sided derivatives at the boundaries. The most general case we consider is

$$\frac{\partial V}{\partial \tau} = \sigma \frac{\partial^2 V}{\partial S^2} + \mu \frac{\partial V}{\partial S} + bV - f, \quad (S, \tau) \in (A, B) \times (0, T),$$

(5.2)

with the initial condition $V(S, 0) = g(S), \quad S \in (A, B)$,
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and supplied with the Robin-type boundary conditions
\[\begin{align*}
\alpha_0 V(A, \tau) + \alpha_1 \sigma \frac{\partial V}{\partial S}(A, \tau) &= g_0(\tau), \\
\beta_0 V(B, \tau) + \beta_1 \sigma \frac{\partial V}{\partial S}(B, \tau) &= g_1(\tau).
\end{align*}\] (5.3)

Let us rewrite (5.2) as a system of the first-order equations as follows
\[\begin{align*}
\sigma \frac{\partial V}{\partial \tau} &= \sigma \frac{\partial \delta}{\partial S} + \mu \delta + \sigma b V - \sigma f, \\
setting \quad \sigma \frac{\partial V}{\partial S} &= \delta,
\end{align*}\]
\[\begin{align*}
V(S, 0) &= g(S), \\
\alpha_0 V(A, \tau) + \alpha_1 \delta(A, \tau) &= g_0(\tau), \\
\beta_0 V(B, \tau) + \beta_1 \delta(B, \tau) &= g_1(\tau).
\end{align*}\] (5.4)

Our goal is to approximate \(V\) and \(\delta\) and after this procedure we will have the values for both the option price and its delta. A scheme for approximating (5.4) was proposed by Keller in 1970, [13]. Assume that \(\sigma\) and \(\mu\) are constants, \(f \equiv 0, b \equiv 0\) and \(\alpha_0 = \beta_0 = 1, \alpha_1 = \beta_1 = 0\). Define the quantities
\[\begin{align*}
S_{j+1/2}^n &= \frac{1}{2} (S_j^n + S_{j+1}^n), \\
\tau_{n+1/2} &= \frac{1}{2} (\tau_n + \tau_{n+1}), \\
V_{j+1/2}^n &= \frac{1}{2} (V_j^n + V_{j+1}^n), \\
V_{j+1}^{n+1/2} &= \frac{1}{2} (V_j^n + V_{j+1}^{n+1}), \\
D^+_h V_j^n &= (V_{j+1}^n - V_j^n) / h, \\
D^+_k V_j^n &= (V_{j+1}^n - V_j^n) / k.
\end{align*}\]

The so-called, Keller box scheme is defined as
\[\begin{align*}
&-\sigma D^+_\tau V_{j+1/2}^n + \sigma D^+_x \delta_{j+1/2}^{n+1/2} + \mu D^+_x \delta_{j+1/2}^{n+1/2} = 0, \\
\sigma D^+_h V_j^n &= \delta_{j+1/2}^n.
\end{align*}\] (5.5)
The main advantages of this scheme are

- it has the second order accuracy in time and space,
- it is unconditionally stable,
- it requests the data and the coefficients in the equation only to be piecewise smooth
- it is $A$-stable (a method is called $A$-stable, if all numerical solutions tend to zero, as $n \to \infty$, when the method applied with the fixed positive $h$ to any differential equation $dx/dt = qx$, where $q$ is a complex constant with negative real part), \cite{14}.

Discretising (5.4) in the space direction first by the centered difference and then later in time we obtain the motivation for (5.5). After spatial discretization we have

\begin{align}
  -\sigma dV_{j+1/2}^{t} + \sigma D_{h}^{+} \delta_j + \mu \delta_{j+1/2} &= 0, \quad j = 0, \cdots, J - 1, \quad (5.7) \\
  \sigma D_{h}^{+} V_j &= \delta_{j+1/2}, \quad (5.8) \\
  V_j(0) &= g(S_j), \quad j = 0, 1, \cdots, J, \quad \text{and} \quad (5.9)
\end{align}

We now combine the terms in (5.7) to give a system of ODEs which will be shown to be $A$-stable.

**Lemma 2** There are the following relations for difference operators $D_{h}^{0}$, $D_{h}^{+}$ and $D_{h}^{-}$

\begin{align}
  (a) \quad D_{h}^{0} \delta_j &= \sigma D_{h}^{+} D_{h}^{-} V_j \\
  (b) \quad \delta_{j+1/2} + \delta_{j-1/2} &= 2\sigma D_{h}^{0} V_j \\
  (c) \quad D_{h}^{+} \delta_j + D_{h}^{-} \delta_{j-1} &= 2D_{h}^{0} \delta_j
\end{align}

We now write (5.7) at two consecutive mesh points $S_j$ and $S_{j-1}$ and add the results to give

\begin{align}
  -\sigma \frac{d}{dt} (V_{j+1/2} + V_{j-1/2}) + \sigma \left(D_{h}^{+} \delta_j + D_{h}^{+} \delta_{j-1}\right) + \mu \left(\delta_{j+1/2} + \delta_{j-1/2}\right) &= 0
\end{align}
Using Lemma 2 the above equation can be written as

$$\frac{1}{4} \frac{d}{dt} (V_{j-1} + 2V_j + V_{j+1}) + \sigma D_h^+ D^+_h V_j + \mu D_h^0 V_j = 0.$$  

Finally, we can rewrite this equation as a system of ODEs

$$C \frac{dU}{d\tau} + AU = 0, \quad U(0) = U_0,$$  

(5.10)

where

$$C = \begin{pmatrix} 2 & 1 & 0 \\ 1 & \ddots & \ddots \\ \ddots & \ddots & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

and A is the known matrix. If we will use fitting factor in equation (5.10) we see that this system is an unconditionally stable. In conclusion, there are the advantages of the box scheme for risk [5]:

- it has the second order accuracy in time and space
- it can deal with the discontinuous coefficients
- it can deal with the Neumann and Robin type boundary conditions
- it solves a lot of oscillation problems
Exotic options are widely used today by banks, corporations and institutional investors in their management of risk [10]. The standard call and put options are not suitable to hedge some types of risk and this is the main reason to use exotic options.

The barrier options lie in the class of path-dependent options, which come in various flavours and forms, but their key characteristic is that these types of options are either initiated or exterminated upon reaching a certain barrier level. There are two main types of the barrier options:

- **out-options (knock-out)**, they only pay off if a level is not reached
- **in-options (knock-in)**, they only pay off as long as a level is reached before the expiration date

Depending on the position of the barrier related to the initial value of the underlying asset there are another characteristics of barrier option:

- **up options**, if the barrier is above the initial price,
- **down options**, if the barrier is below the initial price.

Sometimes a rebate is paid if the barrier level is reached. The barrier options can also have an opportunity for an early exercise, like the American style options.

The barrier options are used in the foreign exchanges (FX), the interest rate and option markets. They are used by hedgers to protect the price of the underlying asset above or below some advanced level or by speculators to obtain a somehow less expensive directional play on an underlying asset. In
Using the properties discretize (6.1). First we rewrite equation (6.1) in the conservative form.

In equation (6.2) the flux and source terms are determined as (see [6])

Further integration with respect to \( \tau \) and \( S \) to the order the discretization in time \( 0 = \tau_0 < \tau_1 < \cdots < \tau_N = T \) and in the stock prices \( 0 = S_0 < S_1 < \cdots < S_J = S_{\text{max}} \) gives us the discrete version of (6.1)

where \( V_{jn}^{n+1} \) denotes the approximate value of the option at node \( n+1 \), \( \Delta \tau \) is the time step size, \( F_{jn+1/2} \) and \( F_{jn-1/2} \) are so-called flux terms, \( f_j \) is a a source term and \( \theta \) is a weighting factor.

In equation (6.2) the flux and source terms are determined as (see [6])

where \( \Delta S_j = \frac{1}{2} (S_{j+1} - S_{j-1}) \), \( \Delta S_{j+1/2} = S_{j+1} - S_j \), and

\[
\theta_f^n (V^n_{j+1}) = (-r) V^n_{j+1}.
\]
Note that the corresponding definitions apply at time level $n$. $V_{j-1/2}^{n+1}$ and $V_{j+1/2}^{n+1}$ are the remaining terms from the $(rS\partial V/\partial S)$ term in (6.1). And it equals
\[ V_{j+1/2}^{n+1} = \frac{V_{j+1}^{n+1} + V_{j}^{n+1}}{2}. \] (6.6)

The weighting factor $\theta$ defines the type of the selected scheme:
- $\theta = 0$ - fully explicit
- $\theta = \frac{1}{2}$ - Crank-Nicholson
- $\theta = 1$ - fully implicit

As we know, the explicit method can be unstable, if the time step size is not sufficiently small compared with the stock grid spacing, the Crank-Nicholson method and fully implicit method both are unconditionally stable. Both the fully explicit and the implicit methods are first-order accuracy in time. On the other hand the Crank-Nicholson method is second-order accuracy in time. But the Crank-Nicholson approach may produce large and spurious numerical oscillations and gives very poor estimates of the option value $S$.

The following conditions were used to prevent these spurious oscillations (and were described in [6]). We should take
\[ \Delta S_{j-1/2} < \frac{\sigma^2 S_j^2}{r} \] (6.7)

and
\[ \frac{1}{(1 - \theta)\Delta \tau} > \frac{\sigma^2 S_j^2}{2} \left( \frac{1}{\Delta S_{j-1/2}^2} + \frac{1}{\Delta S_{j+1/2}^2} \right) + r. \] (6.8)

The condition (6.7) is satisfied for $\sigma$ and $r$ away from $S_j$. In the case of the fully implicit method ($\theta = 1$) the condition (6.8) is trivially satisfied. This condition restricts the time step size as the function of the stock grid spacing. Even though the Crank-Nicholson approach is unconditionally stable, it can produce spurious oscillations unless the time step size is no more smaller than twice that required for the fully explicit method to be stable. Above reasons mean that it is preferable to use the fully implicit method for option pricing [6].

Depending on the nature of the constraint we have to choose a suitable strategy for imposing an algebraic constraint on the solution. We will consider an example of a discretely monitored down-and-out option without any rebate. At first we compute $V^{n+1}$ and then apply the constraint
\[ V_j^{n+1} = \begin{cases} 0, & \text{if } S_j \leq h (t^{n+1}, \alpha^{n+1}) H, \\ V_j^{n+1}, & \text{otherwise}, \end{cases} \] (6.9)
where $H$ is the initial level of the barrier, $h$ is a function which allows the barrier to move over time, and $\alpha^{n+1}$ is an arbitrary parameter. For the constant barrier $h$ is equal to one.

In the case of the early exercise opportunity we solve the following differential equation

$$
\frac{\Phi_j^{n+1} - V_j^n}{\Delta \tau} = F_{j-1/2}^{n+1} (V_{j-1}^{n+1}, V_j^{n+1}) - F_{j+1/2}^{n+1} (V_j^{n+1}, V_{j+1}^{n+1}) + f_j^{n+1} (V_j^{n+1})
$$

$$
V_j^{n+1} = \max \left( \Phi_j^{n+1}, S_j - K, 0 \right).
$$

(6.10)

For the down-and-out barrier options with the American style we apply the following conditions

$$
V_j^{n+1} = \begin{cases} 
0, & \text{if } S_j \leq h(t^{n+1}, \alpha^{n+1}) H, \\
\max \left( \Phi_j^{n+1}, S_j - K, 0 \right), & \text{otherwise}.
\end{cases}
$$

(6.11)

This notation allows us to use as constant barriers as discretely ones. It is rather important feature because in real life discretely monitored barriers are applied (daily or weekly).
Chapter 7

The Fitted Finite Volume Scheme

In this chapter we consider a fitted finite volume scheme for the discretization of the Black-Scholes equation. Let us rewrite this equation in the known form, which we used in the previous chapters

\[
\frac{\partial V}{\partial \tau} = \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV,
\] (7.1)

where we employed the time reversal \( \tau = T - t \).

Let us first consider the boundary conditions for equation (7.1). At \( \tau = 0 \), \( V(S, 0) \) is the specified contract payoff. At \( S = 0 \), equation (7.1) reduces to

\[
\frac{\partial V}{\partial \tau} = -rV.
\] (7.2)

For \( S \to \infty \), we have asymptotically the Dirichlet condition

\[
V(S, \tau) = f(S, \tau),
\] (7.3)

where \( f(S, \tau) \) can be determined by financial reasoning. For the computation, we confine the asset region \((0, +\infty)\) to \((0, S_{\text{max}})\), where \( S_{\text{max}} \) is sufficiently large to ensure the accuracy of the solution (7.1). Thus, (7.3) becomes

\[
V(S_{\text{max}}, \tau) = f(S_{\text{max}}, \tau).
\] (7.4)

This localized problem can be solved by the fitted finite volume method. Now, let us consider the discretization of (7.1). We first start with the spatial discretization. For this purpose we transform (7.1) into the following conservative form

\[
\partial_\tau V = \partial_S \left( aS^2V_S + bSV \right) - cV,
\] (7.5)
where
\[ a = \frac{\sigma^2}{2}, \quad b = r + \sigma^2, \quad c = r + b. \] (7.6)

Now, we divide \( I = (0, S_{\text{max}}) \) into \( N \) sub-intervals
\[ I_j = (S_j, S_{j+1}), \quad j = 0, \ldots, N-1, \]
with \( 0 = S_0 < S_1 < \cdots < S_N = S_{\text{max}} \). For each \( j = 0, \ldots, N-1 \), define \( h_j = S_{j+1} - S_j \). Also, we let \( S_{j-1/2} = (S_{j-1} + S_j)/2 \) and \( S_{j+1/2} = (S_j + S_{j+1})/2 \) for each \( j = 2, \ldots, N \).

For each \( j = 1, \ldots, N-1 \), integrating (7.5) over \( J_j = (S_{j-1/2}, S_{j+1/2}) \), we obtain
\[ \int_{J_j} \partial_S V dS = \left. \partial_S \left( aS^2V + bSV \right) \right|_{S_{j-1/2}}^{S_{j+1/2}} - \int_{J_j} cV dS. \] (7.7)

Applying the one-point quadrature rule to the first and the last terms in (7.5) we obtain
\[ \partial_j V_j \ell_j = S_{j+1/2} \rho(V)|_{S_{j+1/2}} + S_{j-1/2} \rho(V)|_{S_{j-1/2}} - cV_j \ell_j, \] (7.8)
for \( j = 0, \ldots, N-1 \), where \( \ell_j = S_{j+1/2} - S_{j-1/2} \) and \( \rho(V) \) is the flux term defined by
\[ \rho(V) := aSV' + bV. \] (7.9)

Let us take a look at the next boundary problem to define an approximation of the \( \rho(V) \) above at the mid-point, on \( S_{j+1/2} \), of the interval \( I_j \)
\[ (aSV' + bV)' = 0, \quad S \in I_j, \]
\[ V(S_j) = V_j, \quad V(S_{j+1}) = V_{j+1}. \] (7.10)

Integrating this equation yields the first-order linear equation
\[ \rho_j(V) = aSV' + bV = C_1, \]
where \( C_1 \) denotes an additive constant. The integrating factor of this linear equation is \( S^\nu \) and the analytic solution to above equation is
\[ V = S^{-\nu} \left( \int S^\nu C_1 aS dS + C_2 \right) = \frac{C_1}{b} + C_2 S^{-\nu}, \]
where \( \nu = b/a \) and \( C_2 \) is also an additive constant.

Applying the boundary conditions to this equation we obtain
\[ V_j = \frac{C_1}{b} + C_2 S_j^{-\nu} \quad \text{and} \quad V_{j+1} = \frac{C_1}{b} + C_2 S_{j+1}^{-\nu}. \]
The solution of this linear system is
\[ \rho_j(V) = C_1 = b \frac{S_{j+1}^{v_j} V_{j+1} - S_j^{v_j} V_j}{S_{j+1}^{v_j} - S_j^{v_j}}. \] (7.11)

By the same way we define the approximation of \( \rho(V) \) at \( S_{j-1/2} \).

According to the fact, that such analysis does not apply to the approximation to \( \rho(V) \) on \( I_0 = (0, S_1) \), we reconsider (7.10) with an extra degree of freedom
\[ (aSV'' + bV)' = C, \quad S \in I_0, \]
\[ V(S_0) = V_0, \quad V(S_1) = V_1. \] (7.12)

This problem has the solution
\[ \rho_0(V) = (aSV' + bV)_{S_{1/2}} = \frac{1}{2} [(a + b)V_1 - (aSV' + b)V_0], \]
\[ V = V_0 + (V_1 - V_0)S/S_1, \quad S \in I_0 = (0, S_1). \] (7.13)

We derive an approximation to \( \rho(V) \) by \( \rho_h(V) \), using (7.11) and (7.13) satisfying
\[ \rho_h(V) = \rho_j(V), \quad \text{if} \quad V \in I_j \]
for \( j = 0, \ldots, N - 1 \).

Substituting (7.11) and (7.13) into (7.7), we obtain
\[ \frac{\partial V}{\partial \tau} = \alpha_j V_{j-1} + \gamma_j V_j + \beta_j V_{j+1}, \] (7.14)
for \( j = 1, \ldots, N - 1 \), where
\[ \alpha_1 = \frac{S_1}{\delta_1} (a + b), \]
\[ \beta_1 = \frac{bS_{1/2}^{v_1} S_1^{v_1}}{(S_1^{v_1} - S_1^{v_1}) \delta_1}, \] (7.15)
\[ \gamma_1 = -\frac{S_1}{\delta_1} (a + b) - \frac{bS_{1/2}^{v_1} S_1^{v_1}}{(S_1^{v_1} - S_1^{v_1}) \delta_1} - c, \]
and
\[ \alpha_j = \frac{bS_{j-1/2}^{v_j} S_{j-1}^{v_j}}{(S_{j-1}^{v_j} - S_{j-1}^{v_j}) \delta_j}, \]
\[ \beta_j = \frac{bS_{j+1/2}^{v_j} S_{j+1}^{v_j}}{(S_{j+1}^{v_j} - S_{j+1}^{v_j}) \delta_j}, \] (7.16)
\[ \gamma_j = -\frac{bS_{j-1/2}^{v_j} S_{j-1}^{v_j}}{(S_{j-1}^{v_j} - S_{j-1}^{v_j}) \delta_j} - \frac{bS_{j+1/2}^{v_j} S_{j+1}^{v_j}}{(S_{j+1}^{v_j} - S_{j+1}^{v_j}) \delta_j} - c, \]
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for \( j = 2, \ldots, N - 1 \).

Thus, the semi-discretization form is

\[
\frac{\partial V_j}{\partial \tau} = \alpha_j V_{j-1} + \gamma_j V_j + \beta_j V_{j+1}.
\]  

(7.17)

Further we consider the time discretization of (7.17). Let \( \tau_j \in (0, T) \) is satisfying \( 0 = \tau_0 > \tau_1 > \cdots > \tau_M = T \), and \( \Delta \tau = \tau_n - \tau_{n-1} > 0 \), where \( M > 1 \) is a positive integer. In our case we apply the fully implicit scheme to (7.17), yielding

\[
\frac{V_j^{n+1} - V_j^n}{\Delta \tau} = \alpha_j V_{j-1}^{n+1} + \gamma_j V_j^{n+1} + \beta_j V_{j+1}^{n+1},
\]  

(7.18)

where \( V_j^n = V(S_j, \tau_n) \) derives the solution at node \( S_j \) and time level \( \tau_n \).

We can rewrite (7.18) in the matrix form

\[
[1 - \Delta \tau M] V^{n+1} = V^n + \Delta \tau R^n,
\]  

(7.19)

where

\[
V^n = [V_1^n, \ldots, V_{N-1}^n]^\top,
\]

\[
\sigma^n = [\sigma_1^n, \ldots, \sigma_{N-1}^n]^\top,
\]

\[
R^n = [\alpha_1 V_0^{n+1}, 0, \cdots, 0, \beta_{N-1} V_N^{n+1}]^\top
\]

and

\[
M = \begin{bmatrix}
\gamma_1 & \beta_1 & & \\
\alpha_2 & \gamma_2 & \beta_2 & \\
& \ddots & \ddots & \ddots \\
& & \alpha_{M-2} & \gamma_{M-2} & \beta_{M-2} \\
& & & \alpha_{M-1} & \beta_{M-1}
\end{bmatrix}
\]

Solving the system (7.19) forward in time we obtain the discrete approximation of the equation (7.1).
Chapter 8

Numerical results

This chapter illustrates numerical solutions to the European option pricing problem. The main purpose is to compare numerical option prices obtained by using the methods of finite differences with the corresponding exact solutions of the Black-Scholes equation. To this end, we created the programs that determine prices of options by using the presented exponentially fitted scheme and the finite volume scheme. The parameter values are provided by Ehrhardt and Mickens [11]. The same values are used for each of the methods mentioned above.

\begin{center}
\begin{tabular}{|l|c|}
\hline
interest rate & $r = 0.08$ \\
volatility & $\sigma = 0.3$ \\
strike price & $K = 100$ \\
initial price & $S = 100$ \\
maximum price & $S_{\text{max}} = 200$ \\
maturity date & $T = 0.5$ \\
\hline
\end{tabular}
\end{center}

The exponentially fitted scheme for the European put option

The first program finds the approximation of the European put option price by the exponentially fitted scheme (4.18). These results are presented in Figure 8.1. In Figure 8.2 the analytic solution obtained by using the Black-Scholes formula (3.2) is shown.
Chapter 8. Numerical Results

Figure 8.1: The approximate solution of the European put option by the exponentially fitted scheme with parameters $r = 0.08$, $\sigma = 0.3$, $K = 100$, $S = 100$, $S_{\text{max}} = 200$, $T = 0.5$.

Figure 8.2: The analytical solution of the Black-Scholes formula for the European put option with parameters $r = 0.08$, $\sigma = 0.3$, $K = 100$, $S = 100$, $S_{\text{max}} = 200$, $T = 0.5$. 
In Figure 8.3 the absolute error of the exponentially fitted scheme to the analytic Black-Scholes solution is shown in the semi-logarithmic plot. Looking at this plot we can note that the obtained results by using the exponentially fitted method are close to the results from the Black-Scholes formula. It means that the exponentially fitted scheme for the European put option is quite accurate.

Figure 8.3: The absolute error of the exponentially fitted scheme in case of the European put option with parameters $r = 0.08$, $\sigma = 0.3$, $K = 100$, $S = 100$, $S_{max} = 200$, $T = 0.5$.

The exponentially fitted scheme for the European call option

Figure 8.4 illustrates the approximation of the European call option price by the exponentially fitted scheme while in Figure 8.5 the exact solution of the Black-Scholes formula is presented.
Chapter 8. Numerical results

Figure 8.4: The approximate solution of the European call option by the exponentially fitted scheme with parameters $r = 0.08, \sigma = 0.3, K = 100, S = 100, S_{max} = 200, T = 0.5$.

Figure 8.5: The analytical solution of the Black-Scholes formula for the European call option with parameters $r = 0.08, \sigma = 0.3, K = 100, S = 100, S_{max} = 200, T = 0.5$. 
Figure 8.6: The absolute error of the exponentially fitted scheme in case of the European call option with parameters $r = 0.08, \sigma = 0.3, K = 100, S = 100, S_{\text{max}} = 200, T = 0.5$.

As shown in Figure 8.6, in case of the European call our results are not so accurate as for the European put option.

From the results of our comparisons shown above we conclude that the European put option prices obtained by numerical methods are significantly more accurate than the European call option prices.

The fitted finite volume scheme for the European put option

In Figure 8.7 we present the European put option prices obtained by the fitted finite volume approximation. Comparing our results for the fitted finite volume scheme with the solution of the Black-Scholes formula we can state that this method produces rather large errors.
Figure 8.7: The approximate solution of the European put option by the fitted finite volume scheme with parameters $r = 0.08$, $\sigma = 0.3$, $K = 100$, $S = 100$, $S_{\text{max}} = 200$, $T = 0.5$.

Figure 8.8: The absolute error of the fitted volume scheme in case of the European put option with parameters $r = 0.08$, $\sigma = 0.3$, $K = 100$, $S = 100$, $S_{\text{max}} = 200$, $T = 0.5$. 
The absolute errors of the option prices obtained by the finite volume scheme are illustrated in Figure 8.8.

The obtained results show that at least in our case the exponentially fitted method gives more appropriate approximations to the option prices than the fitted finite volume scheme.

As we can see further in Figure 8.11 and Figure 8.12 the finite volume scheme depends not so strong on the number of nodes in the grid. Figure 8.9 and Figure 8.10 illustrate that the results by using the exponentially fitted scheme dependence is also rather weak.

Figure 8.9: The approximate solution of the European put option by the exponentially fitted scheme with different grid settings, where $r = 0.08$, $\sigma = 0.3$, $K = 100$, $S = 100$, $S_{\text{max}} = 200$, $T = 0.5$. 
Figure 8.10: The absolute error of the exponentially fitted scheme in case of the European put option with different grid settings, where \( r = 0.08, \sigma = 0.3, K = 100, S = 100, S_{\text{max}} = 200, T = 0.5. \)

Figure 8.11: The approximate solution of the European put option by the fitted finite volume scheme with the different grid settings, where \( r = 0.08, \sigma = 0.3, K = 100, S = 100, S_{\text{max}} = 200, T = 0.5. \)
Figure 8.12: The absolute error of the fitted volume scheme in case of the European put option with the different grid settings, where \( r = 0.08 \), \( \sigma = 0.3 \), \( K = 100 \), \( S = 100 \), \( S_{\text{max}} = 200 \), \( T = 0.5 \).
Chapter 9

Conclusions and Outlook

The Black-Scholes partial differential equation can be solved numerically, for example, using methods of finite differences or analytically. This master thesis is devoted to description some novel finite volume difference methods to apply them for calculation of option prices. In this thesis we have described the main aspects of the Black-Scholes theory and the classical finite difference methods such as an explicit, an implicit and a Crank-Nicolson method.

The main purpose was to consider new classes of accurate and robust methods like the exponentially fitted scheme, the finite volume discretization, the Keller box scheme and the fitted finite volume scheme and try to apply this approaches for the option pricing. We have written MATLAB programs which use the exponentially fitted scheme and the fitted finite volume scheme for some put and call options. In the Chapter 8 we showed the obtained numerical results, described them and compared with the corresponding analytical solution of the Black-Scholes formula.

We will study in the future in more detail the possible occurrence of boundary layers in the Black-Scholes framework and especially how discrete numerical schemes can model this boundary behaviour adequately. For this purpose a deeper study of discrete asymptotic analysis will be needed.
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Appendix

Exponentially fitted scheme for the put/call option
This program finds the approximation of the put or call option price by ex-ponentially fitted scheme and requires following function

```matlab
function price = BlackScholesPrice(S,K,T,r,vol,CallOrPut)

d1 = (log(S/K) + (vol^2/2)*T) / (vol*T^0.5);
d2 = d1 - vol*T^0.5;
if CallOrPut == 'c'
    price = S .* normcdf (d1)-K * exp (-r * T) * normcdf (d2);
    end
if CallOrPut == 'p'
    price = K * exp(-r*T) * normcdf(-d2) - S.* normcdf(-d1);
    end

% PROGRAM CODE

close all
clear all

% Parameters
r = 0.08;    % interest rate
vol = 0.3;   % volatility
K = 100;     % strike price

% Domain
S = 100;     % initial price
Smax = 200;  % maximum price
```

53
T = 0.5; % time interval
CallOrPut = 'p'; % indicator call option = 'c', put option = 'p'

% Discretization
M = 200*2/h; % number of spatial grid points
dx = Smax/M; % space step
stockprice = [0:dx:Smax];
M = length(stockprice)-1;
N = 200*2/h; % number of temporal grid points
dt = T/N; % time step
reversetime = [T:-dt:0];
V = zeros(M+1,N+1);
delprice = zeros(1,M+1);

% Call option price
if CallOrPut == 'c'
    % Terminal condition at time level N+1
    V(:,N+1) = max (0, stockprice - K);
    % Boundary conditions
    V(1,:) = 0;
    V(M+1,:) = stockprice - K*exp (-r * reversetime);
end

% Put option price
if CallOrPut == 'p'
    % Terminal condition at time level N+1
    V(:,N+1) = max(0,K - stockprice);
    % Boundary conditions
    V(1,:) = K*exp(-r * reversetime);
    V(M+1,:) = 0;
end

% Probabilities
b = -r;
a = dt/dx^2; % parabolic mesh ratio
mu = r * stockprice(2:M);
% Define the exponential fitting
vo = 0.5 * vol^2 * stockprice(2:M).^2;
v = mu.* dx.* 0.5.* coth(dx * mu ./ (2*vo));

A = -(a.* v - mu.* dt.* 0.5./ dx);
B = -(-1 + b * dt - 2 * v.* a);
C = -(a.* v + mu.* dt.* 0.5./ dx);

% System matrix
Mat = zeros(M-1,M-1);

for j = 2:M-2
    Mat(j,j-1) = A(j);
    Mat(j,j) = B(j);
    Mat(j,j+1) = C(j);
end
Mat(1,1) = B(1); Mat(1,2) = C(1); % first row
Mat(M-1,M-2) = A(M-1); Mat(M-1,M-1) = B(M-1); % last row

% Right hand side
b = zeros(M-1,1);
b(1,1) = A(1) * V(1,1);
b(M-1,1) = C(M-1) * V(M-1,1);

for n = N:-1:1 % loop backward in time
    f = V(2:M,n+1)-b;
    V(2:M,n) = Mat\f;
end

myindex = find(stockprice < S);
bs = BlackScholesPrice(stockprice,K,T,r,vol,CallOrPut);
delprice = interp1(stockprice(myindex), V(myindex,1) , stockprice);

% Plotting
if CallOrPut == 'c'
    figure
    plot(stockprice, delprice);
xlabel('S')
ylabel('V(S,0)')
title('Call Price')
figure
plot(stockprice,bs);
xlabel('S')
ylabel('V (S,0)')
title('Call Price')
end

if CallOrPut == 'p'
figure
plot(stockprice, delprice);
xlabel('S')
ylabel('V(S,0)')
title('Put Price')
figure
plot(stockprice,bs);
xlabel('S')
ylabel('V (S,0)')
title('Put Price')
end

error = abs(bs- delprice);
figure
semilogy(stockprice,error);
xlabel('S')
ylabel('V(S,0)')
title('Error')

Fitted finite volume scheme for the put option
This program finds the approximation of the put option price by fitted finite volume scheme

close all
clear all
% Parameters
r = 0.08;  \hspace{1cm} \text{\% interest rate}
vol = 0.3;  \hspace{1cm} \text{\% volatility}
K = 100;  \hspace{1cm} \text{\% strike price}
Smax = 200;  \hspace{1cm} \text{\% maximum price}

% Temporal domain
T = 0.5;  \hspace{1cm} \text{\% final time (in years)}

% Discretization parameters
M = 41;  \hspace{1cm} \text{\% number of spatial grid points}
dS = Smax/M;  \hspace{1cm} \text{\% space step}
stockprice = [0:dS:Smax];  \hspace{1cm} \text{\% vector of discrete S values}
M = length(stockprice)-1;
N = 21;  \hspace{1cm} \text{\% number of temporal grid points}
dt = T/N;  \hspace{1cm} \text{\% time step}
reversetime = [T:-dt:0];

% Put option price $P = P(S,t)$
P = zeros(M+1,N+1);
delprice = zeros(1,M+1);

% Initial condition at first time level
P(:,1) = max(0,K-stockprice);  \hspace{1cm} \text{\% payoff}

% 'spatial' boundary conditions
P(1,:) = K*exp(-r*reversetime);
P(M+1,:) = 0;

% Quantities according to (7.6)
a = 0.5*vol^2;
b = r + vol^2;
c = r + b;
nu = b/a;  \hspace{1cm} \text{\% below (7.11)}

Sleft = stockprice(1:M-1);
Scenter = stockprice(2:M);
Sright = stockprice(3:M+1);

% Now $S_{(j \pm 1/2)}$
Sminus = (Sleft+Scenter)./2;
\[
\text{Splus} = \frac{(\text{Sright} + \text{Scenter})}{2}; \\
\text{alpha1} = \frac{\text{Sleft}(1)}{4dS} * (a - b); \\
\text{beta1} = b * \frac{\text{Sminus}(1) * \text{Scenter}(1)^{\nu}}{(\text{Scenter}(1)^{\nu} - \text{Sleft}(1)^{\nu})dS}; \\
\text{gamma1} = -\frac{\text{Sleft}(1)}{4dS} * (a + b) - b * \frac{\text{Sminus}(1) * \text{Sleft}(1)^{\nu}}{(\text{Scenter}(1)^{\nu} - \text{Sleft}(1)^{\nu})dS} - c; \\
\]

% Define the exponential fit
\[
\text{alpha} = b * \frac{\text{Sminus} * \text{Sleft}^{\nu}}{(\text{Sright}^{\nu} - \text{Scenter}^{\nu})dS}; \\
\text{beta} = b * \frac{\text{Splus} * \text{Sright}^{\nu}}{(\text{Sright}^{\nu} - \text{Scenter}^{\nu})dS}; \\
\text{gamma} = -b * \frac{\text{Sminus} * \text{Scenter}^{\nu}}{(\text{Scenter}^{\nu} - \text{Sleft}^{\nu})dS} - b * \frac{\text{Splus} * \text{Scenter}^{\nu}}{(\text{Sright}^{\nu} - \text{Scenter}^{\nu})dS} - c; \\
\]

% Build system matrix \text{Mat}
\[
\text{Mat} = \text{zeros} (M-1,M-1); \\
\text{for} \ j = 2: M-2 \\
\text{Mat}(j,j-1) = dt* \text{alpha}(j); \\
\text{Mat}(j,j) = 1-dt* \text{gamma}(j); \\
\text{Mat}(j,j+1) = dt* \text{beta}(j); \\
\end \\
\text{Mat}(1,1) = 1-dt* \text{gamma1}; \text{Mat}(1,2) = dt* \text{beta1}; \quad \% \text{first row} \\
\text{Mat}(M-1,M-2) = dt* \text{alpha}(M-1); \text{Mat}(M-1,M-1) = 1-dt* \text{gamma}(M-1); \quad \% \text{last row} \\
\]

% Right hand side
\[
\text{b} = \text{zeros} (M-1,1); \\
\text{b}(1,1) = \text{alpha1} * P (1,1); \\
\text{b}(M-1,1) = \text{beta} (M-1) * P (M+1,1); \\
\]

% FORWARD
\[
\text{for} \ n = 2:N \quad \% \text{loop forward in time} \\
\text{f} = P (2:M,n-1)+b; \\
\text{P}(2:M,n) = \text{Mat} f; \\
\end \\
\]

myindex = find (stockprice<Smax); \\
delprice = interp1 (stockprice (myindex), P (myindex,N), stockprice); \\

% Plotting
figure
plot (stockprice, delprice);
xlabel ('S')
ylabel ('P(S,0)')
title ('Put Price')

error = abs(bs - delprice);
figure
semilogy(stockprice, error);
xlabel('S')
ylabel('P(S,0)')
title('Error')