SYMMETRY REDUCTIONS AND EXACT SOLUTIONS FOR NONLINEAR DIFFUSION EQUATIONS

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The symmetry properties of nonlinear diffusion equations are studied using a Lie group analysis. Reductions and families of exact solutions are found for some of these equations.

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1. Introduction

During the last decades a series of new models describing new types of diffusion processes appeared in the literature. They take in account feed back effects and describe strong non linear processes. Such effects are typical for a motion of fluid in a strong nonlinear medium like porous medium with a very small typical gap size or for a nonlinear fluid. These effects appear as well in financial models which describe friction on the market like illiquidity problems or transaction costs. The main difference between these models and well know ones is in the form of the included nonlinearity. Classical models are usually presented by a parabolic equation with a small regular perturbation which contains a small parameter ρ. If the value of the parameter ρ tends to zero, then the equation and its solutions tend correspondingly to a linear parabolic equation and its solutions. So the equations with and without perturbation are from the parabolic type.

In the new type of models we have to do with a singular perturbation. It means that the nonlinearity includes the highest derivative and by vanishing of the small parameter we have to take into account that the structure of the given equation will be changed as well. This means that not all of the solutions to a singular perturbated equation can have any pendant in the linear case, i.e., for $\rho = 0$. A typical example of the regular perturbated diffusion equation is given by the model introduced in Ref. 1,

$$u_t + \frac{a^2(S,t)}{2} u_{SS} + \Big(b(S,t) - \rho \frac{\mu}{\sigma} a(S,t) \Big) u_S + \frac{\gamma}{2}(1-\rho^2) a^2(S,t) u_S^2 = 0. \quad (1)$$

Here $a(S,t), b(S,t)$ are smooth functions of the spatial variable S and the time t; γ, σ, μ are some fixed material constants, ρ is a small parameter. By $\rho \to 0$, $a(S,t) \to \sigma$ and some additional conditions on the function $b(S,t)$ and on the constants γ, μ we reduce this equation to a linear parabolic equation. The models listed below are examples of singular perturbated models because the diffusion coefficient σ is replaced by an expression which includes the second derivative of $u(S,t)$. The model suggested in Ref. 2,

$$u_t + \frac{1}{2}\frac{\sigma^2 S^2 u_{SS}}{(1-\rho\lambda(S)Su_{SS})^2} = 0\,, \tag{2}$$

includes additionally a continuous function $\lambda(S)$ inside of the adjusted diffusion coefficient. The symmetry groups and invariant solutions of (2) for different values of the function $\lambda(S)$ where studied in Refs. 3-5.
A model introduced in Ref. 6,

$$u_t + \frac{1}{2}\frac{\sigma^2(1-\rho u_S)^2}{(1-\rho u_S - \rho S u_{SS})^2} S^2 u_{SS} = 0\,, \tag{3}$$

has diffusion coefficient which depends on both, u_S and u_{SS}. A bit simpler seems to be a model suggested in Ref. 7,

$$u_t + \frac{1}{2}\sigma^2 S^2 u_{SS}(1-\rho S u_{SS})^2 = 0, \tag{4}$$

which is still a non trivial model. The symmetry algebra admitted by this equation was studied in Ref. 8. In the next chapter we describe symmetry properties of the equations (1) and (3).

2. Symmetry Properties of Nonlinear Diffusion Equations

Let us study with methods of the Lie group analysis symmetry properties of equation (3). We use a standard method represented for instance in Ref. 9. On the first step we solve the Lie determining equations and obtain the Lie algebra of symmetry operators admitted by this equations. We obtain the following theorem

Theorem 2.1. *The equation (3) admits a four dimensional Lie algebra L_4 with the following operators*

$$U_1 = \frac{\partial}{\partial t}, \quad U_2 = (S-\rho u)\frac{\partial}{\partial u}, \quad U_3 = \frac{\partial}{\partial u}, \quad U_4 = S\frac{\partial}{\partial S} + u\frac{\partial}{\partial u}. \tag{5}$$

The commutator relations are

$$[U_1, U_2] = [U_1, U_3] = [U_1, U_4] = 0,\ [U_2, U_3] = [U_2, U_4] = 0,\ [U_3, U_4] = U_3. \tag{6}$$

The algebra L_4 has a two dimensional abelian subalgebra $L_2 =< U_1, U_2 >$ spanned by the operators U_1, U_2. L_2 is the center of the algebra L_4. It means we have a decomposable Lie algebra of the type $L_4 = L_2 + < U_3, U_4 >$. All four-dimensional real Lie algebras were classified in the paper Ref. 10. The authors looked for classification of the sub-algebras into conjugacy classes under the group of inner automorphisms. They used as well the idea of normalization which guarantees that the constructed optimal system of subalgebras is unique up to the isomorphisms. This classification allows to divide the invariant solutions into non-intersecting equivalence classes. On this way it is possible to find all essential different invariant solutions of the equation under consideration. We do not provide this classification because it is space consuming. We present the most interesting symmetry reductions and invariant solutions to this equation.

Using the set of invariants we reduce the equation (3) to ordinary differential equations. We chose in the first case two invariants of the following form $z = S\exp\left(\frac{1}{4}bt\right), b \neq 0$ and $w = S^{-2}u - (\rho S)^{-1}$. If we use this invariants as new independent and dependent variables we can reduce the partial differential equation (3) to an ordinary differential equation

$$zw_z((z(z^2w)_{zz})_z)^2 + \frac{2\sigma^2}{b}\ (z^2w)_{zz}((z^2w)_z)^2 = 0. \tag{7}$$

This equation has a solution of the form

$$w(z) = Cz^{\alpha}, \tag{8}$$

where C is an arbitrary constant and α can take following values

$$\alpha_{1,2} = (-\sigma^2 - b \pm \sqrt{\sigma^4 + b^2})/b. \tag{9}$$

It is easy to see that for each value of the parameters σ and b there exist two real solutions (8) to this equation. For the value $\alpha = -2$ the denominator in the equation (3) will be equal to zero and we can exclude this case from further investigations. It is remarkable that after this transformation the reduced equation (7) does not contain any more the parameter ρ. It means all invariant solutions written in S, t variables have the structure similar to the solutions (8) which are now given by

$$u(S,t) = CS^{\alpha-2}e^{\frac{1}{4}\alpha bt} + \frac{S}{\rho}, \tag{10}$$

where the second term is the only term which contains the dependency on the parameter ρ.

If we chose in the second case, in order to reduce the equation (3), $z = \ln S + b/4t$ and $w(z) = (u/S - 1/\rho)S^{\gamma}$ as invariant variables than we obtain

$$\begin{aligned} w_{zz}^2 w_z + w_{zz}(w_z^2(4(1-\gamma)+\kappa) + w_z w(2(1-\gamma)^2 + 2\kappa(1-\gamma)) \\ +w^2\kappa(1-\gamma)^2) + w_z^3(4(1-\gamma)^2 + \kappa(1-2\gamma)) + w_z^2 w(1-\gamma)(4(1-\gamma)^2 \\ +\kappa(2-5\gamma)) + w_z w^2(1-\gamma)^2((1-\gamma)^2 + \kappa(1-4\gamma)) - \kappa\gamma(1-\gamma)^3 w^3 = 0, \end{aligned} \tag{11}$$

where $\kappa = 2\sigma/b$. This second order ordinary differential equation can be reduced in case $w_z \neq 0$ to a first order equation by the substitution $p(w) = w_z(z(w))$ and correspondingly $w_{zz} = p_w p$, i.e., w is an independent variable now,

$$p_w^2 p^3 + p_w p(p^2(4(1-\gamma)+\kappa) + pw(2(1-\gamma)^2 + 2\kappa(1-\gamma)) + w^2\kappa(1-\gamma)^2) \\ +p^3(4(1-\gamma)^2 + \kappa(1-2\gamma)) + p^2 w(1-\gamma)(4(1-\gamma)^2 + \kappa(2-5\gamma)) \\ +pw^2(1-\gamma)^2((1-\gamma)^2 + \kappa(1-4\gamma)) - \kappa\gamma(1-\gamma)^3 w^3 = 0. \quad (12)$$

This equation is quadratic in the first derivative and it is equivalent to two first order equations. For some values of the constants γ and κ it can be explicitly solved. The simplest case we obtain if we chose $\gamma = 1$. The solutions will have a form similar to (10). In other cases the equation can be studied using qualitative methods.

We skip the voluminous formulae and study now the invariant solutions to the Burgers type equation (1). As well in this case we can easily find the corresponding symmetry algebra and using the invariants find symmetry reductions of this equation. Let us introduce the substitutions $z = \log S - \kappa^2/2t$, and $u(S,t) = \gamma^{-1}(1-\rho^2)^{-1}w(z)$, where κ is a constant, and the functions $a(S,t)$ and $b(S,t)$ have special forms $a(S,t) = \kappa S g(z)$, and $b(S,t) = \kappa S g(z)\left(\rho\mu\sigma^{-1} - \kappa g(z) f(z)\right)$, where $g(z), f(z)$ are arbitrary functions and $g(z) \neq 0$. Then the equation (1) will be reduced to the ordinary differential equation

$$w_{zz} + (f(z) - 1 - g(z)^{-2})w_z + w_z^2 = 0. \quad (13)$$

After standard transformations we obtain the solution of this equation

$$u(S,t) = \log\left(C_1 + \int e^{-\Phi(z)}\mathrm{d}z\right) + C_2, \quad \Phi(z) = \int \left(f(z) - 1 - g(z)^{-2}\right)\mathrm{d}z, \quad (14)$$

where C_1, C_2 are arbitrary constants. We obtain that the invariant solutions have the similar structure (14) as we can expect using the Hopf-Cole substitution to the equation (1).

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