Hedging strategy for an option on commodity market

Master’s Thesis in Financial Mathematics

Tkachev Ilya
Hedging strategy for an option on commodity market

Tkachev Ilya

Halmstad University
Project Report IDE1020

Master’s thesis in Financial Mathematics, 15 ECTS credits

Supervisor: Ph.D. Mikhail Nechaev
Examiner: Prof. Ljudmila A. Bordag
External referees: Prof. Volodya Roubtsov

August 20, 2010

Department of Mathematics, Physics and Electrical engineering
School of Information Science, Computer and Electrical Engineering
Halmstad University
Preface

I would like to thank my supervisor Mikhail Nechaev for the interesting theme of research and his help during the work, the Master Program’s coordinator Ljudmila A. Bordag for her advices, and my wife Evgeniya who was always with me and helped me with the project until the final point of this work.
Abstract

In this work we consider the methods of pricing and hedging an option on the forward commodity market described by the multi-factor diffusion model. In the previous research there were presented explicit valuation formulas for standard European type options and simulation schemes for other types of options. However, hedging strategies were not developed in the available literature. Extending known results this work gives analytical formulas for the price of American, Asian and general European options. Moreover, for all these options hedging strategies are presented. Using these results the dynamics of the portfolio composed of options on futures with different maturities is studied on a commodity market.
Contents

1 Introduction 1

2 General theory 5
   2.1 General facts ........................................... 5
   2.2 Assets ............................................... 5
   2.3 Options and a portfolio ................................ 7

3 European type options 11

4 American type options 15
   4.1 The infinite horizon .................................... 16
      4.1.1 A price of the option ................................. 16
      4.1.2 The optimal stopping problem for a Markov processes ... 16
      4.1.3 A price of the option ................................. 18
      4.1.4 The wealth process and the strategy .................... 19
   4.2 The finite horizon .................................... 21

5 Asian options 23
   5.1 Pricing of Asian options .................................. 23
   5.2 Reductions for the arithmetic average case .................. 24
   5.3 Reductions for the geometric average case ................. 26
   5.4 The hedging strategy .................................... 28

6 Portfolio dynamics 29
   6.1 Models description ..................................... 29
   6.2 An illustration of results ................................ 33

7 Conclusions 39

Notation 41

Bibliography 43
Chapter 1

Introduction

Due to the popularity of the derivative market, the theory of the options and forward contracts pricing is very important in the quantitative finance. Moreover, after pricing methods are developed the next problem is to derive appropriate hedging strategies. For the every market model it is very important to give perfect results in application to the real data and provide analytical solutions for the pricing problems if it is possible.

The well-known Black-Scholes model showed, that prices of some derivatives can be easily calculated, but the explanatory power of this model can be not sufficient in some cases. On the other hand, in the previous research it was shown that a more complicated multi-factor model for the forward price

\[ dF(t, T) = F(t, T) \sum_{i=1}^{n} \sigma_i(t, T)dw_i^t \]

has already provided perfect results in application to the real data on a commodity market (for example, in [2]). Obviously, for this model pricing and hedging methods must be developed also. In previous studies in this area there were given explicit formulas for some standard options, but not for hedging strategies. This work extends these results to the wider class of options, providing pricing formulas and hedging methods.

We use the martingale approach which is perfectly described in the book by M. Musiela M. Rutkowski [4]. This approach helps to derive easily pricing equations for options, which are similar to the Black-Scholes equation. The other useful method is reducing optimal stopping problems to the free-boundary ones as it is shown in the book by G. Peskir A. Shiryaev [5]. This method becomes important in considerations of American options.

This work is focused on forward derivatives in the commodity market. Due to the importance of an energy component in the economic development
commodity derivatives models are a popular topic for discussions. There are several main ideas incorporated in models of this market. One of them is to model forward prices directly instead of using models for interest rate, spot prices or other state variables such as convenience yield like it was done in previous models. For example, in the well-known Black-Scholes model the forward price is obtained through the spot price using non-arbitrage arguments.

The other idea is related to the multi-factor model of forward prices. As previous researches have shown, inclusion of only the second factor in the model increases the explanatory power of the model significantly. For example, Cortazar and Schwartz (1994) studied the factor structure of copper futures. The explanatory power of the first two principal components was 97%. There were several studies concerned different markets like oil market, electricity market, petroleum market etc. Among them it is interesting to mention the paper by Jarniven [2] where he consider the factor structure for the price of short-time and long-time forward contracts. He showed, that in this case the explanatory power is lower: for the first two principal components (factors) it is 81% for the oil market and 63% for the pulp market. Moreover, if one would like to get at least 90%, it is necessary to use four-factor models for the both markets.

The general framework for the risk management in commodity derivatives was developed by Clewlow and Strickland [1]. In this paper were obtained analytical formulas for the price of standard European options on forward contracts and described the Monte-Carlo method for pricing American and exotic options. However, hedging strategies are not presented for the multi-factor model. At last, one of the most interesting problem for the forward commodity is a dynamics of the portfolio consisting of options on forwards with different maturity dates. However, in the context of the multi-factor model these questions were not considered.

For all this reasons there are 3 main directions of the investigation in this work. Firstly we will obtain explicit pricing formulas for the different types of options in the case if it is possible. Otherwise the corresponding problems will be formulated as a basis for a numerical solution. Secondly hedging strategies for all these options will be presented. Lastly, we will consider the dynamics of a portfolio comparing three models: the Black-Scholes model, the single-factor model and the multi-factor model.

The structure of this work is following. We use the martingale approach, so in Chapter 2 will be given the shortcut of the theory which is relevant for the thesis project. In Chapter 3 European type options are in the consideration. In that chapter the special theory is given, some pricing methods are developed and hedging strategies are presented for these options. The same
structure have Chapter 4 devoted to American options and Chapter 5 where Asian options are studied. In Chapter 6 we give a study of the portfolio dynamics and in Chapter 7 the results are discussed.
Chapter 1. Introduction
Chapter 2

General theory

2.1 General facts

In this work the key assets are forward contracts. Such a contract is a today agreement between two parties to make a delivery of an underlying asset (e.g. commodity) with a specified price in a future (maturity date). The most popular are futures contracts which are traded on the exchanges and are the most liquid. It is natural to suppose that on the market there are forward contracts with all maturity dates. We assume, that for every maturity date it is just one price of the forward contract, which is called the forward price. We also assume that the risk-free rates are constant, so in this case there is no difference between the price of the forward contract and of the futures contract of the same size with the same maturity.

We will keep in mind, that all forward contracts are on the same commodity asset e.g. oil or gas. In this work some other financial instruments related to the forward contracts will be considered for example spot contracts, a futures index and options.

We are interested in pricing and hedging of options, which underlying will be forward contracts and a futures index. In this work options of different types are considered. The special theory for these cases will be given in the next chapters, and the general theory relevant for all cases will be given in this chapter.

2.2 Assets

Let us introduce the stochastic basis as $(\Omega, \mathcal{F}, \mathbb{F}, P)$ where the filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t \geq 0}$ is supposed to be complete and interpreted as an information flow and $\mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}$ for all $s \leq t$. Note that if some random variable say $\xi_t$ is
$\mathcal{F}_t$-measurable, it means that we know the value $\xi_t$ at the time $s \geq t$, but not earlier. We also assume, that $w^i_t, i = 1, \ldots, n$ are standard Winner processes and $\mathcal{F}_t = \sigma\{w^i_s, s \leq t, 1 \leq i \leq n\}$.

We consider the market where two types of assets are presented. First of them is a forward contract with the price $F(t, T)$ at the time $t$ with a delivery at the time $T$. Assume, that the dynamics of the price for every fixed $T$ is given by the following multi-factor model

$$dF(t, T) = F(t, T) \sum_{i=1}^{n} \sigma_i(t, T) dw^i_t$$

where $w^1, ..., w^n$ are independent Brownian motions related to different sources of uncertainty and the values $\sigma_i(t, T)$ are volatility functions of the forward prices. According to Jarvinen [2] these functions depend only on the time to maturity ($T - t$).

Equation (2.1) has a solution in the closed form

$$F(t, T) = F(0, T) \exp \left\{ -\frac{1}{2} \sum_{i=1}^{n} \int_{0}^{t} \sigma_i^2(u, T) \, du + \sum_{i=1}^{n} \int_{0}^{t} \sigma_i(u, T) \, dw^i_u \right\}.$$ (2.2)

The other asset is risk-free and can be interpreted as a bond price

$$dB(t) = rB(t)dt,$$ (2.3)

or in the exact form

$$B(t) = e^{rt}.$$ (2.4)

Here we assumed, that $B(0) = 1$.

In this model it can be also introduced a spot price $S(t)$ as a special case of the forward price: $S(t) = F(t, t)$ or

$$S(t) = F(0, t) \exp \left\{ -\frac{1}{2} \sum_{i=1}^{n} \int_{0}^{t} \sigma_i^2(u, t) \, du + \sum_{i=1}^{n} \int_{0}^{t} \sigma_i(u, t) \, dw^i_u \right\}.$$ (2.5)

We also work with a futures index. This financial instrument is represented in the following way: its return is calculated as a weighted sum of returns on futures included in this index. We will consider the index which consists of futures contracts on the same underlying, but with different maturities.

Let $T_1, ..., T_m$ be different times to maturity of the forward contracts, which have at time $t$ prices $F(t, t + T_1), ..., F(t, t + T_m)$ respectively.
Let $I_t$ be a price of an index, satisfying the following stochastic differential equation

$$dI_t = I_t \cdot \left( \sum_{j=1}^{m} \alpha_j \frac{dF(t, t + T_j)}{F(t, t + T_j)} \right), \quad (2.6)$$

where $\alpha_j$ is a weight of the forward with the time to maturity which is equal to $T_j$, $\alpha_1 + ... + \alpha_m = 1$ and $I_0 = 1$.

We can replace $\eta_{ij} := \sigma_i(t, t + T_j)$,

and we get

$$dI_t = I_t \cdot \left( \sum_{j=1}^{m} \alpha_j \sum_{i=1}^{m} \eta_{ij} dw_i^t \right). \quad (2.8)$$

It can be easily seen, that if we denote $A_i = \sum_{j=1}^{m} \alpha_j \eta_{ij}$, then

$$dI_t = I_t \cdot \sum_{i=1}^{n} A_i dw_i^t \quad (2.9)$$

Using a condition that $I_0 = 1$, we can integrate (2.9) and obtain for $I_t$ the following expression

$$I_t = \exp \left\{ \sum_{i=1}^{n} A_i w_i^t - \frac{A_i^2}{2} t \right\}. \quad (2.10)$$

Note, that the spot contract and the futures index exist all the time, but the forward contract exists only until the maturity time. This difference will be significant in the further consideration.

2.3 Options and a portfolio

It is convenient to use a notion of the payoff or the gain function $G$. Each type of the option implies its payoff.

In the European case it is defined only for the time $T^*$ and $G^*_T$ is $\mathcal{F}_T^*$-measurable. The price of the option is given by

$$X_0 = B_0 \mathbb{E} \left[ \frac{G^*_T}{B^*_T} \right], \quad (2.11)$$

where an expectation $\mathbb{E}[\cdot]$ is taken with respect to the martingale measure which will be determined below. It also will be shown that the initial measure $P$ is a martingale measure for all cases we will consider.
Before considering the payoff of American options, the stopping times should be introduced.

**Definition 1.** A random variable \( \tau \) is called a stopping time if for all \( t \geq 0 \) the event \( \{ \tau < t \} \in \mathcal{F}_t \) and \( P\{ \tau < \infty \} = 1 \). This means that at the any moment \( t \) we can see if the stopping time happened or not. We denote the set of all stopping times \( \tau \geq t \) as \( \mathcal{M}_t \), and set of all stopping times \( t \leq \tau \leq T \) as \( \mathcal{M}_T^T \).

In the case of the American type options the payoff \( G \) is a process, which is defined by

\[
(G_t)_{0 \leq t \leq T^*} \quad \text{for the standard American options, which can be exercised at any moment before the maturity time and by}
\]

\[
(G_t)_{0 \leq t} \quad \text{for the perpetual American options, which can be exercised at any moment of time without any restrictions.}
\]

The values of the options are given respectively by these formulas

\[
X_0 = B_0 \sup_{\tau \in \mathcal{M}_T^*} \mathbb{E}\left[ \frac{G_{\tau}}{B_{\tau}} \right],
\]

(2.12)

and

\[
X_0 = B_0 \sup_{\tau \in \mathcal{M}_0} \mathbb{E}\left[ \frac{G_{\tau}}{B_{\tau}} \right].
\]

(2.13)

The writer of the option have \( X_0 \) at the moment \( t = 0 \) and must be able to pay payoff if the holder wish to exercise the option. For this reason the writer should develop a hedging strategy, which describes how the initial capital should be invested. Here we consider the replication portfolio for the forward contracts as an underlying. For other underlying assets the strategy will be defined and developed in the related chapters.

We will consider a portfolio consisting of the forward contract with the maturity \( T \) and a risk-free bond. For convenience we use the following notation

\[
F_t = F(t, T)
\]

and

\[
B_t = B(t).
\]

Following Musiela and Rutkowski [4] the forward strategy or the replication portfolio is a pair \( \pi_t = (\gamma_t, \beta_t) \) of the real-valued adapted stochastic processes. Since it costs nothing to enter in a forward contract, the capital of the replication portfolio is given by

\[
X_{\pi}^t = \beta_t B_t.
\]

(2.14)
We say that the portfolio $\pi$ is self-financing or $\pi \in SF$ if and only if
\begin{equation}
    dX_t^\pi = \gamma_t dF_t + \beta_t dB_t.
\end{equation}
(2.15)

Further we will not use subscript $\pi$ for the capital of the portfolio.
Under the initial measure the process $F_t$ is a martingale. We show now, that a discounted portfolio $\left(\frac{X_t}{B_t}\right)$ is a martingale too with respect to the initial measure.

**Proposition 1.** The discounted portfolio is a martingale with respect to the initial measure. Moreover
\begin{equation*}
    d \left( \frac{X_t}{B_t} \right) = d\beta_t = \gamma_t \frac{dF_t}{B_t}.
\end{equation*}

**Proof.** By the simple calculations we obtain
\begin{equation*}
    d \left( \frac{X_t}{B_t} \right) = -r \left( \frac{X_t}{B_t} \right) dt + \frac{1}{B_t} dX_t = -r \beta_t dt + \gamma_t \frac{dF_t}{B_t} + \beta_t \frac{dB_t}{B_t}.
\end{equation*}

Using that $dB_t = r B_t dt$, we obtain
\begin{equation*}
    d \left( \frac{X_t}{B_t} \right) = \gamma_t \frac{dF_t}{B_t}.
\end{equation*}

\square

**Corollary 1.** The initial measure is a martingale measure for the considered model.
Chapter 3

European type options

Suppose, we have some European-type contingent claim $G_{T^*}$, $T^* < T$. As it was mentioned if $X_{T^*} = G_{T^*} \text{ P-a.s.}$, then

$$X_0 = B_0 \mathbb{E} \left[ \frac{G_{T^*}}{B_{T^*}} \right]. \quad (3.1)$$

Clewlow and Strickland [1] got an analytical solution for the European call using this fact and the distribution of $G_{T^*}$. We use the martingale approach which also allows us to find a hedging strategy for such type of options.

Suppose, $G_{T^*}$ is some function of $F_{T^*}$. For example, in the case of the European call we have

$$G_{T^*} = (F_{T^*} - K)^+. \quad (3.2)$$

So, let $V_t = X_t$ be a price of an option which is some function of time $t$ and the forward price at this time which is equal to $F_t$. Since the discounted portfolio is a martingale, an expression $\left( \frac{V(t,F_t)}{B_t} \right)$ must be a martingale.

Proposition 2. The option price $V(t,F)$ admits an equation

$$\frac{\partial V}{\partial t} + \frac{1}{2} \tilde{\sigma}^2(t)F^2 \frac{\partial^2 V}{\partial F^2} - rV = 0, \quad (3.3)$$

where $\tilde{\sigma}(t) = \sqrt{\sum_{i=1}^{n} \sigma^2_i(t,T)}$.

Moreover, a hedging strategy for this option is given by

$$\gamma_t = \frac{\partial V}{\partial F} \quad (3.4)$$

and

$$\beta_t = \frac{V}{B_t}. \quad (3.5)$$
**Proof.** To prove these results we should check the conditions for \( \left( \frac{V(t,F_t)}{B_t} \right) \) to be a martingale. The stochastic differential of this expression is given by

\[
d \left( \frac{V(t,F_t)}{B_t} \right) = \left( -r \frac{V}{B_t} + \frac{1}{B_t} \frac{\partial V}{\partial t} \right) dt + \frac{1}{B_t} \frac{\partial V}{\partial F} dF_t + \frac{1}{B_t} \frac{1}{2} \frac{\partial^2 V}{\partial F^2} dF_t^2.
\]

Since

\[
d F_t^2 = \sum_{i=1}^{n} \sigma_i^2(t,T) dt = \tilde{\sigma}(t) F_t^2 dt,
\]

we have

\[
d \left( \frac{V(t,F_t)}{B_t} \right) = \frac{1}{B_t} \left[ \frac{\partial V}{\partial t} + \frac{1}{2} \tilde{\sigma}^2(t) F_t^2 \frac{\partial^2 V}{\partial F^2} - rV \right] dt + \frac{\partial V}{\partial F} F_t \sum_{i=1}^{n} \sigma_i(t,T) dw^i_t.
\]

Since \( \left( \frac{V(t,F_t)}{B_t} \right) \) is a martingale, the drift must be zero, so

\[
\frac{\partial V}{\partial t} + \frac{1}{2} \tilde{\sigma}^2(t) F_t^2 \frac{\partial^2 V}{\partial F^2} - rV = 0.
\]

Moreover, in this case

\[
d \left( \frac{V(t,F_t)}{B_t} \right) = \frac{\partial V}{\partial F} dF_t = \frac{\partial V}{\partial F} d\beta_t = \gamma_t dF_t.
\]

but \( X_t = V \) and we showed, that \( d \left( \frac{X_t}{B_t} \right) = d\beta_t = \gamma_t d\beta_t. \) For this reason we obtain

\[
\gamma_t = \frac{\partial V}{\partial F},
\]

and from the balance equation

\[
\beta_t = \frac{V}{B_t}.
\]

\( \Box \)

**Remark 1.** *We can also see from (3.2) that it is the usual Black-Scholes equation with a time-dependent volatility and a dividend yield which is equal to the risk-free rate. This means, that results from the theory of solutions of the Black-Scholes equation can be used.*
Let $V(t, F)$ be a solution of (3.2) with terminal condition (payoff)

$$V_{T^*} = \Lambda(F_{T^*})$$

and $V^{BS}(t, S; \sigma, r)$ be a solution of the usual Black-Scholes equation with the payoff

$$V_{T^*}^{BS} = \Lambda(S_{T^*}),$$

and the volatility $\sigma$ and the risk-free rate $r$.

In this case

$$V(t, F) = \exp\{-r(T^* - t)\} \cdot V^{BS}(t, F; \sigma(t), 0),$$

where

$$\bar{\sigma}^2(t) = \frac{1}{T^* - t} \sum_{i=1}^{n} \int_{t}^{T^*} \sigma^2(u, T) \, du.$$ 

**Example 1.** For the European call option we have

$$V(t, F) = \exp\{-r(T^* - t)\} \left( FN(d) - KN(d - \bar{\sigma}(t)\sqrt{T^* - t}) \right),$$

which is the same formula as Clewlow and Strickland [1] obtained. Here $N(x)$ is a cumulative function of the Gaussian distribution and

$$d = \frac{\log \frac{F}{K} + \frac{1}{2} \bar{\sigma}^2(t)(T^* - t)}{\bar{\sigma}(t)\sqrt{T^* - t}}.$$ 

In this case

$$\frac{\partial V}{\partial F} = \exp\{-r(T^* - t)\} N(d),$$

so

$$\gamma_t = \exp\{-r(T^* - t)\} N(d)$$

and

$$\beta_t = \exp\{-rT^*\} V(t, F_t).$$
Chapter 4

American type options

In this chapter the case of American-type options will be considered. This case is more difficult than the European one, because we have to deal with the optimal stopping problems: the holder of an American option can choose the moment of an execution and it is the key factor for the pricing of the American type options.

It is necessary to distinguish two cases: the finite time horizon and the infinite time horizon. The second one is more idealistic and allows to find solutions in an analytical way, but fortunately some financial instruments are well described exactly with the infinite horizon. Usually it is the perpetual options, but we can not consider these options on a forward contract because the forward has the finite horizon itself. Up to this reason the futures index will be considered as an underlying for a perpetual option.

The case of the finite horizon is a more complicated one, because the time to maturity of the option must be taken into account. It does not allow to find the explicit solution even in the case of the ordinary Black-Scholes model with the constant coefficients, and proper numerical methods have to be used for the pricing in this case.

Both these cases will be considered in this work: at first the infinite horizon and then the finite horizon.
4.1 The infinite horizon

4.1.1 A price of the option

Let us consider the American type perpetual payoff for a Put option $G$ which is given by

$$G_t = e^{-\lambda t}(K - I_t)^+ \quad t \geq 0.$$  \hfill (4.1)

Here

- $\lambda$ is the discounting factor;
- $K$ is a strike price of the option;
- $x^+ := x \lor 0 = \max\{0, x\}$.

The price of the option is given by

$$X_0 = B_0 \sup_{\tau \in \mathcal{M}_0} \mathbb{E}\left[\frac{G_\tau}{B_\tau}\right].$$  \hfill (4.2)

In our case the problem takes the form

$$X_0 = \sup_{\tau} \mathbb{E}[e^{-(\lambda+r)\tau}(K - I_\tau)^+] ,$$  \hfill (4.3)

which is the optimal stopping problem for the Markov process $I = (I_t)_{t \geq 0}$.

4.1.2 The optimal stopping problem for a Markov processes

To solve the problem (4.3) we should recall the theory for optimal stopping problems for Markov processes.

Let $Y = (Y_t)_{t \geq 0}$ be a strong Markov process taking value in the set $E$. We consider a general stopping problem in the following formulation

$$V(y) = \sup_{\tau} \mathbb{E}_y[G(Y_\tau)],$$  \hfill (4.4)

where the expectation is taken with respect to the measure $P_y$ such that $P_y\{Y_0 = y\} = 1$.

**Definition 2.** The infinitesimal operator of the process $Y$ acts on the functions $V : \mathbb{R} \to \mathbb{R}$ and is given by

$$(\mathbb{L}_Y V)(y) = \lim_{t \to 0} \frac{\mathbb{E}_y[V(Y_t)] - V(y)}{t}.$$
Definition 3. A measurable function $f = f(y)$ is superharmonic or excessive for a process $Y$ if

$$
\mathbb{E}_y[f(Y_\tau)] \leq f(y), \quad \forall \tau \in \mathcal{M}_0 \text{ and } y \in E.
$$

(4.5)

In other words, every stopping strategy will not give us the value of $f$ greater, then the current value. When we consider the problem (5.8), we should at the every time moment decide if it is optimal to stop now, or it is optimal to wait. Since the process $Y$ is a Markov process, all its probabilistic characteristics depend only on its current value, say $y$.

At each time moment $t$ there are two possibilities:

- to stop now and get the value $G_t(y)$;
- to wait and get the expected value $V(y)$.

Due to the strong Markov property of $Y$, the decision to stop or to wait depends only on $y$. According to [5], we introduce the continuation set

$$
C = \{y \in E : V(y) > G(y)\},
$$

(4.6)

and the stopping set

$$
D = \{y \in E : V(y) = G(y)\}.
$$

(4.7)

Until $y \in C$ the expected value of waiting for the gain in the future is greater since $V(y) > G(y)$, but if $y \in D$ the current value of the gain function $G(y)$ is equal to the expected value of the future gain $V(y)$. The theorem below will show, that the stopping time

$$
\tau_D = \inf\{t \geq 0 : Y_t \in D\}
$$

(4.8)

is optimal for the problem (5.8). It should be mentioned, that if $V$ is *lower semicontinuous* (lsc) and $G$ is *upper semicontinuous* (usc) then $C$ is an open set and $D$ is closed, so the first entry time $\tau_D$ is a stopping time of the process $Y$.

Theorem 1. Let $\tau^*$ be the optimal stopping time to the problem (5.8) i.e. let

$$
V(y) = \mathbb{E}_y[G(Y_{\tau^*})]
$$

(4.9)

for all $y \in E$. Then the value function $V$ is the smallest superharmonic function which dominates the gain function $G$ on $E$.

If we additionally assume that $V$ is lsc and $G$ is usc, then
1. the stopping time $\tau_D$ satisfies $\tau_D < \tau^*$ $P_y$-a.s. for all $y \in E$ and is optimal in (5.8);

2. the stopped process which is given by $(V(Y_{t\wedge \tau_D}))_{t \geq 0}$ is a right-continuous martingale under $P_y$ for all $y \in E$.

The prove of this theorem can be found in [5] pp. 37-39.

Now the problem (5.8) can be reformulated in terms of the free-boundary problems. In the next section we consider such reduction in the case of (4.3).

### 4.1.3 A price of the option

It can be shown by the Ito formula that

$$L_I = \frac{1}{2} y^2 A^2 \frac{d^2}{dy^2},$$

(4.10)

where $A^2 = \sum_{i=1}^n A_i^2$.

So, there exists a point $b \in (0, K)$ such that

$$\tau_D = \inf \{ t \geq 0 : I_t \leq b \}$$

(4.11)

is an optimal stopping time for the problem formulated in (4.3), i.e.

$$X_0 = \mathbb{E} \left[ e^{-\lambda \tau_D} (K - I_{\tau_D})^+ \right].$$

The standard arguments based on the strong Markov property of $I$ lead to the following free-boundary problem

$$L_I V(y) = (\lambda + r) V(y), \quad y > b,$$

$$V(y) > (K - y)^+, \quad y > b,$$

$$V(y) = (K - y)^+, \quad y \leq b,$$

$$V'(y) = -1, \quad y = b.$$

(4.12)

Now we formulate the following proposition

**Proposition 3.** The problem (4.12) has following explicit solution

$$V(y) = \begin{cases} \frac{K-b}{r} y^q, & y > b, \\ K - y, & y \leq b, \end{cases}$$

(4.13)

where

$$q = \frac{1}{2} \left( 1 - \sqrt{1 + \frac{8}{A^2}(\lambda + r)} \right),$$

(4.14)

and

$$b = \frac{qK}{1 - q}.$$

(4.15)
Proof. At first we solve $LIP(y) = (λ + r)P(y)$ on some (unknown) interval $(b, \infty)$. We find solution in the form $P(y) = y^q$. After substituting it in the equation, we get

$$\frac{1}{2}A^2q(q-1)y^q = (λ + r)y^q,$$

so we get

$$q^2 - q - \frac{2}{A^2}(λ + r),$$

where $q_1 < 0 < q_2$.

Then the general solution will have the form

$$V(y) = C_1y^{q_1} + C_2y^{q_2},$$

where $C_1$ and $C_2$ are undetermined constants.

Since $V(y)$ should be less then $K$ for $y > 0$, it is clear that $C_2 = 0$. The boundary conditions in the point $b$ from (4.12) are used to determine values of $b$ and $C_1$.

$$b = \frac{qK}{1 - q},$$

$$C_1 = \frac{K - b}{b^q},$$

which completes the proof.

4.1.4 The wealth process and the strategy

To find the wealth process $X_t$ we use the following result from [3]

$$X_t = B_t \mathbb{E} \left[ \left. \frac{G_t}{B_t} \right| \mathcal{F}_t \right].$$

Due to the strong Markov property of $I_t$ we get

$$X_t = \mathbb{E} \left[ \left. e^{-rT}G_t^{\tau_T} \right| \mathcal{F}_t \right]$$

$$= e^{-\lambda t} \mathbb{E} \left[ \left. e^{-rT}G_t^{\tau_T} \cap \theta_t \right| I_t \right]$$

$$= e^{-\lambda t} \mathbb{E} \left[ \left. e^{-rT}G_t^{\tau_T} \cap \theta_t \right| I_t \right],$$

(4.17)
or

$$X_t = e^{-\lambda t} V(I_t),$$  \hspace{1cm} (4.18)

where the function $V$ is given by (4.13).

Here $\theta_t$ is the usual transition operator

$$\theta_t(\omega(s)) = \omega(t + s) \text{ for all } \omega \in \Omega, \ s \geq 0, \ t \geq -s.$$

Due to the non-arbitrage and self-financing property of the strategy $\pi$ we write

$$\left( \frac{X_t}{B_t} \right) = X_0 + \int_0^t \gamma_s \frac{dI_s}{B_s} - C_t,$$ \hspace{1cm} (4.19)

where $C_t$ is a non-negative consumption process.

On other hand from (4.18) we see, that

$$\left( \frac{X_t}{B_t} \right) = X_0 + \int_0^t e^{-(r+\lambda)s} V'(I_s) \, dI_s - \int_0^t ((r+\lambda)V(I_s) - L_t V(I_s)) \, ds \hspace{1cm} (4.20)$$

Noting, that the Doob’s decomposition for the supermartingales

$$\left( \frac{X_t}{B_t} \right) = X_0 + M_t - C_t$$

is unique, we get the hedging strategy for the American perpetual Put option

**Proposition 4.** The hedging strategy for the American perpetual Put option on the futures commodity index is given by

$$\gamma_t = e^{-\lambda t} V'(I_t),$$

$$\beta_t = e^{-(r+\lambda)t} V(I_t).$$  \hspace{1cm} (4.21)
4.2 The finite horizon

In this chapter we consider the case of the American type options with the finite horizon. Even in the case of the Black-Scholes model with the constant coefficients there is no analytical solution for the price of such options.

We consider options on forward contracts. Let the payoff for the American vanilla put \( G \) be given by

\[ G_t = (K - F_t)^+. \tag{4.22} \]

The following result from \[3\] will help us to find price of this option and the wealth process.

The wealth process is given by

\[ X_t = B_t \text{ess sup}_{\tau \in \mathbb{M}_t^*} \mathbb{E} \left[ \frac{G_\tau}{B_\tau} \bigg| \mathcal{F}_t \right]. \tag{4.23} \]

In particular, the price of the option is equal to

\[ X_0 = \sup_{\tau \in \mathbb{M}_0^*} \mathbb{E} \left[ \frac{G_\tau}{B_\tau} \right]. \tag{4.24} \]

The key difference between the perpetual options and options with the finite maturity is a strong time-dependence in the latter case. Moreover, the process \( F_t \) is a non-homogenous Markov process because its distribution depends both on the current value \( F_t \) and time \( t \). For this reason we consider the homogenous Markov process \( Y_t = (t, X_t) \). In this case the continuation set \( C \) and the stopping set \( D \) lie in the set \( E = [0, T] \times \mathbb{R}_+ \) and we use results from the theory of the optimal stopping problems for Markov processes.

Now, because the process \((t, F_t)\) is a homogenous strong Markov process, the function \( \frac{X_t}{B_t} = \frac{V(t, F_t)}{B_t} \) must be a martingale in the continuation set. Since

\[ d \left( \frac{V(t, F_t)}{B_t} \right) = \frac{1}{B_t} \left[ \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2(t) F_t^2 \frac{\partial^2 V}{\partial F^2} - rV \right] dt + \frac{\partial V}{\partial F} dF_t \tag{4.25} \]

we get

\[ (\mathcal{L}_F V)(t, F) = \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2(t) F_t^2 \frac{\partial^2 V}{\partial F^2} - rV = 0. \tag{4.26} \]

This leads to the following free-boundary problem in \((t, F) \in [0, T] \times \mathbb{R}_+\),
(\mathbb{L}_F V)(t, F) = 0 \quad \text{if } F \in (B^*(t), \infty),
V(t, F) = (K - F)^+ \quad \text{if } F \in [0, B^*(t)],
V(T, F) = (K - F)^+,
V(t, B^*(t)) = (K - B^*(t))^+, \tag{4.27}
\frac{\partial V}{\partial F}(t, B^*(t)) = -1.

**Remark 2.** The problem (4.27) cannot be solved in the analytical way even in the case of the constant coefficients like in the Black-Scholes model. However, there are many well-developed numerical methods for solving free-boundary problems of such type. With the help of them, the option’s price and the wealth process values can be calculated with a good power of accuracy.

**Remark 3.** The free-boundary problem is formulated for both optimal stopping problems (4.23) and (4.24) in such a way: if the function \( V^*(t, F) \) is a solution of the (4.27), then \( V^*(0, F_0) = X_0 \) as well as \( V^*(t, F_t) = X_t \). Moreover, the continuation and stopping sets are following

\[ C = \{(t, F) \in [0, T] \times \mathbb{R}_+ : F > B^*(t)\}, \]

and
\[ D = \{(t, F) \in [0, T] \times \mathbb{R}_+ : F \leq B^*(t)\}. \]

Remark [3] provides the formula for the wealth process in the case if the function \( V^*(t, F) \) is calculated. On the other hand, the decomposition (4.25) helps to determine the function \( \gamma_t \). Summarizing all arguments from above we formulate the following proposition

**Proposition 5.** The hedging strategy for an American put option is given by
\[ \gamma_t = \frac{\partial V^*}{\partial F}(t, F_t), \]
\[ \beta_t = e^{-r t} V^*(t, F_t), \tag{4.28} \]
where \( V^*(t, F) \) is a solution of the free-boundary problem (4.27).
Chapter 5

Asian options

Asian options are path-dependent options, which means that their values depend not only on the current price of the underlying, but also on its dynamics in the previous time period. We consider Asian options of the European type, without any possibility of an early exercise, so the arguments from Chapter 3 will be used.

5.1 Pricing of Asian options

We consider an Asian option of the European type with the following payoff

$$G_{T^*} = \Lambda(J_{T^*}, F_{T^*}),$$

where $J$ is a path-dependent variable,

$$J_t = \int_0^t f(s, F_s) \, ds.$$  \hspace{1cm} (5.2)

For example, the Asian average strike call payoff is given by

$$C_{T^*} = \left( F_{T^*} - \frac{J_{T^*}}{T^*} \right)^+$$

and for the Asian average strike put is given by

$$P_{T^*} = \left( \frac{J_{T^*}}{T^*} - F_{T^*} \right)^+.$$  \hspace{1cm} (5.4)

The function $f$ in (5.2) defines the type of the average in an Asian option. There are some standard cases like

$$f(t, F_t) = F_t.$$
for the arithmetic average and

\[ f(t, F_t) = \log F_t \]

for the geometric average.

Due to the path-dependence, the value of the option \( V_t \) at the time \( t \) should be a function of three variables, i.e. \( V(t, J, F_t) \).

Before we derive the equation for the function \( V \) we should calculate the differential of \( J_t \).

\[ dJ_t = d\left( \int_0^t f(s, F_s) \, ds \right) = f(t, F_t) \, dt, \]

if the function \( f \) is a continuous deterministic function.

Now, we look for conditions for \( \frac{V(t, J_t, F_t)}{B_t} \) to be a martingale. We calculate the stochastic differential of this expression

\[
d\left( \frac{V(t, J_t, F_t)}{B_t} \right) = \left( -r \frac{V}{B_t} + \frac{\partial V}{\partial J} \right) dt \frac{1}{B_t} \frac{\partial V}{\partial J} dJ_t + \frac{1}{B_t} \frac{\partial V}{\partial F} dF_t + \frac{1}{2} \frac{1}{B_t^2} \frac{\partial^2 V}{\partial F^2} dF_t^2 = \]

\[
\frac{1}{B_t} \left[ \frac{\partial V}{\partial t} + f(t, F) \frac{\partial V}{\partial J} + \frac{1}{2} \tilde{\sigma}^2(t) F_t^2 \frac{\partial^2 V}{\partial F^2} - rV \right] dt + \frac{\partial V}{\partial F} dF_t. \tag{5.5}
\]

Since \( F_t \) is a martingale, the function \( V(t, J, F) \) must satisfy the following equation

\[
\frac{\partial V}{\partial t} + f(t, F) \frac{\partial V}{\partial J} + \frac{1}{2} \tilde{\sigma}^2(t) F_t^2 \frac{\partial^2 V}{\partial F^2} - rV = 0. \tag{5.6}
\]

with a terminal condition

\[ V(T^*, J, F) = \Lambda(J, F). \tag{5.7} \]

**Remark 4.** The latter problem \((5.6)-(5.7)\) is to be solved in \( \mathbb{R}^3 \) which is a complicated case for both analytical solutions and numerical methods. As it will be showed in the next chapter, in some special cases this problem can be reduced to the equivalent one in \( \mathbb{R}^2 \).

### 5.2 Reductions for the arithmetic average case

As it can be seen from \((5.3)\) and \((5.4)\) in some cases the payoff function \( \Lambda \) can be represented as

\[
\Lambda(J, F) = F^\alpha L(R) \tag{5.8}
\]
Hedging strategy for an option on com. market

where \( R = J \) and \( \alpha \) is usually either 1 or 0.

Let us assume that the function \( f(t, F) = F \), i.e. we consider the case of an arithmetic average. If the payoff function admits the symmetry (5.8), we can use reductions presented in [7].

The new function will be

\[ H(t, R) = F^\alpha V(t, J, F). \]  

(5.9)

The problem (5.6)-(5.7) takes the form

\[
\frac{\partial H}{\partial t} + \frac{1}{2} \tilde{\sigma}^2(t) R^2 \frac{\partial^2 H}{\partial R^2} + \left[ 1 + \tilde{\sigma}^2(t)(1 - \alpha)R \right] \frac{\partial H}{\partial R} + \left[ \frac{1}{2} \tilde{\sigma}^2(t) \alpha (\alpha - 1) - r \right] H = 0
\]

(5.10)

with a terminal condition

\[ H(T^*, R) = L(R). \]  

(5.11)

For example, in the case \( \alpha = 1 \) which corresponds to the call and put options this equation takes the form

\[
\frac{\partial H}{\partial t} + \frac{1}{2} \tilde{\sigma}^2(t) R^2 \frac{\partial^2 H}{\partial R^2} + \frac{\partial H}{\partial R} - rH = 0.
\]

(5.12)

**Remark 5.** The equation (5.12) is similar to the Black-Scholes equation with some difference in the coefficient before \( \frac{\partial H}{\partial R} \). However, this difference makes solution of this equation much more difficult. The Black-Scholes equation can be reduced to the heat equation using appropriate substitutions derived from the symmetry properties of the equation. The key difference is that Black-Scholes equation admits a wide class of symmetries, but it is not the case for the equation (5.12).

Using the theory of Lie point symmetries, our aim is to find the symmetry algebra for the (5.12). We look for the infinitesimal generator in the form

\[
U = \xi(t, R, H) \frac{\partial}{\partial t} + \eta(t, R, H) \frac{\partial}{\partial R} + \Phi(t, R, H) \frac{\partial}{\partial H}.
\]

(5.13)

For the equation of interest the general form of the generator is given by

\[
U_H = (c_1 e^{\xi} H + c_2 \beta) \frac{\partial}{\partial H}.
\]

(5.14)

where \( \beta \) is a solution of (5.12) itself.

We make the following substitutions

\[
\xi = R - t,
\]

\[ u(\xi, t) = e^{-rt} H(R, t). \]

(5.15)
For this function $u$ an equation (2.1) takes the form

$$\frac{\partial u}{\partial t} + \frac{1}{2} \sigma^2(t)(\xi + t)^2 \frac{\partial^2 u}{\partial \xi^2} = 0,$$

with a terminal condition

$$u(T^*, \xi) = e^{-rT^*} L(\xi + T^*),$$

The equation (5.16) has the symmetry algebra with the following general form of the generator

$$U = (c_1 u + c_2 \beta') \frac{\partial}{\partial u}.$$ (5.18)

where $\beta'$ is a solution of (5.16) itself.

So, Lie symmetry group can not help to solve the equation (5.16) analytically. But it was useful to reduce (5.6)-(5.7) to a simpler form (5.16)-(5.17) and appropriate numerical methods can be applied to solve it.

5.3 Reductions for the geometric average case

The other case when it is possible to reduce the problem (5.6)-(5.7) to an equivalent one in $\mathbb{R}^2$ is the case of the geometric average

$$f(t, F) = \log F,$$

when the payoff is given in form

$$\Lambda(J, F) = \Lambda(J).$$

For example, these conditions are satisfied by the rate call and the rate put options with the payoff respectively equal to

$$C(T^*, J, F) = (J - K)^+$$

and

$$P(T^*, J, F) = (K - J)^+.$$ 

In this case we make the following substitution (see [7])

$$y = \frac{J + (T^* - t) \log F}{T},$$

$$W(t, y) = V(t, J, F).$$ (5.19)
In the new variables equation (5.6) is written as

\[
\frac{\partial W}{\partial t} + \frac{1}{2} \tilde{\sigma}^2(t) \left( 1 - \frac{t}{T^*} \right)^2 \frac{\partial^2 W}{\partial y^2} - \frac{1}{2} \tilde{\sigma}^2(t) \left( 1 - \frac{t}{T^*} \right) \frac{\partial W}{\partial y} - rW = 0, \quad (5.20)
\]

with the terminal conditions in the form

\[
W(T^*, y) = \Lambda(yT^*). \quad (5.21)
\]

This problem is similar to one in [7], which helps us to derive following step-to-step procedure.

1. Suppose that the Asian option has a payoff \( \Lambda(J) \). Calculate the price of the option with the Black-Scholes equation with the constant coefficients and a payoff \( \Lambda(T^* \log F) \). Let us call it \( V_{BS}(F, t, r, \sigma) \).

2. Replace \( \sigma^2 \) in \( V_{BS}(F, t, r, \sigma) \) through the following expression

\[
\frac{1}{T^* - t} \int_t^{T^*} \tilde{\sigma}^2(s) \left( 1 - \frac{s}{T^*} \right)^2 \, ds.
\]

3. Replace \( r \) in the obtained formula through the following expression

\[
-\frac{1}{T^* - t} \int_t^{T^*} \frac{1}{2} \tilde{\sigma}^2(s) \left( 1 - \frac{s}{T^*} \right) \, ds.
\]

4. The obtained formula must be multiplied by

\[
\exp \left\{ -\int_t^{T^*} r + \frac{1}{2} \tilde{\sigma}^2(s) \left( 1 - \frac{s}{T^*} \right) \, ds \right\}
\]

5. In the resulting formula \( F \) should be replaced by

\[
e^{J/T^*} F(T^* - t)/T^*.
\]

Then as a result we obtain the formula for the Asian option with the payoff \( \Lambda(J) \).
5.4 The hedging strategy

Lastly we shortly consider how to construct the hedging strategy for the considered asian options on the forward contracts. Since these options were of the European type, the theory from Chapter 3 can be used.

1. We assume, that the properties of the Asian option such as the form of the function \( f(t, F) \) and the payoff are known.

2. Using the appropriate methods we obtain the function \( V(t, J, F) \) from equation (5.6) either in an explicit or in a numerical form.

3. The portfolio capital and its increments are given by (2.14) and (2.15) respectively.

4. According to the equation (5.5) we see, that again

\[
\gamma_t = \frac{\partial V}{\partial F} \\
\beta_t = \frac{V}{B_t}
\]  

(5.22)

and these formulas provide the hedging strategy for Asian type options.
Chapter 6

Portfolio dynamics

One of the most interesting and complicated problems on the futures market is the pricing and the hedging of the portfolio which is composed of options on futures with different maturities. In this last chapter we consider an application of the obtained results to study the dynamics of such portfolio. We consider the portfolio of two options on futures with different maturity dates \( T_1 \) and \( T_2 \) which prices are \( F(t, T_1) \) and \( F(t, T_2) \) respectively. Our main interest is to show how different models describe this dynamics.

One of them states that there is a deterministic dependence between \( F(t, T_1) \) and \( F(t, T_2) \) meanwhile other assumes a stochastic dependence. In the first case it is enough to know the price of one futures to determine the price of the portfolio. However, in other model such technics leads to the significant errors in pricing when using the standard approach.

We compare three models

1. The Black-Scholes model with the constant volatility;
2. The single-factor model with a time-dependent volatility;
3. The multi-factor model in the form (2.1).

To illustrate this difference, we consider graphs of the portfolio price against the price of the futures \( F(t, T_1) \). To make it we at first consider connections between \( F(t, T_1) \) and \( F(t, T_2) \) in these models.

6.1 Models description

1. The Black-Scholes Model. We recalling, that in the Black-Scholes model the futures price is given by

\[
F(t, T) = e^{(T-t)}S(t),
\]

(6.1)
where $S(t)$ is an asset (or spot) price. Under the martingale measure this takes a form

$$F(t, T) = e^{rT + \sigma w_t - \frac{1}{2} \sigma^2 T}. \quad (6.2)$$

From the latter formula we see, that if

$$F_1(t) := F(t, T_1)$$

and

$$F_2(t) := F(t, T_2)$$

then

$$F_2(t) = e^{r(T_2 - T_1)} F_1(t). \quad (6.3)$$

So, in this case we have a deterministic dependence between the prices of futures written on the same underlying asset with different maturities.

2. The single-factor model

The single-factor model is represented by the following stochastic differential equation

$$dF(t, T) = F(t, T) \sigma(t, T) dw_t. \quad (6.4)$$

From this equation we obtain that

$$F_j(t) = F_j(0) \exp \left\{ -\frac{1}{2} \int_0^t \sigma^2(u, T_j) \, du + \int_0^t \sigma(u, T_j) \, dw_u \right\}, \quad (6.5)$$

for $j = 1, 2$.

It will be more convenient to consider the logarithm of the forward price

$$L_j(t) := \log F_j(t) = \log F_j(0) - \frac{1}{2} \int_0^t \sigma^2(u, T_j) \, du + \int_0^t \sigma(u, T_j) \, dw_u. \quad (6.6)$$

We see that $L_j(t)$ is a normally distributed random variable with an expectation

$$\mu_j(t) = \log F_j(0) - \frac{1}{2} \int_0^t \sigma^2(u, T_j) \, du$$

and a variance defined by

$$\sigma_j^2(t) = \int_0^t \sigma^2(u, T_j) \, du.$$
Moreover, the covariance of $L_1(t)$ and $L_2(t)$ is equal to

$$s_{12}(t) = \text{Cov}(L_1(t), L_2(t)) = \int_0^t \sigma(u, T_1)\sigma(u, T_2) \, du.$$ 

We also denote by $\rho$ a correlation

$$\rho(t) = \frac{s_{12}(t)}{s_1(t)s_2(t)}.$$ 

Let us fix the time $t$. In this case the joint density function of $(L_1(t), L_2(t))$ will be

$$f_{12}(x,y) = \frac{1}{2\pi s_1(t)s_2(t)\sqrt{1-\rho^2(t)}} \exp\left\{-\frac{1}{2(1-\rho^2(t))} Q(x,y) \right\}, \quad (6.7)$$

where

$$Q(x,y) = \frac{(x-\mu_1(t))^2}{s_1^2(t)} + \frac{(y-\mu_2(t))^2}{s_2^2(t)} - \frac{2\rho(t)(x-\mu_1(t))(y-\mu_2(t))}{s_1(t)s_2(t)}. \quad (6.8)$$

We want to get the distribution of $L_2(t)$ by the condition that $L_1(t) = x$. We use the fact that

$$f_{L_2|L_1}(y|x) = \frac{f_{12}(x,y)}{f_1(x)}, \quad (6.9)$$

where $f_1(x)$ is a density function of $L_1(t)$ and

$$f_1(x) = \frac{1}{s_1(t)\sqrt{2\pi}} \exp\left\{-\frac{(x-\mu_1(t))^2}{2s_1^2(t)} \right\}.$$ 

Using (6.9) we get, that

$$\text{Law}\{L_2(t)|L_1(t)\} = \mathcal{N}\left(\hat{\mu}(t), \hat{s}^2(t)\right), \quad (6.10)$$

where $\hat{\mu}(t)$ is a conditional mean

$$\hat{\mu}(t) = \mu_2(t) + \frac{s_{12}(t)}{s_1^2(t)}(L_1(t) - \mu_1(t)) \quad (6.11)$$

and $\hat{s}^2(t)$ is a conditional variance given by

$$\hat{s}^2(t) = s_2^2(t)(1-\rho^2(t)). \quad (6.12)$$
Remark 6. From the latter equation \((6.12)\) it follows, that the smaller the absolute value of the correlation between the price logarithms, the greater the conditional variance.

Now, we can estimate the price \(F_2(t)\) through the price \(F_1(t)\) using the distribution \((6.10)\). Indeed, we have that

\[
P\{|L_2(t) - \hat{\mu}(t)| \leq 3\hat{s}(t)\} > 0.995, \tag{6.13}
\]
or

\[
P\{\exp\{\hat{\mu}(t) - 3\hat{s}(t)\} < F_2(t) < \exp\{\hat{\mu}(t) + 3\hat{s}(t)\}\} > 0.995. \tag{6.14}
\]

The last estimation will be used further to construct an estimation of the portfolio price.

3. The multi-factor model. The estimations for the multi-factor model can be obtained in the same way as for the single-factor model. Due to this fact, it is possible to use results from the previous consideration with slight differences. More precisely, only the following quantities change their form

- the logarithm of the price has now the following form
  \[
  L_j(t) = \log F_j(0) - \frac{1}{2} \sum_{i=0}^{n} \int_{0}^{t} \sigma_i^2(u, T_j) \, du + \sum_{i=0}^{n} \int_{0}^{t} \sigma_i(u, T_j) \, dw_i; \tag{6.15}
  \]

- the expectation of the logarithm is equal to
  \[
  \mu_j(t) = \log F_j(0) - \frac{1}{2} \sum_{i=0}^{n} \int_{0}^{t} \sigma_i^2(u, T_j) \, du; \tag{6.16}
  \]

- the variance of the logarithm is given by
  \[
  s_j^2(t) = \sum_{i=0}^{n} \int_{0}^{t} \sigma_i^2(u, T_j) \, du; \tag{6.17}
  \]

- the covariance is
  \[
  s_{12}(t) = \text{Cov}(L_1(t), L_2(t)) = \sum_{i=0}^{n} \int_{0}^{t} \sigma_i(u, T_1)\sigma_i(u, T_2) \, du. \tag{6.18}
  \]

In these new terms the estimations \((6.13)\) and \((6.14)\) will hold for this multi-factor model taking into account that the conditional mean and the conditional variance are given by \((6.11)\) and \((6.12)\) respectively.
6.2 An illustration of results

Lastly we show figures illustrating the difference in dynamics of the portfolio within considered models. There are presented graphs of the portfolio dynamics within the Black-Scholes model and the single-factor model due to the significant difference between these models. The graphs for the portfolio dynamics within the multi-factor model are similar to the single-factor model, but with the greater explanatory power.

We consider the single-factor model with following parameters

- \( F_1(0) = 10, F_2(0) = 11. \)
- \( T_1 \) is four months or \( 4/12 \), \( T_2 \) is five months or \( 5/12 \), \( T = 0.32 \) - just before the expiration of the first forward.
- \( \sigma = 1.5 \) for the single-factor model.
- the portfolio consists of the long put on \( F_1 \) with the strike price \( K = 11 \) and the short put on \( F_2 \) with the same strike price.
- the risk-free rate \( r = 0.07. \)

Let us consider some helpful graphs for the single-factor model: the conditional variance and the correlation between prices in the time (Figure 6.1)

![Figure 6.1: The conditional variance (red curve) and \( 1 - \rho^2 \) (green curve). Correlation is an decreasing function of the time and the conditional variance is a increasing one.](image)

and the price behavior at the maturity time of the option (Figure 6.2).
Figure 6.2: The prices of underlying futures at the maturity time of the options $T = 0.32$ against the price of the first futures: the price of $F_1$ (red curve), the mean price of the $F_2$ (green curve) and the upper and lower prices for $F_2$ (dashed curves).

Now we proceed to compare the graphs for the portfolio dynamics within the Black-Scholes model and the single-factor model. We compare the portfolio price against the price of $F_1$ at time $t = 0.1$ (Figure 6.3 and Figure 6.4), $t = 0.2$ (Figure 6.5 and Figure 6.6) and $t = 0.25$ (Figure 6.7 and Figure 6.8). It can be seen, that at the start point of the graph the confidential domain between the upper and the lower price for the portfolio is small, but with the time it increases rapidly tending to the up and low payoff.
Figure 6.3: The Black-Scholes model: the payoff (red line) and the price of the portfolio at time $t = 0.1$ (green line).

Figure 6.4: The single-factor model: lower (red) and upper (green) bounds for the payoff, the confidential domain for the price of the portfolio at time $t = 0.1$ (between the dashed lines).
Figure 6.5: The Black-Scholes model: the payoff (red line) and the price of the portfolio at time $t = 0.2$ (green line).

Figure 6.6: The single-factor model: lower (red) and upper (green) bounds for the payoff, the confidential domain for the price of the portfolio at time $t = 0.2$ (between the dashed lines).
Figure 6.7: The Black-Scholes model: the payoff (red line) and the price of the portfolio at time $t = 0.25$ (green line).

Figure 6.8: The single-factor model: lower (red) and upper (green) bounds for the payoff, the confidential domain for the price of the portfolio at time $t = 0.25$ (between the dashed lines).
Chapter 7

Conclusions

In this work we consider derivatives on the forward market where the forward price admits the multi-factor dynamics (2.1). The study was focused on the developing of the pricing and hedging methods using the martingale approach. Three main types of options were in consideration: an European, an American and an Asian ones. For all these types the explicit pricing formulas were obtained for a wide class of options, when it was possible. In other cases were presented analytical results and some recommendations for the numerical calculations or an estimation such as for problems (4.27) for an American Put with the finite horizon and (5.16) for the arithmetic average Asian options.

For all considered options corresponding hedging strategies were presented in an explicit form in the cases of the explicit formulas for the options prices. Otherwise, there were given formulas for the hedging strategy expressed by the price of the option.

Lastly, using the pricing results it was shown the significant difference between the well-known Black-Scholes model and the multi-factor model in describing the dynamics of the portfolio of options.

Though the wide class of options was in the consideration, of course there are a lot of options which are not priced in the framework of the multi-factor model. The further research can be extended to take into consideration other exotic options like barrier options or options which payoff depends on the forward prices with different times to maturity, e.g. swaptions.

The other direction of the further research is related to the model itself. The interesting way is to add to the multi-factor diffusion model a jump component. In this case the real market data should be used to calculate the explanatory power of the new model, as well as the pricing methods and hedging strategies should be developed for the new model. The latter problem involves the theory of Levy processes.
Notation

$(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ A filtered probability space with a set of outcomes $\Omega$, $\sigma$-algebra $\mathcal{F}$, a flow or a filtration $\mathbb{F}$ and a probability measure $\mathbb{P}$.

$\mathbb{P}$ A probability measure.

$\mathbb{E}[X]$ An expectation of the random variable $X$.

$\text{Cov}(X, Y)$ A covariance of the random variables $X$ and $Y$.

$\text{Law}(X)$ A probability law of the random variable $X$.

$\pi = (\beta, \gamma)$ An investment portfolio.

$F(t, T)$ A forward price at the time $t$ for the delivery at time $T$.

$\sigma_i(t, T)$ An $i$-th volatility function.

$w_i^t$ An $i$-th brownian motion.

$\tau$ A stopping time.

$\mathcal{M}_i^T$ The set of stopping times taking their values in $[t, T]$.

$\mathcal{M}_i$ The set of stopping time taking their values in $[t, \infty)$.

$G$ A payoff function (contingent claim).

$V$ A value function.

$X^\pi$ A capital of a investment portfolio $\pi$.

$L_Y$ An infinitesimal operator of the process $Y$. 

41
Bibliography


*Essays on Pricing Commodity Derivatives.* Helsinki School of Economics, HeSE print, Finland.  


