Abstract

Spliting out the valuation of the options into two cases - discrete and continuous - we present the closed-form solution for the discrete case and the method of the Laplace transformation for the continuous case where the option values can be computed only numerically. The applied method of the Laplace transformation leads to using then the numerical method of the inverse Laplace transformation. We consider Gaver-Stehfest method of the inverse Laplace transformation that was used by Kimura to valuate the continuous installment options and Kryzhnyi method that was never applied before to valuate the options. Both methods are compared.
Chapter 1

Introduction

Starting from the ancient age people tried to hedge their trading risks. We can find the predecessors of trading options looking at the history of ancient Rome, Phoenicia, Greece. With time passing options became more and more popular, drawing not only the hedgers, but the speculators also. In 1848, the new page of options’ history was written as ”Chicago Board of Trade” (CBOT) was set up and the options started being traded officially. Developing rather slowly the option market then got into the boom in the end of 1960’s - middle 1970’s caused by the opening of the ”Chicago Board Options Exchange” (CBOE) and the appearing of the well-known Black-Scholes model. This was the time when the modern history of the options started. Since that moment the interest in options was growing: the volumes of trading have increased, variety of new options types has appeared. Troubles with the analytical valuation of some options caused a new way of the valuation, followed the technical progress, - the numerical one. Today the numerical valuation of the options is very important part of the Financial Mathematics. Some types of options can not be valuated analytically, but by numerical computation only. One of the problems that looks interesting to study - the numerical valuation of the installment options - underlies the current master thesis.

Installment options are the financial derivatives where the small initial premium is paid up-front and the other part of the premium is divided into the installments to be paid during the lifetime of the contract up to maturity. At each installment date the investor has the right to decide if he continues to pay for the contract or he terminates paying, allowing the option to lapse. In this work we consider both cases of the installments payment: discrete and continuous. The continuous case means that the holder pays a stream of the installments at a given rate per unit time. In real life it looks like accumulating of the premium sum by a certain continuous rate, afterwards
paid by the holder in the case of exercising. The holder has a choice to stop the contract at any time before the maturity. Nowadays the installment options are rather widely traded in the financial markets. For instance, installment warrants on Australian stocks listed on the Australian Stock Exchange; the Deutsche Bank’s 10-year warrant, etc. Taking in mind the nature of the installment options we can find a number of other contracts similar to them: some life insurance contracts and capital investment projects might be considered as installment options (see Dixit and Pindyck [8]). Thomassen and Van Wouwe [20] applied the installment option in pharmacy comparing the development of a new drug, evolving 6 stages, with a 6-variate installment option; MacRae [17] modeled the employee stock option as an installment option; and so on.

The research on installment options seems to be extremely needed. Despite this fact, just several studies of the installment options exist. The first public paper was an article by Karsenty and Sikorav [12]. Later on, Davis et al. [6], [7] applied the martingale approach, derived no-arbitrage bounds for the price of the installment option and considered the static hedging strategies. Ben-Ameur et al. [3] developed a dynamic-programming procedure to value American installment options. This approach was applied to installment warrants in the Australian Stock Exchange. The study of the continuous installment options was presented in the works of Ciurlia and Roko [4] and Alobaidi [1]. Ciurlia and Roko studied the American case applying the ”multipiece exponential function” (MEF) method to derive an integral form of the initial premium. Their applied technique suffers the serious drawback, because the MEF method generates the discontinuity in the optimal stopping and early exercise boundaries. Alobaidi analyzed the European case using the Laplace transformation to solve the free boundary problem. However, the method used is rather specific and is not appropriate for a numerical computation.

Being interested, first of all, in the numerical valuation of the installment options, we consider mainly the papers of Griebsch et al. [11], where the closed-form solution for the discrete installment options is deduced, and Kimura [13], where the method of a Laplace transformation for the numerical valuation of the continuous installment options is applied.

The main idea of this work is a valuation of the installment options. Several objectives should be achieved then. Some misunderstanding in the literature about the features of the installment option and its difference from the compound option forces us to set the first goal - make the ideas clear as for these types of options: to specficate them, describe their essentials and carry out the classification. The second goal is to consider the discrete case of the installment options, presenting the closed-form formula for their valuation.
As the third aim we set an investigation of the numerical valuation of the options in the continuous case through the Laplace transform method, where we develop Matlab code to deal with it. The computation of Greeks of the continuous installment options implementing the Matlab software is the next objective. The final goal is to compare two methods of the numerical inverse Laplace transformation - the Gaver-Stehfest and the Kryzhnyi methods, respectively - that we use for the valuation of the options and their Greeks.

The work is composed in the following way. In Chapter 2 we introduce the compound and the installment options, consider their essentials, showing the differences between them. The closed-form solution for the discrete installment options and the formulation of the valuation problem of the continuous installment options are also presented in Chapter 2. In Chapter 3 we describe the Laplace-Carson transformation method and the methods of the inverse Laplace transformation, that are used for the numerical valuation of the continuous installment options. The theoretical and computational results of the valuation of the stopping boundary, the values of the options and their Greeks are included in Chapter 4. In Conclusion we summarise the ideas and results mentioned in above chapters. The program code is attached to the work in the Appendix.
Chapter 2

Installment and Compound options

The first study of the installment type options was related to the investigation of compound options. Generally, the compound option is the particular case of an installment option. As it is mentioned in some papers (e.g. Alobaidi [1], Davis et al. [6]) the compound option is a simple installment option in the case of two discrete payments, meaning thereby an option on an option. But having a look through the derivatives history we see that the compound option appeared before the installment options and started to be an initial point for their further investigation. It leads us to the necessity to distinguish between these two types of options that look rather similar.

2.1 Compound options

The compound option is an option on an option. Subsequently the compound option has two strike prices and two expiration dates. Paying the initial premium the holder buys the compound option, on the first expiration date he can choose either to buy the option or not. In this moment the compound option turns into the European vanilla option, that can be exercised or not on the second expiration date (see Figure 2.1).

The extra opportunity to make a choice at time \( t_1 \) always makes the total amount of the premium to be paid for a compound option higher than the price of the plain vanilla option. However, obviously, the price of a compound option at \( t_0 \) is smaller than the vanilla option price, since the premium is pulled apart in time. Basically, the compound options may have either the European type or the American type.

The investigation of the compound options was initiated by Geske [10] in
Figure 2.1: The lifetime of a compound option, \( t_0 \) is a compound option inception date, \( t_1 \) is the first expiration date, \( T \) is the time of maturity, \( z_0 \) is a compound option initial premium, \( z_1 \) is the first strike price, \( K \) is the strike price in the time of maturity.

1979. In the framework of the Black-Scholes model we consider the European call option \( c(t,T,S_t,K) \) with a strike price \( K \) maturing at time \( T \), where \( S_t \) is the spot price of the underlying. The payoff of such plain vanilla option is equal to \( \max(S_T - K, 0) \). The initial premium \( z_0 = c_{\text{call}} \) to acquire the call on a call option is paid at the time \( t_0 \), the exercise price \( z_1 \) to obtain the European call option is paid at the time \( t_1 \) and the final exercise is at the time \( T \). Taking the decision at the moment \( t_1 \) either to terminate the option or to continue to hold it, the investor concerns the relation between the value of the European call option and the exercise price to be paid for it. If the exercise price \( Z_1 \) is higher than the option value the holder terminates option, if this is not the case, the compound option will be exercised. So the value of a compound call on a call option at the time \( t_1 \) is given by

\[
c_{\text{call}} = \max(c(t_1,T,S_{t_1},K) - z_1, 0).
\]

The price of the compound call on a call option derived by Geske:

\[
\begin{align*}
c_{\text{call}} &= S_te^{-\delta(T-t)}\Phi_2\left(d_+(S_t,S_{t_1}^*,t_1-t),d_+(S_t,K,T-t);\sqrt{(t_1-t)/(T-t)}\right) \\
&- Ke^{-r(T-t)}\Phi_2\left(d_-(S_t,S_{t_1}^*,t_1-t),d_-(S_t,K,T-t);\sqrt{(t_1-t)/(T-t)}\right) \\
&- z_1e^{-r(t_1-t)}\Phi_2(d_-(S_t,S_{t_1}^*,t_1-t)),
\end{align*}
\]

where

\[
d_\pm(a,b,\tau) = \frac{\log(a/b) + (r - \delta \pm \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}.
\]

Here are

- \( S_{t_1}^* \) - the critical price of the asset such that \( c(t_1,T,S_{t_1},K) = z_1 \);
- \( \delta \) - the dividend yield;
- \( \sigma \) - the volatility;
\( r \) - the interest rate;
\( \Phi_2(x, y; \rho) \) - the bivariate cumulative normal distribution function, where
\( \rho = \sqrt{(t_1 - t)/(T - t)} \) is the correlation coefficient for overlapping Brownian increments.

Davis et al. [6] recommended an alternative way of looking at the compound call on a call option. Actually at time \( t_0 \) the holder buys the underlying call for the total amount of premium \( z = z_0 + z_1 e^{-r(t_1-t_0)} \), i.e. the initial premium and the discounted value of the second premium. At the same time the holder has the right to get rid of this option at time \( t_1 \) selling it for the price \( z_1 \). Therefore the total premium of the compound call on a call option might be presented as the underlying call option plus a put on the call with exercise at the time \( t_1 \) and the strike price \( z_1 \)

\[
z_0 + z_1 e^{-r(t_1-t_0)} = c(t_0, T, S_{t_0}, K) + p_{\text{call}}(t_0, t_1, S_{t_0}, z_1), \quad (2.1)
\]

where \( p_{\text{call}} \) denotes the compound put on a call option.

Such decomposition shows that the total premium for a call on a call exceeds the European call option price by the value of the put on the call option.

There exist 4 types of the compound options: a call on a call, a call on a put, a put on a call and a put on a put. Their prices are given through the following formulas [15]

\[
c_{\text{put}} = Ke^{-r(T-t)}\Phi_2\left(-d_-(S_t, S_{t_1}^*, t_1-t), -d_-(S_t, K, T-t); \sqrt{\frac{(t_1-t)}{(T-t)}} \right)
\]
\[
- S_t e^{-\delta(T-t)}\Phi_2\left(-d_+(S_t, S_{t_1}^*, t_1-t), -d_+(S_t, K, T-t); \sqrt{\frac{(t_1-t)}{(T-t)}} \right)
\]
\[
- z_1 e^{-r(t_1-t)}\Phi_2(-d_-(S_t, S_{t_1}^*, t_1-t));
\]

\[
p_{\text{put}} = S_t e^{-\delta(T-t)}\Phi_2\left(d_+(S_t, S_{t_1}^*, t_1-t), -d_+(S_t, K, T-t); -\sqrt{\frac{(t_1-t)}{(T-t)}} \right)
\]
\[
- Ke^{-r(T-t)}\Phi_2\left(d_-(S_t, S_{t_1}^*, t_1-t), -d_-(S_t, K, T-t); -\sqrt{\frac{(t_1-t)}{(T-t)}} \right)
\]
\[
+ z_1 e^{-r(t_1-t)}\Phi_2(d_-(S_t, S_{t_1}^*, t_1-t));
\]
The evolution of the compound option theory is presented in Table 2.1.

<table>
<thead>
<tr>
<th>Reference</th>
<th>Approach</th>
<th>Put-call alternating</th>
</tr>
</thead>
<tbody>
<tr>
<td>Geske (1977, 1979)</td>
<td>PDE</td>
<td>Put/Call</td>
</tr>
<tr>
<td>Agliardi and Agliardi (2003)</td>
<td>PDE</td>
<td>Call</td>
</tr>
</tbody>
</table>

Table 2.1: The evolution of compound option theory [16].

2.2 Installment options

The definition of the installment options can be formulated in the following way: the option where the premium is divided into different parts and is paid during the option lifetime. Every installment date presents the moment when the holder takes the decision either to continue to pay the premiums or allow the contract to lapse. As in the case of the compound options the total premium of the installment option is always higher than the vanilla options premium. This property can be explained by the additional opportunities to terminate the contract without paying the whole sum of the premium. The installment option is interesting for the investors who are ready to overpay for the advantage to terminate the payments and reduce the losses if their investment position goes wrong (see Figure 2.2).

Dealing with the installment options we can separate two cases of the installment payments: discrete and continuous.

**Discrete case** means that the installment option has a finite number of exercise dates, e.g. 3, 6, 8. Note that the case of a 2-payments installment option is exactly the compound option.

Griebsch et al. [11] give the following example of a discrete installment option in the Foreign Exchange market.

**Example.** A company from the Euro-zone wants to hedge receivables from an export transaction in USD due in 12 months time. The goal of the company...
Figure 2.2: Two scenarios of an installment option [11]. The left-side figure: the continuation of payments until the maturity. The right-side figure: the termination of the contract after the first installment date.

is to be able to buy EUR at a lower spot price if EUR goes down on the one side, but on the other side to be hedged against a stronger EUR. The future income in USD will be under the review at the end of each quarter. To achieve the target the company buys a EUR installment call option with 4 equal quarterly premium payments (Table 2.2).

<table>
<thead>
<tr>
<th>Spot reference</th>
<th>1.150 EUR-USD</th>
</tr>
</thead>
<tbody>
<tr>
<td>Maturity</td>
<td>12 months</td>
</tr>
<tr>
<td>Notional</td>
<td>USD 100000</td>
</tr>
<tr>
<td>Premium per quarter of the installment option</td>
<td>USD 12500</td>
</tr>
<tr>
<td>The total amount of the premium</td>
<td>USD 50000</td>
</tr>
<tr>
<td>Premium of the corresponding European vanilla call</td>
<td>USD 4000000</td>
</tr>
</tbody>
</table>

Table 2.2: Example of an installment call option [11]. The total premium for an installment option exceeds the premium of the corresponding vanilla option.

The company pays 12500 USD on the trade date. After one quarter, the company has the right to prolong the installment option. To do this the company must pay another 12500 USD. Such decisions have also to be taken after 6 months and 9 months. If at one of these three decision days the company does not pay, then the contract terminates. If all the premium payments are made, then in 9 months the contract turns into a European vanilla call option.

If the EUR-USD exchange rate is above the strike at maturity, then the company buys EUR at maturity at a rate of 1.150. If the EUR-USD exchange
rate is below the strike at maturity the option expires worthless.

**Continuous case** means that the holder pays a stream of installments at a given rate per unit time. The holder makes a choice to stop the contract at any time before the maturity. This opportunity turns the valuation of the installment options into a free boundary problem. For the continuous case there exist two types of the installment options: European and American.

The first published paper devoted to the installment options was written by Karsenty and Sikorav [12] in 1993. However, earlier in 1984 Geske and Johnson introduced so-called "sequential compound option" (SCO) or "multi-fold compound option".

A multi-fold compound option is the composition of the European vanilla options presenting simply an option on an option on an option and so on. Each fold option may be either call or put. Actually a multi-fold compound

![Diagram of installment options]

**Figure 2.3:** The classification of installment options.
option is nothing else than the discrete installment option. In installment options the premiums paid on each installment date can be also presented as exercise prices of every new option, so meaning then the multi-fold compound option. The type of every option - if it is a call or a put - is defined in advance arbitrarily. Different authors call these options in different ways - either an installment options or a multi-fold compound options. To avoid the problem of misunderstanding we present the scheme (Figure 2.3) distinguishing types of options.

The problems of the valuation of discrete and the European continuous installment options are presented in the next sections and chapters.

2.2.1 The discrete case

In this section we present the closed-form formula for the valuation of the discrete installment options.

We consider the standard Black-Scholes model, where the asset price $S_t$ follows the geometric Brownian motion

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t,$$

(2.2)

where $\mu = (r - \delta)$, $r$ and $\delta$ denote the interest rate and the continuous dividend yield respectively. $\sigma$ is volatility and $dW_t$ is a standard Brownian motion on a risk-neutral probability space.

![n-variate installment option](image)

Figure 2.4: The lifetime of an installment option, $t_0$ is an installment option inception date, $t_i, i = 1, ..., n$ is an installment date, $z_0$ is an installment option initial premium, $z_i, i = 1, ..., n$ is an installment option premium.

The installment option has $n$ installment dates which are denoted in the Figure 2.4 by $t_1, t_2, ..., t_n = T$. On every of these dates the holder has to pay the premium $z_1, z_2, ..., z_{n-1}$ if he wants to continue the contract. On the inception date $t_0$ the holder buys the installment option by the price $V_0$ equal to the initial premium $z_0$. Computing the value of the installment option $V_0$
to enter the contract, we start with the option payoff at the time of maturity $T$

$$V_n = \max(\varphi_n(s - z_n), 0) = (\varphi_n(s - z_n))^+,$$

where $s = S_T$ is the spot price of the underlying at $T$, $z_n$ the exercise price. The coefficient $\varphi_n$ is equal to $+1$ if the underlying option is the vanilla call, and $-1$ if the underlying option is the vanilla put. Discounting expectation we can define the value of the underlying option at time $t_{n-1}$. Going the same procedure we can find the payoff function of this option. At the time $t_i$ the holder can stop paying the premiums, terminating the contract, or pay $z_i$ to keep it alive. In the case of continuation the value is

$$e^{-r(t_{i+1} - t_i)}\mathbb{E}[V_{i+1}(S_{t_{i+1}})|S_{t_i} = s].$$

Then the value at time $t_i$ which we obtain by the backward recursion is given by

$$V_i(s) = \begin{cases} \max[e^{-r(t_{i+1} - t_i)}\mathbb{E}[V_{i+1}(S_{t_{i+1}})|S_{t_i} = s] - z_i, 0] & \text{for } i = 1, \ldots, n - 1, \\ V_n(s) & \text{for } i = n. \end{cases}$$

The unique arbitrage-free price of the installment option is

$$V_0(s) = z_0 = e^{-r(t_1 - t_0)}\mathbb{E}[V_1(S_{t_1})|S_{t_0} = s].$$

Griebsch et al. [11] derived the closed form-solution to valuate the installment option. They applied the Curnow and Dunnett integral reduction technique to solve the equation (2.3).

Denote by $\vec{z} = (z_1, \ldots, z_n)$ the exercise price vector, $\vec{t} = (t_1, \ldots, t_n)$ the vector of the exercise dates and $\vec{\varphi} = (\varphi_1, \ldots, \varphi_n)$ the vector of the Put/Call coefficients of the n-variate installment option. Then the closed-form formula of the
n-variate installment option value reads

\[
V_n(S_0, z^T, \bar{t}, \bar{\varphi}) = \\
e^{-rt_n} S_0 \varphi_1 \cdots \varphi_n \\
\times \Phi_n \left( \frac{\ln S_0^{S_t} + \mu^+(t_1)}{\sigma \sqrt{t_1}}, \frac{\ln S_0^{S_t} + \mu^+(t_2)}{\sigma \sqrt{t_2}}, \cdots, \frac{\ln S_0^{S_t} + \mu^+(t_n)}{\sigma \sqrt{t_n}}; R_n \right) \\
- e^{-r t_n} z_n \varphi_1 \cdots \varphi_n \\
\times \Phi_n \left( \frac{\ln S_0^{S_t} + \mu^-(t_1)}{\sigma \sqrt{t_1}}, \frac{\ln S_0^{S_t} + \mu^-(t_2)}{\sigma \sqrt{t_2}}, \cdots, \frac{\ln S_0^{S_t} + \mu^-(t_n)}{\sigma \sqrt{t_n}}; R_n \right) \\
- e^{-r t_{n-1}} z_{n-1} \varphi_1 \cdots \varphi_{n-1} \\
\times \Phi_{n-1} \left( \frac{\ln S_0^{S_t} + \mu^-(t_1)}{\sigma \sqrt{t_1}}, \frac{\ln S_0^{S_t} + \mu^-(t_2)}{\sigma \sqrt{t_2}}, \cdots, \frac{\ln S_0^{S_t} + \mu^-(t_{n-1})}{\sigma \sqrt{t_{n-1}}}; R_{n-1} \right) \\
\vdots \\
- e^{-r t_2} z_2 \varphi_1 \varphi_2 \Phi_2 \left( \frac{\ln S_0^{S_t} + \mu^-(t_1)}{\sigma \sqrt{t_1}}, \frac{\ln S_0^{S_t} + \mu^-(t_2)}{\sigma \sqrt{t_2}}; \rho_{12} \right) \\
- e^{-r t_1} z_1 \varphi_1 \Phi_1 \left( \frac{\ln S_0^{S_t} + \mu^-(t_1)}{\sigma \sqrt{t_1}} \right) \\
= e^{-r t_n} S_0 \prod_{i=1}^{n} \varphi_i \Phi_n \left[ \left( \frac{\ln S_0^{S_t} + \mu^+(t_m)}{\sigma \sqrt{t_m}} \right)_{1, \ldots, n} \right] \\
- \sum_{i=1}^{n} e^{-r t_i} z_i \prod_{j=1}^{i} \varphi_j \Phi_j \left[ \left( \frac{\ln S_0^{S_t} + \mu^-(t_m)}{\sigma \sqrt{t_m}} \right)_{1, \ldots, j} \right],
\]

(2.4)

where \( S_t^{*} \) is such a spot price \( S_t \) that \( V_t(S_t^{*}) = z_i \). \( \mu^{(\pm)} \) is equal to \( r \pm \frac{1}{2} \sigma^2 \). The correlation coefficients \( \rho_{ij} \) for overlapping Brownian increments are defined as \( \sqrt{t_i/t_j} \).

In comparison to other methods (see [3]), the presented closed-form formula suggested by Griebsch et al. [11] seems to be the most convenient way to value the discrete installment options.

### 2.2.2 The continuous case

In this section we consider the case of the continuous installment options and define the problem we face valuating them.
We assume that the price of the underlying asset \( S_t \) obeys the geometric Brownian motion described by the stochastic differential equation (2.2). The value \( V_t = V(t, S_t; q) \) of the continuous installment option depends on the time \( t \), the spot price of the underlying \( S_t \) and the continuous installment rate \( q \). In time \( dt \) the holder pays the premium \( qdt \) to continue the contract. Using the Itô’s Lemma to derive the dynamics for the value of continuous installment option, we get

\[
dV_t = \left( \frac{\partial V_t}{\partial t} + (r - \delta)S_t \frac{\partial V_t}{\partial S} + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 V_t}{\partial S^2} - q \right) dt + \sigma S_t \frac{\partial V_t}{\partial S} dW_t. \tag{2.5}
\]

We make a portfolio that includes the continuous installment option and \(-\Delta\) amount of the underlying asset

\[\Pi_t = V_t - \Delta S_t,\]

with dynamics

\[
d\Pi_t = dV_t - \Delta dS_t - \Delta(S_t \delta dt). \tag{2.6}
\]

Plugging (2.2) and (2.5) into (2.6), we obtain

\[
d\Pi_t = \left( (r - \delta) \left( \frac{\partial V_t}{\partial S} - \Delta \right) + \frac{\partial V_t}{\partial t} + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 V_t}{\partial S^2} - q - \Delta S_t \delta \right) dt \\
+ \sigma S_t \left( \frac{\partial V_t}{\partial S} - \Delta \right) dW_t.
\]

To remove the risk of uncertainty we choose \( \Delta = \frac{\partial V_t}{\partial S} \). It makes the portfolio riskless now and it has to yield the return \( r \) to avoid the arbitrage opportunities

\[
r \left( V_t - \frac{\partial V_t}{\partial S} S_t \right) = \left( \frac{\partial V_t}{\partial t} + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 V_t}{\partial S^2} - q - \frac{\partial V_t}{\partial S} S_t \delta \right).
\]

Finally, we get an inhomogeneous Black-Scholes partial differential equation (PDE) for the valuation of the continuous installment options

\[
\frac{\partial V_t}{\partial t} + (r - \delta)S_t \frac{\partial V_t}{\partial S} + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 V_t}{\partial S^2} - rV_t = q. \tag{2.7}
\]

\( q \) should be greater than zero, if it is equal to zero the Black-Scholes PDE turns into the homogeneous type.

**The Call case**

We consider the European installment call option \( c(t, S_t; q) \) with the maturity
$T$ and the exercise price $K$. The payoff at the maturity is $\max(S_T - K, 0)$. The opportunity to terminate the contract at any time $t \in [0, T]$ makes the valuation of a continuous installment option an optimal stopping problem. In other words, we need to find such points $(t, S_t)$ that the termination of the option is optimal.

Denote the domain $= [0, T] \times [0, +\infty]$ as $D$, the stopping region and the continuation region as $S$ and $C$, respectively. Then the stopping region is given by

$$S = \{(t, S_t) \in D | c(t, S_t; q) = 0\},$$

the optimal stopping time $\tau^*_c$ is defined by

$$\tau^*_c = \inf \{u \in [t, T] | (u, S_u) \in S\}.$$

Being the complement of $S$ in $D$ the continuation region $C$ has the following representation

$$C = \{(t, S_t) \in D | c(t, S_t; q) > 0\}.$$

The boundary that lies between regions $S$ and $C$ is called stopping boundary, and is defined by

$$S_t = \inf \{S_t \in [0, +\infty) | c(t, S_t; q) > 0\}.$$

The stopping boundary $(S_t)_{t \in [0, T]}$ is essentially the lower critical asset price below which it is necessary to terminate the contract.

In the continuation region $C$, where $S > S_t$, the call value $c(t, S; q)$ can be determined from the inhomogeneous Black-Scholes PDE

$$\frac{\partial c}{\partial t} + (r - \delta)S \frac{\partial c}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 c}{\partial S^2} - rc = q,$$

supplied with the boundary conditions

$$\lim_{S \downarrow S_t} c(t, S; q) = 0,$$

$$\lim_{S \downarrow S_t} \frac{\partial c}{\partial S} = 0,$$

$$\lim_{S \uparrow \infty} \frac{\partial c}{\partial S} < \infty,$$

and the terminal condition

$$c(T, S; q) = \max(S - K, 0).$$
The following integral representation is the value function of the continuous installment call option

\[ c(t, S_t; q) = c(t, S_t) - q \int_t^T e^{-r(u-t)} \Phi(d_-(S_t, S_u, u-t)) du, \quad (2.8) \]

where

\[ d_\pm(a, b, \tau) = \frac{\log(a/b) + (r - \delta \pm \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}. \]

\( c(t, S_t) = c(t, S_t; 0) \) is the value of the European vanilla call option

\[ c(t, S_t) = S_t e^{-\delta(T-t)} \Phi(d_+(S_t, K, T - t)) - Ke^{-r(T-t)} \Phi(d_-(S_t, K, T - t)). \]

The proof is given in the work of Kimura [13].
From expression (2.8) we can see that the price of the continuous installment option is the difference between the European vanilla call option and the expected discounted value of the installment premiums along the optimal stopping boundary. Actually due to the conditions, the optimal stopping boundary \((S_t)_{t \in [0, T]}\) obeys the integral equation

\[ c(t, S_t) - q \int_t^T e^{-r(u-t)} \Phi(d_-(S_t, S_u, u-t)) du = 0. \]

To find the values of the options and, therefore, the optimal stopping boundaries, we need to use a numerical approach. In our current work we consider the Laplace-Carson transformation method for computation. The Laplace-Carson transformation method and inverse Laplace transformation methods are presented in the next chapter.

**The Put case**

We act in this case analogously to the call case approach. Consider the European installment put option with the maturity date \(T\) and the exercise price \(K\). Now \(\overline{S}_t\) is the upper asset price above which the holder has to terminate the contract. The stopping boundary \((\overline{S}_t)_{t \in [0, T]}\) also divides \(D\) into 2 regions: a continuation region \(C = \{(t, S_t) \in [0, T] \times [0, \overline{S}_t]\}\) and a stopping region \(S = \{(t, S_t) \in [0, T] \times [\overline{S}_t, \infty]\}\).

In the continuation region \(C\), where \(S < \overline{S}_t\), the value of the continuous installment put option \(p(t, S; q)\) can be found from the inhomogeneous Black-Scholes PDE

\[ \frac{\partial p}{\partial t} + (r - \delta)S \frac{\partial p}{\partial S} + \frac{1}{2}\sigma^2S^2 \frac{\partial^2 p}{\partial S^2} - rp = q, \]
supplied with the boundary conditions

\[ \lim_{S \uparrow S_t} p(t, S) = 0, \]
\[ \lim_{S \downarrow 0} \frac{\partial p}{\partial S} = 0, \]
\[ \lim_{S \downarrow 0} \frac{\partial p}{\partial S} < \infty, \]

and the terminal condition

\[ p(T, S; q) = \max(K - S, 0). \]

The value function of the continuous installment put option has the following integral expression [13]

\[ p(t, S_t; q) = p(t, S_t) - q \int_t^T e^{-r(u-t)} \Phi(-d_- (S_t, S_u, u - t)) du, \quad (2.9) \]

where \( p(t, S_t) = p(t, S_t; 0) \) is the value of the European vanilla put option

\[ p(t, S_t) = Ke^{-r(T-t)} \Phi(-d_-(S_t, K, T - t)) - S_t e^{-\delta(T-t)} \Phi(-d_+(S_t, K, T - t)). \]

A decomposition of the total premium

Returning to the Section 2.1 we find the decomposition of the compound option (see Formula (2.1)). There the total premium of a compound option was the sum of the underlying call option plus a put on the call. Following the same idea we also suppose that the premium sum of the continuous installment option is equal to the respective European vanilla option plus the right to leave at any time at a pre-determined rate.

Considering the limiting case of the discrete installment options and using the risk-neutral approach Griebsch et al. proved this idea (see [11]). They observed that the total premium of the continuous installment call option is the European vanilla call option plus an American put option on this European call

\[ c(t, S_t; q) + K_t = c(t, S_t) + P(t, S_t; q), \quad (2.10) \]

where \( K_t = \frac{q}{r}(1 - e^{-r(T-t)}) \) is the discounted sum of the premiums not to be
paid if the contract is terminated at the moment $t$, and for the set $S_{t,T}$ of stopping times with values in $[t,T]$ (a.s.)

$$P_c(t, S_t; q) = \text{ess}\sup_{s \in S_{t,T}} \mathbb{E}[e^{-r(s-t)} \max(K_s - c(s, S_s), 0)]|\mathcal{F}_t]$$

is the value of the American compound put option with the maturity at $T$ written on the European vanilla call option.

This decomposition will be used in the next chapter to obtain the Greeks formulas.
Chapter 3

Methods

3.1 The Laplace Transform

3.1.1 Definitions

It was in the beginning of the 20th century when Bateman [2] (1882-1944) was the first to consider the Laplace transform as a tool for solving integral equations. Nowadays integral transforms are very actively used in various ways to solve problems of the mathematical modeling. Among other things Cohen [5] presents a couple of applications of the Laplace transform in heat conduction in a rod, laser anemometry and exotic options valuing.

**Definition 1** Assume that a real valued function \( f(t) \) is defined for all positive \( t \) in the range \((0, \infty)\). Then the Laplace transform of the function \( f(t) \) is defined by

\[
\mathcal{L}\{f(t)\} = \int_0^\infty e^{-\lambda t} f(t) \, dt, \quad (3.1)
\]

if the integral \( \int_0^\infty e^{-\lambda t} f(t) \, dt \) is converges. It is straightforward to see that applying the Laplace transform for a partial differential equation with two variables (in our case it’s time and the asset price) will reduce it to an ordinary differential equation, which is a much simpler problem. In our work some generalization of the Laplace transform by Carson is used, called ”the Laplace-Carson transform”. The only reason for using it is that it generates more simple formulas for the transformed values.
Definition 2 For the same assumptions as above the Laplace-Carson transform of the function \( f(t) \) is defined by

\[
\mathcal{LC}\{f(t)\} = \lambda \int_0^\infty e^{-\lambda t} f(t) \, dt.
\]

3.1.2 The Basic Properties
It follows from the definition 2 that if \( f(t) \) and \( g(t) \) are any two functions satisfying the conditions of the definition 3.1 then

\[
\mathcal{LC}\{af(t) + bg(t)\} = \lambda \int_0^\infty e^{-\lambda t} (af(t) + bg(t)) \, dt = a\mathcal{LC}\{f(t)\} + b\mathcal{LC}\{g(t)\}.
\]

Lemma 1 Assuming that \( f(t) \) is continuous and differentiable and \( f'(t) \) is continuous except a finite number of points in any finite interval \((0, T)\) then

\[
\mathcal{LC}\{f'(t)\} = \lambda \mathcal{LC}\{f(t)\} - \lambda f(0).
\]

Proof: The proof is taken from Cohen [5] and applied to the Laplace-Carson transform. In any finite interval \((0, T)\) we can write

\[
\lambda \int_0^T e^{-\lambda t} f(t) \, dt = \sum_{i=0}^{n-1} \lambda \int_{t_i}^{t_{i+1}} e^{-\lambda t} f(t) \, dt,
\]

where \( t_0 = 0, t_n = T \) and \( t_1, t_2, \ldots, t_{n-1} \) are points of discontinuity of \( f'(t) \) on interval \((0, T)\). For each term on the right hand side we can apply the integration by parts,

\[
\int_{t_i}^{t_{i+1}} e^{-\lambda t} f(t) \, dt = e^{-\lambda t_i} f(t_i) + \lambda \int_0^{t_{i+1}} e^{-\lambda t} f(t) \, dt - e^{-\lambda t_{i+1}} f(t_{i+1}) + \lambda \int_0^{t_{i+1}} e^{-\lambda t} f(t) \, dt.
\]

Since \( f(t) \) is continuous we obtain

\[
\lambda \int_0^T e^{-\lambda t} f(t) \, dt = \lambda e^{-\lambda T} f(T) - \lambda f(0+) + \lambda^2 \int_0^T e^{-\lambda t} f(t) \, dt.
\]

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By $T \to \infty$ we get

$$\mathcal{L}\{f(t)\} = \lambda \mathcal{L}\{f'(t)\} - \lambda f(0).$$

\[\square\]

### 3.2 The Inverse Laplace Transform

#### 3.2.1 The Definition and Properties

For a function $F(\lambda) = \mathcal{L}\{f(t)\}$ we denote the inverse Laplace transform as $\mathcal{L}^{-1}\{F(\lambda)\}$, i.e.

$$\mathcal{L}^{-1}\{F(\lambda)\} = f(t).$$

Actually from the definition of the Laplace transform 3.1 it can be seen that the inverse Laplace transform cannot be unique in the class of piecewise continuous functions. If the functions $f(t)$ and $g(t)$ differ only in a finite set of values of $t$, then

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{g(t)\}.$$

Hence, for applying the Laplace transform to our problem it is necessary to be in the area of uniqueness, which is defined by the Lerch’s theorem.

**Theorem 1** (*Lerch’s theorem [5]*). If for a continuous function $f(t)$

$$F(\lambda) = \int_{0}^{\infty} e^{-\lambda t} f(t) \, dt, \quad \lambda > \gamma, \quad (3.2)$$

then there is no other continuous function satisfying (3.2)

Now if we have an ODE solution for the corresponding transformed PDE, and an exact formula for determining $\mathcal{L}^{-1}\{F(\lambda)\}$ we can easily produce a continuous solution for our PDE. Of course for elementary functions the Laplace transform can be computed directly by computing the integral (3.1), so if your original function is one of these, you can find it in the tables of Laplace transforms. For $\mathcal{L}\{f(t)\}$ defined by a rational function, the inverse transform can be computed easily using the expansion theorem. In the general case an analytical formula for the Laplace transform inversion is proved by the Bromowich theorem.
**Theorem 2** [5] Let $f(t)$ have a continuous derivative and let $|f(t)| < Ae^{\gamma t}$, where $\gamma$ and $A$ are positive constants. Define $F(\lambda) = \mathcal{L}\{f(t)\}$, then

$$f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{ut} F(u) du$$

Actually this integral is too special for computing directly, so various numerical methods are applied for computing the function values from its Laplace transform.

### 3.2.2 The Numerical Inverse Laplace Transform

**The Post-Widder formula**

Post and Widder [18] managed to present the original function $f(t)$ as a limit of some sequence, involving the n-th derivative of $F(\lambda)$ on the real axis, which is more convenient for the numerical computation of the inverse Laplace transform than trying to compute the integral on the complex plane. The result is formulated in the following theorem.

**Theorem 3** *(Post and Widder theorem [5]).* If for a continuous function $f(t)$ the integral

$$F(\lambda) = \int_0^\infty e^{-\lambda t} f(t) \, dt,$$

converges for every $\lambda > \gamma$, then

$$f(t) = \lim_{n \to \infty} \left(\frac{-1}{n}\right)^n \frac{1}{n!} \left(\frac{n}{t}\right)^{n+1} F^{(n)}\left(\frac{n}{t}\right).$$

**The Gaver-Stehfest methods**

There are two major problems of using this formula for the Laplace transform inversion. The first problem is differentiating $F(\lambda)$ a large number of times. It can be a big obstacle if $F(\lambda)$ is a complicated function, even if Maple or Mathematica differentiating routines are used. Besides, high order derivatives are sensible to the round-off errors causing thereby instabilities. The second problem is that the convergence to limit is very slow. However the convergence can be speeded up using an appropriate extrapolation technics. That is how a group of the numerical Laplace transform inversion methods called ”akin to Post-Widder formula” were developed.
Another inversion formula can also be obtained from the following arguments. Let

\[ I_n = \int_0^\infty \delta_n(t, u)f(t)du, \quad (3.3) \]

where the functions \( \delta_n(t, u) \) converge to the delta function as \( n \) tends to infinity, and thus

\[ \lim_{n \to \infty} I_n = f(t). \]

The right-hand side of equation (3.3) can be presented as some function multiplied by the \( n \)-th derivative of the Laplace transform of \( f(t) \). For example the Post-Widder formula can be obtained from (3.3) letting

\[ \delta_n(t, u) = \frac{(nu/t)^n \exp(-nu/t)}{(n-1)!}. \]

Using similar arguments Gaver [9] suggested using the functions

\[ \delta_n(t, u) = \frac{(2n)!}{n!(n-1)!} a(1 - e^{-au})^n e^{-nau}, \]

where \( a = ln(2/t) \), which leads to

\[ f(t) = \lim_{n \to \infty} I_n(t) = \lim_{n \to \infty} \frac{(2n)!}{n!(n-1)!} a \Delta^n F(na). \]

It is similar to the Post-Widder formula, but instead of \( F^{(n)}(\lambda) \) we have the \( n \)-th finite difference \( \Delta^n F(\lambda) \). Still, the convergence of \( I_n \) to \( f(t) \) is too slow. But Gaver showed that \( (I_n - f(t)) \) can be expanded asymptotically in powers of \( (1/n) \), and Stehfest improved the Gaver’s method [19] and presented an algorithm based on approximating \( f(t) \) by the sum

\[ a \sum_{n=1}^{N} K_n F(na), \]

where

\[ K_n = (-1)^{n+N/2} \sum_{k=\lfloor(n+1)/2\rfloor}^{\min(n,N/2)} \frac{k^{N/2}(2k)!}{(N/2-k)!k!(k-1)!(n-k)!(2k-n)!}. \]

This algorithm is called the Stehfest algorithm or ”the Gaver-Stehfest algorithm”. 23
The Kryzhnyi method

Kryzhnyi suggests in his work [14] that the algorithms, which are based on choosing different delta convergent sequences can be compared by analysing the ‘focusing’ abilities of the numerical and the exact inverse transforms of $e^{\lambda t}$. Focusing abilities means how does the peakness of a delta approximating function is kept while increasing $t$. Of course this function flattens with the time, it happens because the kernel of the integral (3.3) satisfies a scaling property.

Focusing on this qualitative characteristics Kryzhnyi developed another algorithm of approximating the original function from its Laplace transform. Firstly, he applied the Mellin transform to equation (3.1) and got a solution in terms of the Mellin transform, which can be inverted after multiplying it by a suitable chosen factor. The result can be expressed by two equations,

$$f_R = \int_0^\infty f(tu) \frac{\sqrt{u}}{u+1} \frac{sin(R \ln u)}{u-1} du,$$

$$f_R = \int_0^\infty F(u) \Pi(R, tu) du,$$

where $\gamma$ is a regularization parameter and $R(\gamma) \to \infty$, while $\gamma \to 0$.

Here, instead of some number $N$, after which we stop the computation we have a value of some function $R$ in point $\gamma$.

After some generalization we have

$$\Pi(R, u) = \frac{1}{\pi \varphi(1)} \mathcal{L}^{-1} \left[ \frac{sin(R \ln p)}{p-1} \varphi(p) \right],$$

where $\varphi(u)$ is an arbitrary continuous function $\varphi(1) \neq 0$. From this equation follows that various kernels can be constructed in this way by choosing the function $\varphi(u)$. However, we can choose $\varphi(u)$ in such a way that the kernel can be expressed analytically using known transforms from tables.

Actually this approach by Kryzhnyi will be more tunable for different types of problems, because we can vary the regularization parameter $\gamma$ and choose different functions $\varphi(u)$. There are some limitations on $R$ proved by Kryzhnyi in [14]:

- $\lim_{R \to \infty} \Pi(R, x)$ does not exist,

- for the fixed precision arithmetic the value of parameter $R > 0$ cannot be increased infinitely without loss of the accuracy, which is explained by the next limitation,
the optimal value of the parameter $R$ is close to a linear function of number $n$ of correct digits in the input data: $n/2 < R_{opt} < n$.

Nevertheless, the technic of choosing these parameters $R$ and $\gamma$ is quite complicated.
Chapter 4

Results

4.1 The analytical expression for transformed variables

Our next goal is to apply the Laplace transform on equation (2.7) and solve it in the transformed variables. For convenience we are reverting the direction of time by change of the variable \( \tau = T - t \) and defining \( \tilde{c}(\tau, S; q) = c(T - \tau, S; q) = c(t, S_t; q) \) and \( \tilde{S}_\tau = S_{T-\tau} = S_t \) for \( \tau \leq 0 \). The Laplace-Carson transform of this variables follows from Definition 2

\[
\begin{align*}
\tilde{c}^*(\lambda, S; q) &= \mathcal{LC}\{\tilde{c}(\tau, S; q)\} \\
&= \lambda \int_0^\infty e^{-\lambda \tau} \tilde{c}(\tau, S; q) d\tau,
\end{align*}
\]

\[
\tilde{S}^*(\lambda) = \mathcal{LC}\{\tilde{S}(\tau; q)\} = \lambda \int_0^\infty e^{-\lambda \tau} \tilde{S}_\tau d\tau.
\]

Again, we prefer the Laplace-Carson transform to the Laplace transform because the constant values do not change the transformation and the Laplace-Carson approach generates simpler formulas for our problem. Applying the Laplace-Carson transform to the inhomogeneous PDE (2.7) we will get an inhomogeneous ODE of the same order. For solving this type of ODE we need to solve the corresponding homogeneous ODE, so it makes sense first to consider the transformation of the original Black-Scholes PDE for the plain vanilla options, where the parameter \( q \) is absent.

**Lemma 2** Let \( c^*(\lambda, S) = \mathcal{LC}\{\tilde{c}(\tau, S)\} \) define a Laplace-Carson transform of...
a value of a vanilla call option with the reversed time. Then

\[
c^*(\lambda, S) = \begin{cases} 
  \frac{K}{\theta_1 - \theta_2} \lambda \left( 1 - \frac{r - \delta}{\lambda + r} \right) \left( \frac{S}{K} \right)^{\theta_1}, & \text{if } S < K, \\
  \frac{K}{\theta_1 - \theta_2} \lambda \left( 1 - \frac{r - \delta}{\lambda + r} \right) \left( \frac{S}{K} \right)^{\theta_1} + \frac{\lambda S}{\lambda + \delta} - \frac{\lambda K}{\lambda + r}, & \text{if } S \geq K,
\end{cases}
\]  

(4.1)

where \( \theta_1 \) and \( \theta_2 \) depend on \( \lambda \) and are real roots of the quadratic equation

\[
\frac{1}{2} \sigma^2 \theta^2 + (r - \delta - \frac{1}{2} \sigma^2) \theta - (\lambda + r) = 0. 
\]  

(4.2)

It can be seen that when putting \( \theta = 1 \) and \( \theta = 0 \) we get negative values on the left hand side of equation (4.2). This means that both roots are outside the interval \((0, 1)\), so we numerate it in such a way that \( \theta_1 > 1 \) and \( \theta_2 < 0 \).

**Proof:** The original proof can be found in [13]. After changing variables the Black-Scholes PDE reads

\[
-\frac{\partial \tilde{c}}{\partial \tau} + (r - \delta)S \frac{\partial \tilde{c}}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 \tilde{c}}{\partial S^2} - r \tilde{c} = 0, \quad S > 0, 
\]  

(4.3)

supplied with the boundary conditions

\[
\lim_{S \downarrow 0} \tilde{c}(t, S) = 0, \\
\lim_{S \uparrow \infty} \frac{d \tilde{c}}{dS} < \infty,
\]

and the initial condition

\[
\tilde{c}(0, S) = (S - K)^+. 
\]

After transforming equation (4.3) we obtain a corresponding ODE

\[
\frac{1}{2} \sigma^2 S^2 \frac{d^2 \tilde{c}^*}{dS^2} + (r - \delta)S \frac{d \tilde{c}^*}{dS} - (\lambda + r) \tilde{c}^* + \lambda (S - K)^+ = 0, \quad S > 0, 
\]  

(4.4)

with the boundary conditions

\[
\lim_{S \downarrow 0} \tilde{c}^*(\lambda, S) = 0, \\
\lim_{S \uparrow \infty} \frac{d \tilde{c}^*}{dS} < \infty,
\]

Equation (4.4) is a linear homogeneous ODE of Euler type and can be reduced to a linear ODE with constant coefficients by substituting \( S = e^y \) and solved easily yielding (4.1).
Theorem 4 \[13\] If $S > S^*$, 
\[
c^*(\lambda, S; q) = c^*(\lambda, S) + \frac{q}{\lambda + r \theta_1 - \theta_2} \left( \frac{S}{S^*} \right)^{\theta_1}
\]
and $c^*(\lambda, S; q) = 0$ otherwise. The stopping boundary is given by 
\[
S^*(\lambda) = \left[ \frac{2(\lambda + \delta)q}{\lambda(1 - \theta_2)K \sigma^2} \right]^{-\theta_1} K.
\]

**Proof:** It is straightforward that the solution for this equation is a sum of solutions for the homogeneous equation and a particular solution of the inhomogeneous equation. It can be easily seen that the second part of the formula for $c^*(\lambda, S; q)$, without $c^*(\lambda, S)$ is a solution for the corresponding inhomogeneous ODE. For a detailed proof we refer the reader to Kimura [13].

Additionally, the same approach can be used for proceeding the solution for the put case. The result can be formulated by the following theorem.

Theorem 5 \[13\]. If $S < S^*$, 
\[
p^*(\lambda, S; q) = p^*(\lambda, S) + \frac{q}{\lambda + r \theta_1 - \theta_2} \left( \frac{S}{S^*} \right)^{\theta_1},
\]
and $p^*(\lambda, S; q) = 0$ otherwise. With
\[
p^*(\lambda, S) = \begin{cases} 
\frac{K \lambda}{\theta_1 - \theta_2 \lambda + \delta} \left( 1 - \frac{r - \delta}{\lambda + r} \theta_2 \right) \left( \frac{S}{K} \right)^{\theta_1}, & \text{if } S \geq K, \\
\frac{K \lambda}{\theta_1 - \theta_2 \lambda + \delta} \left( 1 - \frac{r - \delta}{\lambda + r} \theta_2 \right) \left( \frac{S}{K} \right)^{\theta_1} + \frac{\lambda S}{\lambda + \delta} - \frac{\lambda K}{\lambda + r}, & \text{if } S < K.
\end{cases}
\]
The stopping boundary is given by 
\[
S^*(\lambda) = \left[ \frac{2(\lambda + \delta)\lambda q}{\lambda(\theta_1 - 1)K \sigma^2} \right]^{-\theta_1} K.
\]
4.1.1 Transformed Greeks

In Section 2.2.2, devoted to the continuous case of installment options we mentioned the limiting case proved by Griebsch. This decomposition of the option in a vanilla call option and an American compound option was shown by Kimura [13] to be very valuable when trying to approximate the installment options Greeks. We have

\[ c(t, S_t; q) + K_t = c(t, S_t) + P_c(t, S_t; q), \]

with

\[ K_t = \frac{q}{r}(1 - e^{-r(T-t)}). \]

Using the integral representation (2.8) we obtain

\[ K_t - P_c(t, S_t; q) = q \int_t^T e^{-r(u-t)} \Phi(d_-(S_t, S_u, u-t)) \, du. \]

Substituting \( \Phi(x) = 1 - \Phi(-x) \) we get an integral representation for the American compound option

\[ P_c(t, S_t; q) = q \int_t^T e^{-r(u-t)} \Phi(-d_-(S_t, S_u, u-t)) \, du. \]

Due to the linearity of the Laplace-Carson transform we get for the time-reversed values

\[ \mathcal{L}C\{\tilde{c}(t, \tilde{S}_t; q)\} + \mathcal{L}C\{\tilde{K}_t\} = \mathcal{L}C\{\tilde{c}(t, \tilde{S}_t)\} + \mathcal{L}C\{\tilde{P}_c(\tau, \tilde{S}; q)\} \]

From Theorem 4 we see that

\[ P^*_c(t, S_t; q) - K^*_t = \frac{q}{\lambda + r} \frac{\theta_1}{\theta_1 - \theta_2} \left( \frac{S}{S^*} \right)^{\theta_2} - \frac{q}{\lambda + r}. \] (4.8)

Here, the inverse Laplace-Carson transform of the term \( \frac{q}{\lambda + r} \) can be computed analytically

\[ \mathcal{L}C^{-1}\left[ \frac{q}{\lambda + r} \right] = q \int_0^\tau e^{-ru} \, du = \frac{q}{r} (1 - e^{-r(T-t)}) = K_t, \]

thus for the transformed value of a American put on a call we have

\[ P^*_c(t, S_t; q) = \frac{q}{\lambda + r} \frac{\theta_1}{\theta_1 - \theta_2} \left( \frac{S}{S^*} \right)^{\theta_2}. \] (4.9)
Hence the Greeks of the continuous installment call option can be expressed by Greeks of the vanilla call and Greeks of the American put on a vanilla call with a floating strike price $K_t$.

\[
\Delta_{c(t,S;q)} = \frac{\partial c}{\partial S} = \Delta_{ct,S} + \mathcal{LC}^{-1}[\Delta_{p^*}],
\]

\[
\Gamma_{c(t,S;q)} = \frac{\partial^2 c}{\partial S^2} = \Gamma_{ct,S} + \mathcal{LC}^{-1}[\Gamma_{p^*}],
\]

\[
\Theta_{c(t,S;q)} = -\frac{\partial c}{\partial \tau} = \Theta_{ct,S} + qe^{-r\tau} \mathcal{LC}^{-1}[\Theta_{p^*}].
\]

Now using (4.9) we find explicit formulas for the transformed values of American compound option greeks,

\[
\Delta_{p^*} = \mathcal{LC} \left[ \frac{\partial P^*}{\partial S} \right] = \frac{\partial P^*}{\partial S},
\]

\[
\Gamma_{p^*} = \mathcal{LC} \left[ \frac{\partial^2 P^*}{\partial S^2} \right] = \frac{\partial^2 P^*}{\partial S^2},
\]

\[
\Theta_{p^*} = -\mathcal{LC} \left[ \frac{\partial P^*}{\partial \tau} \right] = -\lambda (P^*(\lambda, S; q) - P^*(0, S; q)) = -\lambda P^*(\lambda, S; q).
\]

Using the same arguments for the installment put case we obtain

\[
\Delta_{p(t,S;q)} = \frac{\partial p}{\partial S} = \Delta_{pt,S} + \mathcal{LC}^{-1}[\Delta_{p^*}],
\]

\[
\Gamma_{p(t,S;q)} = \frac{\partial^2 p}{\partial S^2} = \Gamma_{pt,S} + \mathcal{LC}^{-1}[\Gamma_{p^*}],
\]

\[
\Theta_{p(t,S;q)} = -\frac{\partial p}{\partial \tau} = \Theta_{pt,S} + qe^{-r\tau} \mathcal{LC}^{-1}[\Theta_{p^*}].
\]

Correspondingly

\[
\Delta_{p^*} = \mathcal{LC} \left[ \frac{\partial P^*}{\partial S} \right] = \frac{\partial P^*}{\partial S},
\]

\[
\Gamma_{p^*} = \mathcal{LC} \left[ \frac{\partial^2 P^*}{\partial S^2} \right] = \frac{\partial^2 P^*}{\partial S^2},
\]

\[
\Theta_{p^*} = -\mathcal{LC} \left[ \frac{\partial P^*}{\partial \tau} \right] = -\lambda (P^*(\lambda, S; q) - P^*(0, S; q)) = -\lambda P^*(\lambda, S; q).
\]
4.1.2 The Numerical Results

During our work we developed a set of Matlab functions for valuing continuous installment options and its Greeks via the inverse Laplace transform methods. The algorithm is based on results of Kimura [13], in which the author uses two algorithms for the inverse Laplace transform: the Euler summation and the Gaver-Stehfest method. We decided to use the Gaver-Stehfest and the Kryzhnyi algorithms, described in Chapter 3.

Our algorithm of the continuous installment option valuing consists of two numerical procedures. It is finding the stopping boundary and the numerical integration of the integral in (2.8) or (2.9). The difference in comparison to Kimura's algorithm is that we use the popular integration Matlab routine \texttt{quad}, which uses the Simpson formula for the integration and determines integration nodes automatically and then evaluates the stopping boundary in each node.

![Figure 4.1: Stopping boundaries for put and call with $T = 1, t = 0, \delta = 0.03, r = 0.02$.](image)

The results of approximation of the stopping boundaries for the put and the call cases and its sensitivity to the installment rate $q$ and the dividend yield $\delta$ are presented in Figure 4.1 and Figure 4.2. From the Figures it can be seen that in dependence on parameters $q$ and $\delta$, the stopping boundary can be either a monotonic or non-monotonic function, unlike the exercise boundaries of the American style options. This non-monotonic behavior also appears in some types of Asian options and draws great interest of researchers.

The values of the continuous installment options obtained by our Matlab programme are given in last two columns of Tables 4.1 and 4.2. The results of the Euler method are taken from Kimura [13]. Kimura also notes that every value obtained by the Gaver-Stehfest algorithm is smaller than the values obtained by the Euler summation. These values differ significantly, which
Figure 4.2: Stopping boundaries for the put and the call with $T = 1$, $t = 0$, $q = 10$, $r = 0.02$.

<table>
<thead>
<tr>
<th>$q$</th>
<th>$S$</th>
<th>Euler-based (Kimura)</th>
<th>Gaver-Srehfest</th>
<th>Kryzhnyi</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>95</td>
<td>3.7071</td>
<td>3.7072</td>
<td>3.7069</td>
</tr>
<tr>
<td></td>
<td>105</td>
<td>8.3994</td>
<td>8.3995</td>
<td>8.3993</td>
</tr>
<tr>
<td>3</td>
<td>95</td>
<td>2.2280</td>
<td>2.2283</td>
<td>2.2266</td>
</tr>
<tr>
<td></td>
<td>105</td>
<td>6.6385</td>
<td>6.6388</td>
<td>6.6379</td>
</tr>
<tr>
<td></td>
<td>115</td>
<td>12.9687</td>
<td>12.9690</td>
<td>12.9686</td>
</tr>
<tr>
<td>6</td>
<td>95</td>
<td>0.6754</td>
<td>0.6761</td>
<td>0.6703</td>
</tr>
<tr>
<td></td>
<td>105</td>
<td>4.2745</td>
<td>4.2752</td>
<td>4.2723</td>
</tr>
<tr>
<td></td>
<td>115</td>
<td>10.2533</td>
<td>10.2540</td>
<td>10.2527</td>
</tr>
</tbody>
</table>

Table 4.1: Values of call with $t = 0$, $T = 1$, $K = 100$, $r = 0.03$, $\delta = 0.05$, $\sigma = 0.2$ computed by different algorithms.

caused the author to mistrust the Gaver-Stehfest algorithm. But as for our results, it can be seen from the tables that all three algorithms produce very close results. On Figure 4.3 you can see a 3D plot of the call and the put values in dependence on time and asset price.

Now the values for different Greeks are presented in Figures 4.4, 4.5, 4.6. Actually not all values for Greeks presented here make sense. We can only evaluate Greeks if we are above the stopping boundary in the call case and below a stopping boundary in the put case. On Figure 4.5 you can see an unexpected blow up of the put gamma in case of $q = 15$. On the figure black markers define the value of the stopping boundary in each case. So this unexpected behavior is not important for us, because it happens after reaching the stopping boundary.

Kimura [13] noticed that the Gaver-Stehfest method behaves bad for valu-
Figure 4.3: The option value for the put and the call with $T = 1$, $t = 0$, $\delta = 0.03$, $r = 0.02$, $q = 10$ and stopping boundaries.

Figure 4.4: The Delta value for the put and the call, in dependence on $q$ where $q = 5, 10, 15$ with $T = 1$, $t = 0$, $r = 0.02$, $\delta = 0.04$. 
Figure 4.5: The Gamma value for the put and the call, in dependence on $q$ where $q = 5, 10, 15$ with $T = 1, t = 0, r = 0.02, \delta = 0.04$.

Figure 4.6: The Theta value for the put and the call, in dependence on $\delta$ where $\delta = 0.08, 0.04, 0.02$ with $T = 1, t = 0, r = 0.02, \delta = 0.04$. 
Table 4.2: Values of put with $t = 0$, $T = 1$, $K = 100$, $r = 0.03$, $\delta = 0.05$, $\sigma = 0.2$ computed by different algorithms.

<table>
<thead>
<tr>
<th>q</th>
<th>S</th>
<th>Euler-based (Kimura)</th>
<th>Gaver-Srehfest</th>
<th>Kryzhnyi</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>85</td>
<td>16.9438</td>
<td>16.9439</td>
<td>16.9439</td>
</tr>
<tr>
<td></td>
<td>95</td>
<td>10.3046</td>
<td>10.3047</td>
<td>10.3047</td>
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<tr>
<td></td>
<td>105</td>
<td>5.5703</td>
<td>5.5704</td>
<td>5.5705</td>
</tr>
<tr>
<td>3</td>
<td>85</td>
<td>15.0001</td>
<td>15.0008</td>
<td>15.0009</td>
</tr>
<tr>
<td></td>
<td>95</td>
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<td>8.4286</td>
<td>8.4289</td>
</tr>
<tr>
<td></td>
<td>105</td>
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<td>3.8489</td>
<td>3.8497</td>
</tr>
<tr>
<td>6</td>
<td>85</td>
<td>12.1253</td>
<td>12.1259</td>
<td>12.1263</td>
</tr>
<tr>
<td></td>
<td>95</td>
<td>5.7647</td>
<td>5.7652</td>
<td>5.7666</td>
</tr>
<tr>
<td></td>
<td>105</td>
<td>1.7010</td>
<td>1.7018</td>
<td>1.7051</td>
</tr>
</tbody>
</table>

The results for both used inverse methods look quite reasonable in the whole region where the stopping boundary is not reached, even for a time very close to expiration. Trying to compare the two algorithms used for the inverse Laplace transform we used a comparison method, proposed by Kryzhnyi [14]. The method is based on inverting the function $e^{\lambda x}$, which analytical inverse transform is the delta function. The better the algorithm approximates the delta function while inverting $e^{\lambda x}$ and preserves its peakness while increasing $t$, the better it will approximate other functions too. On Figure 4.7 you can see the results of the reconstructing the delta function. The Kryzhnyi method shows more peaked values and more slowly flattens with time, but it is difficult to say which one is better because of these fast oscillations of the curve obtained by the Kryzhnyi method. When reconstructing monotonic functions from its Laplace transform both Kryzhnyi and the Stehfest methods show good results, but when dealing with a damped oscillating function it occurs that the Stehfest algorithm cannot compete with the Kryzhnyi method. On Figure 4.8 we see that the curve, produced by the Stehfest algorithm flattens much faster than the one produced by the Kryzhnyi method. But in our case it is difficult to say which of the methods is more precise because we are reconstructing the non oscillating functions, but as for computational costs it is much more convenient to use the Stehfest algorithm.
Figure 4.7: The reconstruction of the delta function by Stehfest and Kryzhnyi algorithms.

Figure 4.8: The reconstruction of the damped oscillating function by Stehfest and Kryzhnyi algorithms. The dashed line - the exact values, the red line - values obtained by the Gaver-Stehfest method, the blue line - values obtained by the Kryzhnyi method.
Chapter 5

Conclusions

The study of the installment and compound options theory led us to the idea that the discrete installment options are the same derivative products as the multi-fold compound options, just presented in other ways by different authors. Meanwhile, the compound options are included as the special case of the above options (see the classification in Fig. 2.3).

The valuation of installment options was split into two cases: discrete and continuous ones. In the discrete case it is possible to deduce the closed-form solution presented by Griebsch et al., using the backward recursion and then applying the Curnow and Dunnett integral reduction technique. Considering the continuous case we touched only on the European case, where we faced the stopping boundary problem. To simplify the stopping boundary expression we used the Laplace-Carson transformation as it was done by Kimura [13]. The computation of the transformed values was followed by the inverse Laplace transformation procedure. We considered two methods of the inverse Laplace transformation: the Gaver-Stehfest method that was used by Kimura to valuate the continuous installment options and the Kryzhnyi method that was never applied before as for a valuation of the options. Eventually, we developed the program code for Matlab to compute the stopping boundaries, values and Greeks of the European continuous installment options, implementing above methods. The results obtained as for options and the comparison of the methods were presented in the tables and graphs.
Bibliography


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*The generalized sequential compound options pricing and sensitivity analysis.* Mathematical Social Sciences, 55, 38–54.


[18] E.L. Post (1930)


Chapter 6

Appendix

6.1 Matlab Program Listing

function f = theta1fun(sigma,delta,lyam,r)
    \%roots of quadratic equation in lemma 1\
    f=-.5*(-2.*delta-1.*sigma^2+2.*r-1.*(4.*delta^2+4.*delta*sigma^2-8...
    ...*delta*r+sigma^4+4.*sigma^2*r+4.*r^2+8.*sigma^2*lyam).^(1/2))/sigma^2;

function f = theta2fun(sigma,delta,lyam,r)
    \%roots of quadratic equation in lemma 1
    f=-.5*(-2.*delta-1.*sigma^2+2.*r+(4.*delta^2+4.*delta*sigma^2-8....
    *delta*r+sigma^4+4.*sigma^2*r+4.*r^2+8.*sigma^2*lyam).^(1/2))/sigma^2;

function y=d1(S,K,r,delta,sigma,T,t)
    y=(log(S./K)+(r-delta+((sigma^2)./2)).*(T-t))./(sigma.*sqrt((T-t)));

function y=d2(S,K,r,delta,sigma,T,t)
    y=d1(S,K,r,delta,sigma,T,t)-(sigma*sqrt(T-t));

function y = stopping_boundary(lyam)
    \%using global varibles
    global T;
    global K;
    global sigma;
global r;
global q;
global delta;
Kex=K;
Tex=T;
Rex=r;
Sigex=sigma;
Qex=q;
Deltaex=delta;

y=(Kex./lyam).*((2*(lyam+Deltaex)*Qex)./(lyam.*(1-...
    theta2fun(Sigex,Deltaex,lyam,Rex))*Kex.*(Sigex^2))... 
   *(1./theta1fun(Sigex,Deltaex,lyam,Rex)));

function y = opvalue(q,sigma,r,T,K,delta,S_t,t)
    f=@integrated;
    y=S_t*exp(-delta*(T-t))*normcdf(d1(S_t,K,r,delta,sigma,T,t))-
    K*exp(-r*(T-t))*normcdf(d2(S_t,K,r,delta,sigma,T,t))-... 
    q*quad(f,t+0.0000001,T,1.e-6,0,sigma,r,T,K,delta,S_t,t);

function y = stopping_derivative(lyam)
    global K;
y=lyam*stopping_boundary(lyam) - K*lyam;

function f = integrated(u,sigma,r,T,K,delta,S_t,t)
    if(u<T)
        f=exp(-r*(u-t)).*normcdf((log(S_t... 
            /stehfest(’stopping_boundary’,T-u,16))...
            +(r-delta-0.5*sigma^2).*exp(-delta*(T-t))*(0,1);
    else
        f=exp(-r*(u-t)).*normcdf((log(S_t./K)+(r-delta-0.5*sigma^2)...
            *(0,1);
    end;

function y=greek_delta(S,K,r,delta,sigma,T,t)
    y=normcdf(d1(S,K,r,delta,sigma,T,t),0,1)*exp(-delta*(T-t));
function f=greek_delta_p(lyam)
    % value for transformed delta/lyambda
    % using global variables ;S_tex,Kex,Rex,Deltaex,Sigex,Tex,t,
    global T;
    global K;
    global sigma;
    global r;
    global q;
    global delta;
    global S;
    global t
    Kex=K;
    Tex=T;
    Rex=r;
    Sigex=sigma;
    Qex=q;
    Deltaex=delta;
    S_tex=S;
    tex=t;
    f=(Qex/((lyam+Rex)*lyam))*((theta1fun(Sigex,Deltaex,lyam,Rex)...
        *theta2fun(Sigex,Deltaex,lyam,Rex))/(theta1fun(Sigex,Deltaex,lyam,Rex)-...
        theta2fun(Sigex,Deltaex,lyam,Rex)))*(1/(S_tex))*((S_tex/(lyam*...
        stopping_boundary(lyam)))^theta2fun(Sigex,Deltaex,lyam,Rex));

function y = greek_gamma(S,K,r,delta,sigma,T,t)
y=(Nder(S,K,r,delta,sigma,T,t)*exp(-delta*(T-t)))/(S*sigma*sqrt(T-t));

function f = greek_gamma_p(lyam) % value for transformed greek/lyambda
% using global variables
    global T;
    global K;
    global sigma;
    global r;
    global q;
    global delta;
    global S;
    global t;
Kex=K;  
Tex=T;  
Rex=r;  
Sigex=sigma;  
Qex=q;  
Deltaex=delta;  
S_tex=S;  
tex=t;  
theta1=theta1fun(sigma,delta,lyam,r);  
theta2=theta2fun(sigma,delta,lyam,r);  
f=((S_tex/stehfest('stopping_boundary',Tex-tex,16))^theta2)...  
   *(1/(S_tex^2*lyam))*(Qex/(lyam+Rex))*(((theta1*theta2)*...
   (theta2-1))/(theta1-theta2));

function y=greek_theta(S,K,r,delta,sigma,T,t)
    y=-((S*Nder(S,K,r,delta,sigma,T,t)*sigma*exp(-delta*(T-t)))...  
        /(2*sqrt(T-t)))-(r*K*exp(-r*(T-t))*...
        normcdf(d2(S,K,r,delta,sigma,T,t)))...  
        delta*S*normcdf(d1(S,K,r,delta,sigma,T,t))...
        *exp(-delta*(T-t))

function f=greek_theta_p(lyam)
    global T;  
    global K;  
    global sigma;  
    global r;  
    global q;  
    global delta;  
    global S;  
    Kex=K;  
    Tex=T;  
    Rex=r;  
    Sigex=sigma;  
    Qex=q;  
    Deltaex=delta;  
    S_tex=S;  
f=-((lyam*Qex)/(lyam+Rex))*...
(\text{theta1fun}(\text{Sigex}, \text{Deltaex}, \text{lyam}, \text{Rex})... \\
/ (\text{theta1fun}(\text{Sigex}, \text{Deltaex}, \text{lyam}, \text{Rex}) -... \\
\text{theta2fun}(\text{Sigex}, \text{Deltaex}, \text{lyam}, \text{Rex})))... \\
* ((\text{S\_tex}/\text{stopping\_boundary}(\text{lyam}))^{... \\
\text{theta2fun}(\text{Sigex}, \text{Deltaex}, \text{lyam}, \text{Rex}));