Comparative analysis of the SVJJ and the Hyperbolic models on the Swedish market

Master's Thesis in Financial Mathematics

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Preface

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Halmstad, May 2008
Ekaterina Anisimova, Tomasz Lapinski
Abstract

In this thesis we investigate and compare two recently developed models of the option valuation according to the Swedish market. The first model is the Stochastic Volatility model with jumps in the stock price and the volatility (SVJJ) and the second is the Hyperbolic model. First of all we make brief introduction about the valuation of derivatives and considered models. Then we introduce methods for the estimation of parameters for each model. To solve this problem for the SVJJ model we use the Empirical Characteristic Function Estimation and for the Hyperbolic we use the Maximum Likelihood Method. Before explicit calculations (with estimated parameters) we describe the derivation of the pricing formula which is based on characteristic functions and densities. In conclusion we made numerical valuations of the call option prices for the OMXS30 index on the Swedish Stock Exchange.

The main idea of this thesis is to compare 2 different models using numerical methods and the real data sets. To achieve this goal we firstly, compare the empirical characteristic function obtained from the market and the analytical ones for estimated parameters in case of both models. Secondly, we make a comparison of calculated call option prices and produce the summary.
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Chapter 1

Introduction

Most option traders use the Black-Scholes model for pricing derivative securities. But the classical Black-Scholes assumption about constant volatility for option pricing fails to hold true in practice. Prices calculated by this classical model systematically differ from the real market prices. In response to this fact many new models with different adjustment were developed. For instance, the stochastic volatility assumption in the Heston model or more correct distribution function which fit evolution of a stock price better than the normal distribution which used in the Black-Scholes model.

This theme was and is interesting for researches, see M. Rockinger, M. Semenova [8], P. Carr, D.B. Madan [14], E. Eberlein, U. Keller K. Prause [6] and others.

In our work we consider two models: the Stochastic Volatility Model with jumps in the stock price and in the volatility (SVJJ) described by Tomasz Lapinski and the Hyperbolic Model described by Ekaterina Anismanova. To compare efficiency of these models we used the secondary Swedish market data. For calculations we used data of large Swedish companies ABB Ltd and Tele2 B, and OMX Stockholm 30 index.

Our work consists of three basic parts. In the Chapter 2 of our work we begin study with the short presentation of each model. In the Chapter 3 the estimation of parameters for each model with methods for these estimation are showed. In the Chapter 4 we considered each of models in details and presented results of our calculation for call option prices. Finally we compared the SVJJ and the Hyperbolic models in the Conclusion (see Chapter 5). Proofs of theoretical results and simulation program code are presented in Appendixes.
Chapter 2
Models introduction

2.1 The SVJJ Model

A stochastic volatility model was firstly introduced and described by Heston (1993) [19]. Then, to fit better the evolution of stock prices, especially jump property of stock prices, more complicated models were developed. For example SVJ model which has included jumps in a stock price. Additionally, these jumps can have different distributions, but the most common are normal and exponential. In this work we consider a stochastic volatility model with jumps, both in the stock price and volatility (SVJJ) which was described by Duffie, Pan, and Singleton (2000) [3]. It is given by the following system of equations:

\[
\begin{align*}
    ds_t &= \left(\mu - \lambda m - \frac{1}{2}V_t\right)dt + \sqrt{V_t}dW_t^1 + z_s dq_t,
    \\
    dV_t &= \beta(\alpha - V_t)dt + \sigma\sqrt{V_t}dW_t^2 + z_v dq_t,
\end{align*}
\]

(2.1)

where \( <dW_t^1, dW_t^2> = \rho dt \) is correlation between a Brownian motion in equation for stock price \( s_t \) and volatility \( V_t \). The constants \( \alpha \) and \( \beta \) are responsible for a mean-reverting ability of the process, \( \sigma \) is volatility of volatility \( V_t \), \( \lambda m \) is compensation of jumps in price where \( \lambda \) is the jumps frequency. We have to notice that the height of jumps in the stock price is normally distributed (\( z_s \sim N(\mu_J, \sigma_J) \)) and in volatility it is exponentially distributed (\( z_v \sim \exp(\mu_v) \)).

To compute a price of the call \( C(S_0, T, K, \tau) \) we use a semi-analytical method developed by (Carr and Madan 1999 [14]). Here \( T \) is the maturity time, \( K \) is the strike price and \( \tau = T - t \) is time to maturity. The derivation of the pricing formula and the computation of prices are given in Section 4. However, this method requires knowledge of the model characteristic function, which we will derive in Section 3.
Chapter 2. Models introduction

The parameter vector $\theta = (\mu, \alpha, \beta, \sigma, \lambda, \mu_J, \sigma_J, \rho_J, \sigma_J)$ should be estimated from real data. Semenova (2006)\cite{11} used method of the Empirical characteristic function. The estimation procedure based on the unconditional characteristic function to estimate parameters of the SVJ model, which have jumps only in stock price. It is easy to apply this method to SVJJ model, by some changes in considered characteristic function. We will describe the estimation procedure in the Section 3.

2.2 The Hyperbolic Model

**Definition 1** \cite{2} A Levy process $X = (X_t)_{t \geq 0}$ is a stochastic process satisfying the following conditions

1. $X$ has independent and stationary increments;
2. each $X_0 = 0$ (with probability one);
3. $X$ is stochastically continuous, i.e., for all $a > 0$ and for all $s \geq 0$, $\lim_{t \to s} P(|X(t) - X(s)| > a) = 0$.

A statement about the stationarity of a Levy motion increments means that the probability distribution of any increment $X_s - X_t$ depend only on the length $s - t$ of the time interval. A statement about independence means that the increments $X_s - X_t$ and $X_u - X_v$ are independent random variables whenever the two time intervals do not overlap. The most known examples of a Levy motion are a Wiener process (Brownian motion) and a Poisson process. The increments of such processes are the differences $X_s - X_t$ between their values at different times $t < s$. The probability distributions of the increments of any Levy process are infinitely divisible. There exist a Levy process for any infinitely divisible probability distribution (see also \cite{2}).

It is well-known that in the Black-Scholes model the price process $(S_t)_{t \geq 0}$ is defined by a geometric Brownian motion $W_t$. The local dynamics of this process in terms of a constant drift $\mu$ and driving process $W_t$ describes by a linear stochastic differential equation

$$dS_t = \mu S_t dt + \sigma S_t dW_t,$$  \hspace{1cm} (2.2)

where the constant volatility $\sigma > 0$ and $W_t$ is a standard Brownian motion with a zero mean and a unit standard deviation. The explicit solution to the stochastic differential equation (2.2) is given by

$$S_t = S_0 e^{(\mu - \frac{1}{2} \sigma^2) t + \sigma W_t}.$$  \hspace{1cm} (2.3)
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Eberlein, Keller and Prause (1998) (see [6]) modified equation (2.2) into a more general form

\[ dS_t = a(t, S_t)dt + b(t, S_t)dW_t + \lambda_t S_t dN_t, \]  

(2.4)

where \( W_t \) now is a standard Poisson process with the jumps. Equation (2.4) is non-linear, that is why there exist some technical problems by the estimation of the processes \( a \) and \( b \). These processes can be defined in different ways. For instance, in the stochastic volatility models the process \( b \) is a solution of another stochastic differential equation with another independent Brownian motion [6].

The empirical analysis of financial data of the German secondary market by Eberlein, Keller and Prause (1998) [6] led authors to consider the following reformulation of the classical model (2.2)

\[ dS_t = \mu S_t dt + \sigma S_t dX_t + S_t - (e^{\sigma \Delta X_t} - 1 - \sigma \Delta X_t), \]  

(2.5)

where \( \Delta X_t = X_t - X_{t-} \) denotes the jump at time \( t \) if there is one. Here the jumps are included explicitly into the dynamics by the last part of the equation (2.5).

**Definition 2** Ito’s formula states that for any Levy process \( X_t \) and any twice continuously differentiable function \( f \) on \( R \), we have

\[
f(X_t) = f(X_0) + \frac{\partial}{\partial x} f(X_{t-}) X_t + \frac{1}{2} \frac{\partial^2}{\partial x^2} f(X_{t-}) \langle X^c_t, X^c_t \rangle + \sum_{s \leq t} [f(X_s) - f(X_{s-}) - \frac{\partial}{\partial x} f(X_{s-}) \Delta X_s].
\]

(2.6)

Using the Ito’s formula (2.6) we can obtain the following solution to equation (2.5)

\[ S_t = S_0 e^{(\mu t + \sigma X_t)}. \]

(2.7)

Let \( X = (X_t)_t \) be a hyperbolic Levy motion depending on four parameters \( (\alpha, \beta, \delta, \mu) \). Here \( \alpha \) and \( \beta \) describe the shape of the hyperbolic density, \( \delta \) is a scale parameter and \( \mu \) is a location parameter. For each choice of these parameters there exist a different type of processes.

To model a stock price dynamics (under assumption of arbitrage-free prices) Eberlein, Keller and Prause (1998) choose Esscher transformations. By this approach we can receive a closed option pricing formula that is more exact than the Black-Scholes pricing equation [6]. By Esscher transforms authors obtain the formula for a call option in the following form

\[ C_{hyp} = S_0 \int_{\gamma} f_t(x; \theta + 1) dx - e^{-rT} K \int_{\gamma} f_t(x; \theta) dx. \]

(2.8)
In the equation (2.8) $T$ is time to expiration, $K$ is the strike price, $\gamma = \ln\left(\frac{K}{S_0}\right)$, $H(S_T) = ((S_T - K)_+)$ is the payoff function, $f_t(x; \theta)$ is the density of the distribution of $X_t$ under the risk-neutral measure.

As in the Black-Scholes case in the hyperbolic call price (2.8) there is the usual weighted difference of $S_0$ and $e^{-rT}K$, but now the weights are given by definite probabilities. Note that for the fitting real data into the hyperbolic model we can compute the price of an option employing a fast Fourier transformation (FFT) and a numerical integration.
Chapter 3

The estimation of parameters for the SVJJJ model and for the Hyperbolic model

3.1 The estimation of parameters for the SVJJJ Model

3.1.1 A Characteristic function

The parameter vector \( \theta = (\mu, \alpha, \beta, \sigma, \lambda, \mu_J, \sigma_J, \rho, \sigma_J) \) should be estimated from real data. Semenova (2006) [11] used method of the Empirical characteristic function estimation procedure based on the unconditional characteristic function (UCF).

The conditional characteristic function (CCF) of the SVJJ model (2.1) has the following form

\[
\varphi(u; s_T, V_T, t | s_t, V_t) = \varphi_t(u; s_T, V_T, t) = E[e^{iu s_T | s_t, V_t}].
\]

(3.1)

The CCF (2.5), is a function of the state variables \( s_t \) and \( V_t \). Then we apply Ito’s lemma [16] to the function \( \varphi(s, V, t) \). Like by Heston (1993) [19] we assume that the drift of \( \varphi \) must be equal to zero so it fulfills following condition \( E[df] = 0 \). This condition means the absence of arbitrage. We obtain following equation for the CCF

\[
\frac{\partial \varphi}{\partial t} + (\mu - \lambda m - \frac{1}{2} V) \frac{\partial \varphi}{\partial s} + \beta (\alpha - V) \frac{\partial \varphi}{\partial V} +
\]

\[
\frac{1}{2} V \frac{\partial^2 \varphi}{\partial s^2} + V \sigma \rho \frac{\partial^2 \varphi}{\partial s \partial V} + \frac{1}{2} \sigma^2 V \frac{\partial^2 \varphi}{\partial V^2} +
\]

\[
\lambda E[\varphi(s + z_s, V + z_v, t) - \varphi(s, V, t)] = 0.
\]

(3.2)
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The usual practice by study this kind of equations is to estimate the general structure of the solution first. We suppose that the particular solution has the form (Rockingera, Semenova (2005) [8]).

\[ \varphi(u; s_T, V_T, t|s_t, V_t) = \exp(C(u; \tau) + J(u; \tau) + D(u; \tau)V_t + ius_t). \] (3.3)

where \( \tau = T - t \) and \( T \) is time to maturity.

Additionally, to obtain a PDE from (3.2) we can rewrite jump term as follows

\[
E[\varphi(u; s_T, V_T, t|s_t + z_s, V_t + z_v) - \varphi(u; s_T, V_T, t|s_t, V_t)] \\
= E[\exp(C + J + D(V_t + iu(s_t + z_s)) - \exp(C + J + DV_t + ius_t)] \\
= \exp(C + J + DV_t + ius_t)E[\exp(Dz_v + iuz_s) - 1] \\
= \varphi(u; s_T, V_T, t)[\theta(iu, D) - 1].
\] (3.4)

Functions \( C \) and \( D \) which appear in (3.3) are parts of the solution to the classical Heston model as it shown by Heston (1993) [19]. The function \( J \) corresponds to the response of the system on jumps in equation (2.1).

To obtain exact expressions for functions \( C, D, J \) we derive differential equations for these functions by inserting (3.2) into (3.3). We obtain following system of differential equations

\[
\frac{\partial C}{\partial \tau} = iu\mu + \alpha\beta D, \\
\frac{\partial D}{\partial \tau} = iu\sigma\rho D + \frac{1}{2}D^2\sigma^2 - \beta D - \frac{1}{2}iu(1 - iu), \\
\frac{\partial J}{\partial \tau} = -iu\lambda m + \lambda[\theta(iu, D) - 1].
\] (3.5)

We solve these equations with initial conditions \( C(0; u) = 0 \), \( J(0; u) = 0 \) and \( D(0; u) = iu \), and obtain following expressions

\[
C(u; \tau) = iu\mu\tau - \frac{\alpha\beta}{\gamma^2} \left[ 2\ln \left( 1 - \frac{\gamma + b}{2\gamma} (1 - \exp(-\gamma\tau)) \right) + (\gamma + b)\tau \right], \\
D(u; \tau) = \frac{a}{\gamma + b} \left[ 1 - \frac{2\gamma}{2\gamma - (\gamma + b)(1 - \exp(-\gamma\tau))} \right], \\
J(u; \tau) = -\lambda iu\tau m + \lambda \int_0^\tau [\theta(iu, D(u; y))) - 1] dy,
\]
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\[ b(u) = iu\sigma \rho - \beta, \]

\[ \gamma(u) = \sqrt{\sigma^2 iu(iu - 1) + b(u)^2}, \]

\[ a(u) = iu(1 - iu), \]

\[ m = \frac{\exp(\mu_J + \frac{1}{2}\sigma^2_j)}{1 - \rho_j\mu_v} - 1, \]

\[ \int_0^\tau \left[ \theta(iu, D(u,y)) - 1 \right] dy = \exp(\mu_J iu + \frac{1}{2}\sigma^2_j u^2), \]

\[ \frac{2\mu_v a}{(\gamma c)^2 - (bc - \mu_v a)^2} \ln \left( 1 - \frac{(\gamma + b)c - \mu_v a}{2\gamma c} (1 - \exp(-\gamma \tau)) \right) + \]

\[ \frac{(\gamma - b)\tau}{\mu_v a + \gamma c - bc}, \]

(3.6)

where \( \tau = T - t, u \) are internal variables.

Now we can derive the conditional characteristic function for the log-returns in the asset price \( r_t = s_{t+1} - s_t \). We do this to make estimation more general. We exclude the initial stock price \( s_t \) in order to have possibility to calculate an option price for different stock prices. However we have to mention that the estimation is made only for the particular time to maturity.

\[ \phi(u; r_t, t|s_t, V_t) = \phi(u; (s_{t+1} - s_t), t|s_t, V_t) \]

\[ = \exp(C(u; \tau) + J(u; \tau) + D(u; \tau)V_t + ius_t) - \exp(ius_t) \]  

(3.7)

\[ = \exp(C(u; \tau) + J(u; \tau) + D(u; \tau)V_t) = \phi(u; r_{t+1}, t|V_t). \]

If we take \( t = 0 (\tau = T) \) and \( T = 1 \), the UCF obtain the following form

\[ \phi(u; r, t|V_0) = E[e^{iur\tau}|V_0]. \]  

(3.8)

If we determine it by CCF we obtain

\[ \phi(u; r_T|V_0) = \exp(C(u; \tau) + J(u; \tau)) * E[\exp(D(u; \tau)V_0)]. \]  

(3.9)
where $\tau = T - t$ is time to maturity.

The Variance process in (2.1) consist of a jump part and a square-root process, namely the gamma distribution. These two processes are independent, so the UCF of these processes is just a product of these two parts. In last term of our formula for the UCF and for SVJJ model (3.9) we have volatility $V_t$ in $t = 0$. Therefore the jump process term in the models UCF of the volatility process will be equal to 1 according to Rockingera, Semenova (2005) [8]. So this last term is just equal to the characteristic function of the gamma distribution

$$\varphi_G(u) = \left(1 - \frac{iu\sigma^2}{2\beta}\right)^{-2\alpha\beta/\sigma^2}. \quad (3.10)$$

Finally, UCF which is used for the estimation has the following form

$$\varphi(u; r_T) = \exp(C(u; \tau) + J(u; \tau)) \ast \left(1 - \frac{D(u; \tau)\sigma^2}{2\beta}\right)^{-2\alpha\beta/\sigma^2}. \quad (3.11)$$

However by the calculation of an option price we need a conditioned characteristic function of the form

$$\varphi(u|s_t) = E[e^{ius_T}|s_t]. \quad (3.12)$$

We can rewrite formula (3.11) in the following way

$$\varphi(u; r_T) = \varphi(u; s_T - s_0|s_0) = \varphi(u; s_T|s_0) - \varphi(u; s_0|s_0) =$$

$$\exp(C(u; \tau) + J(u; \tau) + ius_0) \ast \left(1 - \frac{D(u; \tau)\sigma^2}{2\beta}\right)^{-2\alpha\beta/\sigma^2} - \exp(ius_0). \quad (3.13)$$

Therefor we have formula for our model CCF for $t = 0$

$$\varphi(u|s_0) = \exp(C(u; \tau) + J(u; \tau) + ius_0) \ast \left(1 - \frac{D(u; \tau)\sigma^2}{2\beta}\right)^{-2\alpha\beta/\sigma^2}. \quad (3.14)$$

### 3.1.2 The estimation of parameters for the SVJJ model

We estimate the parameter vector $\theta$ using Mathematica[10] software with the function ‘NMinimize’ which defines the global minimum of a function for a set of parameters. We estimate the parameters for the daily data of OMXS 30 index on the Swedish Stock Exchange. For calculations we take the closing prices of the stock during 1338 days, from 30th of December 2002 to 15th of April 2008.
The log-returns in the empirical UCF are scaled by 100 ($r'_t = 100r_t$). It is easier to integrate empirical UCF with the scaled log-returns. In this case boundaries for integral (3.15) with the scaled log-returns are $u = -2.8$ and $u = 2.8$ instead of $u = -280$ and $u = 280$ for the not scaled log-returns. Outside those boundaries the empirical characteristic function vanishes. We take the parameter $\sigma_w = 0.7$ in our weight function.

We estimated the parameters of above UCF by minimizing of the integral

$$I_n(\theta) = \int_{-2.8}^{2.8} (\Re(\varphi(u; \theta) - \varphi_n(u))^2 + (\Im(\varphi(u; \theta) - \varphi_n(u)^2))g(u)\,du. \quad (3.15)$$

The weight function $g(u)$ that we will use is a normal distribution function $g(u) = \frac{1}{\sqrt{2\pi\sigma^2_w}}\exp(-u^2/(2\sigma^2_w))$. The value of $\sigma_w$ is chosen in such way that the weight function will have a similar size as the empirical characteristic function.

The estimation is made for the empirical CF (ECF) of such form

$$\varphi'_n(u'; r_T) = \frac{1}{n}\sum_{j=1}^{n}[\exp(iu100r_j)]. \quad (3.16)$$

The estimated UCF is therefore also scaled by 100 and has the form

$$\varphi(u; r_T) = E[e^{iu100r_T}] = \exp(C(u'100; \tau) + J(u'100; \tau)) \star \left(1 - \frac{D(u'100; \tau)\sigma^2}{2\beta}\right)^{-2\alpha\beta/\sigma^2}. \quad (3.17)$$

However, the computations of an option price is made using CF without scaling. We can obtain it when we apply transformation $r_T = 0.01r'_T$. The Unscaled UCF has the following form

$$\varphi(u; r_T) = E[e^{iu100(0.01r'_T)}] = \exp(C(u0.01; \tau) + J(u0.01; \tau)) \star \left(1 - \frac{D(u0.01; \tau)\sigma^2}{2\beta}\right)^{-2\alpha\beta/\sigma^2}. \quad (3.18)$$

### 3.2 The estimation of parameters for the Hyperbolic Model

The hyperbolic distribution was introduced by Barndorff-Nielsen, Halgreen (1977)(see [13]) and Eberlein, Keller and Prause (1998)(see [6]). The normal
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Figure 3.1: The estimated (grey) and empirical (black) characteristic functions for the OMXS30 index.

<table>
<thead>
<tr>
<th>Index</th>
<th>$\alpha$</th>
<th>$\beta$</th>
<th>$\sigma$</th>
<th>$\sigma_j$</th>
<th>$\mu$</th>
<th>$\mu_j$</th>
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<tr>
<td>OMXS30</td>
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<td>37.4336</td>
<td>53.5489</td>
<td>0</td>
<td>7.01858</td>
<td>1.7772</td>
</tr>
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</table>

<table>
<thead>
<tr>
<th>$\mu_c$</th>
<th>$\rho$</th>
<th>$\rho_j$</th>
<th>$\lambda$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.0877</td>
<td>0.5579</td>
<td>-0.9819</td>
<td>0.0094</td>
</tr>
</tbody>
</table>

Table 3.1: The values of estimated parameters
(gaussian) density is defined by

\[ g(x) = \frac{1}{\sigma \sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}, \tag{3.19} \]

where \( \mu \) is a location parameter, \( \sigma^2 > 0 \) is a scale parameter. The plot of the log density \( g(x) \) is a parabola. For the hyperbolic distribution the density function is given by

\[ f_{(\alpha, \beta, \delta, \mu)}(x) = \frac{\sqrt{\alpha^2 - \beta^2}}{2\alpha \delta K_1(\delta \sqrt{\alpha^2 - \beta^2})} e^{\frac{-\alpha \sqrt{\delta^2 + (x-\mu)^2} + \beta (x-\mu)}{\delta \sqrt{\alpha^2 - \beta^2}}}, \tag{3.20} \]

where \( K_1 \) is the modified Bessel function of the second kind with the index 1, \( \alpha \) determine the shape of density and \( \beta \) is responsible for skewness \((0 \leq \beta < \alpha)\), \( \mu \) is a location parameter, \( \delta \geq 0 \) is a scale parameter. \( I_{\alpha}(x) \) and \( K_{\alpha}(x) \) are the two linearly independent solutions to the modified Bessel’s equation

\[ x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} - (x^2 + \alpha^2) y = 0. \tag{3.21} \]

The modified Bessel functions of the first and second kind are defined as well by following series representations

\[ I_{\alpha}(x) = i^{-\alpha} \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(m + \alpha + 1)} \left( \frac{ix}{2} \right)^{2m+\alpha}, \tag{3.22} \]

\[ K_{\alpha}(x) = \frac{\pi}{2} \frac{I_{-\alpha}(x) - I_{\alpha}(x)}{\sin(\alpha \pi)}, \tag{3.23} \]

where \( \Gamma(z) \) is the gamma function (see also [9]) . Functions (3.22) and (3.23) are well defined for complex arguments \( x \).

The plot of the log density \( f(x) \) is a hyperbola and correspondingly the distribution called the hyperbolic distribution. There exist other parameterizations of the density \( f(x) \) (it is also given by Puig and Stephens (2001) [15]), where the density function \( f(x) \) is given by

\[ f_{(\pi, \zeta, \delta, \mu)}(x) = \frac{1}{2\delta \sqrt{1 + \pi^2 K_1(\zeta)}} e^{-\zeta \left( \sqrt{1 - \pi^2} \sqrt{1 + (\frac{x-\mu}{\delta})^2} - \pi (\frac{x-\mu}{\delta}) \right)}, \tag{3.24} \]

where \( K_1 \) is the modified Bessel function of the second kind with the index 1, \( \mu \) and \( \delta \) are the location and scale parameters as well, \( \zeta \) relates to the peakedness of the distribution, and \( \pi \) relates to the degree of the asymmetry.
Chapter 3. The estimation of parameters for the SVJJ model and for the Hyperbolic model

(for the symmetric distribution $\pi$ is equal to zero, see [4]), $\zeta$ and $\pi$ can be represented using the parameters of (3.20)

$$\zeta = \delta \sqrt{\alpha^2 - \beta^2}, \quad \pi = \frac{\beta^2}{\alpha^2 - \beta^2}$$  \hspace{1cm} (3.25)

and vice versa.

It is well-known that Black-Scholes model is not an exact one for the log returns of prices. Let us consider log returns of prices $x_n = \log(S_n/S_{n-1})$ for $n = 1, ..., t$. The criterion that used for fitting the hyperbolic distribution to the log returns was maximum likelihood. There are several estimation methods in the statistical literature, for example, analogical, moments, maximum likelihood, etc. In this paper we follow Eberlein, Keller and Prause (1995) [6] and focusing on the maximum likelihood method, because it has a lot of advantages, for instance, the estimated parameters converge to the true values as the sample size gets larger, hence if the sample is large enough then the estimated parameters are more exact.

The algorithm of this method is following: let $x = (x_1, x_2, ..., x_n)$ be a continuous random variable with a known density function $f(x; \theta_1, \theta_2, ..., \theta_k)$, where $\theta = (\theta_1, \theta_2, ..., \theta_k)$ is the vector of $k$ unknown constant parameters that we should estimate according to our sampling data $x$. Roughly speaking, the likelihood of a set of $n$ independent observations $x$ is the probability to obtain the particular set of the data which is given by the chosen probability model. The maximum likelihood estimation (MLE) means that those values of the unknown parameter that maximize the sample likelihood will be the best. For the complete data the likelihood function is given by

$$L = L(x_1, x_2, ..., x_n, \theta) = \prod_{i=1}^{n} f(x_i, \theta).$$  \hspace{1cm} (3.26)

Sometimes it is easier to work with the logarithmic likelihood function $\Lambda$ which is introduces by

$$\Lambda = \ln L = \sum_{i=1}^{n} \ln f(x_i, \theta).$$  \hspace{1cm} (3.27)

In both cases the MLE maximize $L$ (or $\Lambda$) and defines parameters $\theta_1, \theta_2, ..., \theta_k$. By the maximizing $\Lambda$ we obtain solutions of $k$ equations of the type

$$\frac{\partial \Lambda}{\partial \theta_j} = 0, \; j = 1, 2, ..., k.$$  \hspace{1cm} (3.28)
In case of a normal distribution, the likelihood function is given by

\[ L = \left( \frac{1}{\sigma \sqrt{2\pi}} \right)^n \exp \left\{ -\frac{1}{2} \sum_{i=1}^{n} \left( \frac{x_i - \mu}{\sigma} \right)^2 \right\} \]  

(3.29)

then

\[ \Lambda = \ln L = -n \ln \sigma - \frac{n}{2} \ln(2\pi) - \sum_{i=1}^{n} \left( \frac{x_i - \mu}{\sigma} \right)^2. \]  

(3.30)

Taking the partial derivatives of \( \Lambda \) with respect to each parameter and setting them equal to zero yields

\[ \frac{\partial \Lambda}{\partial \sigma} = -\frac{n}{\sigma} + \frac{1}{\sigma^3} \sum_{i=1}^{n} (x_i - \mu)^2 = 0 \]  

(3.31)

and

\[ \frac{\partial \Lambda}{\partial \mu} = \frac{1}{\sigma^2} \sum_{i=1}^{n} (x_i - \mu) = 0. \]  

(3.32)

After solving equations (3.31) and (3.32) we obtain

\[ \sigma^2 = \frac{1}{n} \sum_{i=1}^{n} (x_i - \mu)^2, \quad \mu = -\frac{1}{n} \sum_{i=1}^{n} x_i. \]  

(3.33)

In the case of the hyperbolic distribution, the algorithm of MLE will be the same.

For the estimation of the parameters by maximum likelihood method, we used the package R Foundation for Statistical computing (see [18]). We used also the real data of the stock prices of 2 large companies with very liquid behavior and the index that they are acting on the Swedish market: ABB Ltd, Tele2 B and OMX Stockholm 30 Index (OMXS 30). The real data set consists of the daily closing prices from December 30, 2002, to April 15, 2008, resulting in 1326 observations for the log returns.

On the first step, we took the log returns of prices \( x \). We defined \( \sigma \) and \( \mu \) from the normal distribution (3.19). This distribution was carried out using the function \( \text{fitdistr()} \) included in package MASS. On the second step, we specify the data vector \( x = (x_1, x_2, ..., x_n) \) and the type of the density function (for instance, "normal") and this function fitted to the real data vector the parameters \( \sigma \) and \( \mu \) by MLE (see [20]). The obtained results are represented in Figure 3.3 (see also Appendix 5.1 for details).

For the hyperbolic distribution (3.23) in the package R exist a special implemented function \( \text{hyperbFit()} \) which fits the hyperbolic distribution
Chapter 3. The estimation of parameters for the SVJJ model and for the Hyperbolic model

<table>
<thead>
<tr>
<th>Company</th>
<th>$\mu$</th>
<th>$\sigma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>ABB Ltd</td>
<td>-0.0014016424</td>
<td>0.0261516140</td>
</tr>
<tr>
<td>Tele2 B</td>
<td>-0.0003354828</td>
<td>0.0185923290</td>
</tr>
<tr>
<td>OMXS 30</td>
<td>-0.0004937855</td>
<td>0.0119711622</td>
</tr>
</tbody>
</table>

Figure 3.2: Estimated parameters for the normal distribution function $g(\sigma, \mu)(x)$

We specify the data vector $x$ in the following way, that $\text{ThetaStart}$ is equal to zero (it is a starting parameter vector, $\text{Theta}$ take the form $c(\pi, \zeta, \delta, \mu)$), $\text{startValues}$ is equal to "BN" (it is a method of determining starting values for finding parameter vector $\text{Theta}$ by MLE) and the $\text{method}$ is equal to "Nelder-Mead" (here we can define different optimization methods). The "BN" means that the starting values based on Barndorff-Nielsen (1977) (see [13]). The optimization method called "Nelder-Mead" or a simplex method uses function values only and is robust but it is relatively slow. This method work reasonably well for a non-differentiable functions [20]. The parameters for the hyperbolic distribution function are represented in Figure 3.4 (see also Appendix 5.1 for details).

<table>
<thead>
<tr>
<th>Company</th>
<th>$\pi$</th>
<th>$\zeta$</th>
<th>$\delta$</th>
<th>$\mu$</th>
</tr>
</thead>
<tbody>
<tr>
<td>ABB Ltd</td>
<td>-0.0195275</td>
<td>0.1547164</td>
<td>0.0026037</td>
<td>-0.0007277</td>
</tr>
<tr>
<td>Tele2 B</td>
<td>-0.0164758</td>
<td>0.9572916</td>
<td>0.0106779</td>
<td>0.0001552</td>
</tr>
<tr>
<td>OMXS 30</td>
<td>0.096854</td>
<td>0.833414</td>
<td>0.006168</td>
<td>-0.002322</td>
</tr>
</tbody>
</table>

Figure 3.3: Estimated parameters for the hyperbolic distribution function $f(\pi, \zeta, \delta, \mu)(x)$

The hyperbolic distribution must be more exact than the normal distribution in the sense of fitting real data sets, because the hyperbolic density function has higher peak and heavy tails. This effect we demonstrated in Figure 3.5, where the curve of the estimated hyperbolic distribution and the curve of the estimated normal distribution of the log returns are plotted on the same figure as the histogram of the real data values (see Appendix 5.2).

To prove this assumption we found the difference between real values and models values in two following ways: with the areas of this values and with the quantiles of each value.

a. The average standard deviation
Figure 3.4: Fitted densities \( f(\alpha, \beta, \delta, \mu) \) and \( g(\mu, \sigma) \) for the companies ABB Ltd (\( f(59.43, 1.16, 0.0026, -0.0007) \), \( g(-0.0014, 0.02) \)), Tele2 B (\( f(89.66, 1.477, 0.011, 0.00016) \), \( g(-0.0034, 0.0186) \)) and the index OMXS 30 (\( f(135.75, 13.087, 0.0062, -0.0023) \), \( g(-0.00049, 0.012) \)).

The most common measure to compute an error is the average standard deviation, which is defined by

\[
Error = \sqrt{\frac{1}{n} \sum_{i=1}^{n} (x_i - y_i)^2},
\]

(3.34)

where \( x_i \) is the sample of real values, \( y_i \) is the sample of the model values and \( n \) is the length of the sample. For the comparison of the value of the density function \( f(x) \) (or \( g(x) \) as well) with the real data value \( x \) we used areas of this values on the same interval. We consider the interval from the minimal value of \( x \) to the maximal value of \( x \) (which includes \( n \) points) and shared it
on the smaller intervals with the step $h$

$$h = \frac{1}{n - 2}.$$ (3.35)

The size of the real sample $x$ is 1326, hence $h = 0.0007535795$. To start we took all available points to obtain the "ideal" accuracy.

From Figures 3.3 and 3.4 we took the initial values of the parameters for the normal distribution function $g(x)$ from the equation (3.19) and the hyperbolic distribution $f(\pi, \zeta, \delta, \mu)(x)$ from equation (3.24). We transformed the parameters of the density $f(\pi, \zeta, \delta, \mu)(x)$ (see (3.24)) into the more common parameters for $f(\alpha, \beta, \delta, \mu)(x)$ (see (3.20)) using equations (3.25). The transformed parameters have the following form

$$\alpha = \frac{\zeta}{\delta} \sqrt{1 + \pi^2}, \quad \beta = \left| \frac{\pi \zeta}{\delta} \right|,$$ (3.36)

where $\delta$ and $\mu$ be the same as in (3.24). The transformed parameters are shown in Figure 3.6.

<table>
<thead>
<tr>
<th>Company</th>
<th>$\alpha$</th>
<th>$\beta$</th>
<th>$\delta$</th>
<th>$\mu$</th>
</tr>
</thead>
<tbody>
<tr>
<td>ABB Ltd</td>
<td>59.43307</td>
<td>1.160358</td>
<td>0.0026037</td>
<td>-0.0007277</td>
</tr>
<tr>
<td>Tele2 B</td>
<td>89.66384</td>
<td>1.477083</td>
<td>0.0106779</td>
<td>0.0001552</td>
</tr>
<tr>
<td>OMXS 30</td>
<td>135.7513</td>
<td>13.08682</td>
<td>0.006168</td>
<td>-0.002322</td>
</tr>
</tbody>
</table>

Figure 3.5: Parameters for the hyperbolic distribution $f(\alpha, \beta, \delta, \mu)(x)$

Then for each small interval we took the area of the real data called $Sx$, where the number of all $x$-values in each interval are multiplied on the step-size $h$. The area of the density function on the same intervals is equal to the integral of density (3.19) (or (3.20)). The final step is to determine the error by using equation (3.34), where as real value $x_i$ we took the area $Sx$ and as the models value $y_i$ we took the area of the density $g(x)$ or $f(x)$ (see Appendix 5.3). The errors that we obtained for each of the distributions are represented in Figure 3.7.

In these results of computer program we can see that the accuracy of the hyperbolic model is better than normal distribution by one order of magnitude.

b. A deviation in the quantiles
### Comparative analysis of the SVJJ and the hyperbolic models on the Swedish market

<table>
<thead>
<tr>
<th>Company</th>
<th>Error for normal distribution</th>
<th>Error for hyperbolic distribution</th>
</tr>
</thead>
<tbody>
<tr>
<td>ABB Ltd</td>
<td>0.01099971</td>
<td>0.00324802</td>
</tr>
<tr>
<td>Tele2 B</td>
<td>0.01477369</td>
<td>0.003742189</td>
</tr>
<tr>
<td>OMXS 30</td>
<td>0.02041190</td>
<td>0.002391773</td>
</tr>
</tbody>
</table>

Figure 3.6: Approximation errors between real and model values (using areas)

There exist a lot of definitions on quantiles (or percentiles). For instance, quantile $x_\alpha$ is the number such that if $X$ is a random variable, then the probability $P(X \leq x_\alpha) = \alpha$. To our mind the most common definition is the following.

**Definition 3 [17]** The quantile of a distribution is defined as

$$Q(p) = \inf \{ x : F(x) \geq p \}, 0 < p < 1,$$

where $F(x)$ is the distribution function. Sample quantiles provide nonparametric estimators based on a set of independent observations $x_1, ..., x_n$ from the distribution $F$.

To compute the quantiles of the sample data for the normal distribution and for the hyperbolic distribution we used two following properties

$$P \left( x_{\frac{1-\alpha}{2}} \leq X \leq x_{\frac{1+\alpha}{2}} \right) = \alpha$$

and

$$F_X(x_\alpha) = \alpha = \int_{-\infty}^{x_\alpha} f(t)dt.$$

We calculate quantiles using equations for the density functions (3.19) and (3.20), we also used Figure 3.6 with the estimated parameters and equations (3.37)-(3.39). We compute the Error from equation (3.34) where as a sample value we took quantiles of real data and as a model value we took quantiles of the normal or the hyperbolic densities (for details see Appendix 5.4). We obtain following results, which are represented in Figure 3.8.

Here we see that the results obtained by using the hyperbolic density function of log returns are better than the results obtained by using the normal density function. To make the results more visible we created the Quantile - Quantile plots for each of studied companies and for the index. See Figures 3.9 - 3.11, where on the $X$-axis theoretical quantiles (to the normal or to the hyperbolic distribution function) and on the $Y$-axis sample quantiles.
Chapter 3. The estimation of parameters for the SVJJ model and for the Hyperbolic model

<table>
<thead>
<tr>
<th>Company</th>
<th>Error for the normal distribution</th>
<th>Error for the hyperbolic distribution</th>
</tr>
</thead>
<tbody>
<tr>
<td>ABB Ltd</td>
<td>0.1373122</td>
<td>0.004049403</td>
</tr>
<tr>
<td>Tele2 B</td>
<td>0.09721975</td>
<td>0.001846221</td>
</tr>
<tr>
<td>OMXS 30</td>
<td>0.03820144</td>
<td>0.000801534</td>
</tr>
</tbody>
</table>

Figure 3.7: Approximation errors between the real and the model values (using quantiles)

are situated. As usual the Quantile-Quantile plots show the deviation of the real values from the normality. We see that the empirical hyperbolic quantiles are similar to the sample quantiles, but the empirical normal quantiles are deviated from the sample quantiles (specially in the tails).

Figure 3.8: The Quantile-Quantile plots for the log returns of the ABB Ltd company stocks in the time period from December 30, 2002, to April 15, 2008.
Figure 3.9: The Quantile-Quantile plots for the log returns of the Tele2 B company stocks in the time period from December 30, 2002, to April 15, 2008.

Figure 3.10: The Quantile-Quantile plots for the log returns of the OMXS 30 index stocks in the time period from December 30, 2002, to April 15, 2008.
CHAPTER 3. THE ESTIMATION OF PARAMETERS FOR THE SVJJ MODEL AND FOR THE HYPERBOLIC MODEL
Chapter 4

Option Pricing using the SVJJ model and the Hyperbolic model

4.1 The valuation of a call option price in the frame of the SVJJ Model

4.1.1 The general formula

The general formula for a call option price is given by

\[ C(S_t, t, T, K) = E[e^{-rT} \max\{S_T - K, 0\}|S_t]. \]  \hspace{1cm} (4.1)

where \( T \) is maturity time, \( K \) is strike price, \( r \) is interest rate and \( S \) is a stock price.

When we separate the expectations we obtain

\[ C(S_t, t, T, K) = E[e^{-rT}S_T 1_{\{S_T > K\}}|S_t] - e^{-rT}KE[1_{\{S_T > K\}}|S_t]. \]  \hspace{1cm} (4.2)

The Second expectation is equal to

\[ E[1_{\{S_T > K\}}|S_t] = P(S_T > K|S_t). \]  \hspace{1cm} (4.3)

For the first expectation we define a change of measure (theorem in Appendix), sometimes called a forward measure

\[ \frac{dP^S}{dP} |_{S_t} = G_t^S = \frac{e^{-rt}S_t}{S_0}. \]  \hspace{1cm} (4.4)
Chapter 4. Option Pricing using the SVJJ model and the Hyperbolic model

Then in general, when $X_T$ is a random variable we obtain

$$E^{S}[X_T|F_t] = E[G^{S}_t X_T|S_t] = \frac{1}{S_t} E[e^{-rT} S_T X_T|S_t].$$  \hspace{1cm} (4.5)

Then our first expectation can be represented as

$$E[e^{-rT} S_T 1_{\{S_T > K\}}|S_t] = S_tE[G^{S}_t 1_{\{S_T > K\}}|S_t] = S_tE^{S}[1_{\{S_T > K\}}|S_t] = S_tP^S(S_T > K|S_t).$$ \hspace{1cm} (4.6)

where $P^S$ is the forward measure.

That gives us formula for the call price

$$C(S_t, t, T, K) = S_tP^S(S_T > K|S_t) - e^{-rT} K P(S_T > K|S_t).$$ \hspace{1cm} (4.7)

### 4.1.2 Expressing probability by a characteristic function

In our case the problem is to calculate probabilities in (4.7) by using a characteristic function. We do it in the following way.

The characteristic function has a form:

$$\varphi(u) = \int_{-\infty}^{\infty} e^{iux} f(x) dx.$$ \hspace{1cm} (4.8)

where $f(x)$ is a probability density function.

To obtain $f(x)$ which is used in calculation of an option price we use the inverse Fourier transform \cite{21}

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iux} \varphi(u) du.$$ \hspace{1cm} (4.9)

The probability $P(X > a)$ can be expressed as

$$P(b > X > a) = \int_{a}^{b} f(x) dx.$$ \hspace{1cm} (4.10)

We insert into the above equation our formula for the probability density function and we obtain a usual statement of Fourier’s integral theorem

$$P(\infty > X > a) = \frac{1}{2\pi} \int_{a}^{\infty} dx \int_{-\infty}^{\infty} e^{-iux} \varphi(u) du.$$ \hspace{1cm} (4.11)
Gil-Pelaez [7] gives a simplified formula for the probability

\[ P(X \leq a) = \frac{1}{2} + \frac{1}{2\pi} \int_0^\infty \frac{e^{iux}\varphi(-u) - e^{-iux}\varphi(u)}{iu} du \]

Next, note that for an arbitrary complex number \( z \), \( z + \bar{z} = 2\text{Re}[z] \) and the characteristic function is a hermitian function which means that has following proprety \( \varphi(u) = \varphi(-u) \):

\[ \frac{1}{2\pi} \int_0^\infty \frac{e^{iux}\varphi(-u)du}{iu} - \frac{1}{2\pi} \int_0^\infty \frac{e^{-iux}\varphi(u)du}{iu} = \]

\[ -\frac{1}{\pi} \int_0^\infty \text{Re}[\frac{e^{-iux}}{iu}\varphi(u)]du. \quad (4.13) \]

After substitution of (4.13) in (4.12) we obtain a formula for the probability which we looked for by using characteristic function

\[ P(X > a) = 1 - P(X \leq a) = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \text{Re}[\frac{e^{-iux}}{iu}\varphi(u)]du. \quad (4.14) \]

### 4.1.3 The call option formula based on the characteristic function

The general formula for call was given by the expression

\[ C(S_t, t, T, K) = S_tP^S(S_T > K|S_t) - e^{-rT}KP(S_T > K|S_t). \quad (4.15) \]

We have expressed the probability with the characteristic function (4.14), now we can apply this to our probabilities.

For the measure \( P^S \) the characteristic function is different from 3.11

\[ \varphi^S(u) = E^S[e^{iuz}|s_t] = E[G^S_te^{iuz}|s_t] = E[^1_{S_t}e^{-rT}S_Te^{iuS_T}|s_t] \]

\[ = E[e^{-rT}\ln(S_t)e^{iuS_T}|s_t] = E[e^{-rT}e^{-s_t}(1+iu)S_T|s_t] \]

\[ = e^{-rT-s_t}E[e^{i(uu-2iu)|s_t}] = e^{-rT-s_t}\varphi(u-i|s_t). \quad (4.16) \]

where

\[ \varphi(u|s_t) = E[e^{iu\ln(K)}|s_t]. \quad (4.17) \]
Finally, our first probability is equal to

\[ P_S(s_T > \ln(K)|s_t) = \frac{1}{2} + \frac{1}{\pi} \int_0^{\infty} \text{Re}\left[ \frac{e^{-iu \ln(K)}}{iu} e^{-r T - s_t} \varphi(u - i|s_t) \right] du. \] (4.18)

where \( s_t = \ln(S_t) \). And the second probability is equal to

\[ P(s_T > \ln(K)|s_t) = \frac{1}{2} + \frac{1}{\pi} \int_0^{\infty} \text{Re}\left[ \frac{e^{-iu \ln(K)}}{iu} \varphi(u|s_t) \right] du. \] (4.19)

### 4.1.4 Computations of the call option price

We valuate an European call option on OMXS 30 index on Nordic Stock Exchange where \( r = 0.025 \).

The SVJJ model characteristic function was calibrated for stock prices between 30th of December 2002 and 15th of April 2008. The initial time of valuation of the option is at 29th of April 2008 (\( S_0 = 990.7 \)) and at the 2nd of May 2008 (\( S_0 = 1008.2 \)).

The complete formula for the call option has the following form

\[
C(S_0, \tau, T, K) = S_0 \left( \frac{1}{2} + \frac{1}{\pi} \int_0^{\infty} \text{Re}\left[ \frac{e^{-iu \ln(K)}}{iu} e^{-r T - s_0} \varphi(u - i|s_0) \right] du \right) \\
- e^{-r T} K \left( \frac{1}{2} + \frac{1}{\pi} \int_0^{\infty} \text{Re}\left[ \frac{e^{-iu \ln(K)}}{iu} \varphi(u|s_0) \right] du \right) .
\] (4.20)

where the characteristic functions \( \varphi(u - i|s_0) \) and \( \varphi(u|s_0) \) have forms

\[
\varphi(u - i|s_0) = E[e^{i(u - i)100(0.01 \tau^*)}|s_0] e^{i(u - i)s_0} = \exp(C((u - i)0.01; \tau) +
J((u - i)0.01; \tau) + i(u - i)s_0) *
\left(1 - \frac{D((u - i)0.01; \tau) \sigma^2}{2\beta} \right)^{-2\alpha/\sigma^2} .
\] (4.21)

\[
\varphi(u|s_0) = E[e^{iu100(0.01 \tau^*) + ius_0}|s_0] = \exp(C(u0.01; \tau) + J(u0.01; \tau) +
+ ius_0) *
\left(1 - \frac{D(u0.01; \tau) \sigma^2}{2\beta} \right)^{-2\alpha/\sigma^2} .
\] (4.22)

All prices are in Swedish Crowns currency (SEK).

### 4.1.5 Conclusions

The option prices which we obtained using SVJJ model (Table 4.1) are significantly smaller than the prices on the real market for the considered options.
Comparative analysis of the SVJJ and the hyperbolic models on the Swedish market

<table>
<thead>
<tr>
<th>$S_0$</th>
<th>$K$</th>
<th>$\tau = T(t = 0)$</th>
<th>Market</th>
<th>Computed</th>
</tr>
</thead>
<tbody>
<tr>
<td>990.27</td>
<td>1080</td>
<td>45/252</td>
<td>4.75</td>
<td>1.8679</td>
</tr>
<tr>
<td>990.27</td>
<td>990</td>
<td>45/252</td>
<td>33.25</td>
<td>7.5099</td>
</tr>
<tr>
<td>990.27</td>
<td>940</td>
<td>45/252</td>
<td>65.875</td>
<td>52.9466</td>
</tr>
<tr>
<td>1008.2</td>
<td>1080</td>
<td>42/252</td>
<td>6.375</td>
<td>1.7897</td>
</tr>
<tr>
<td>1008.2</td>
<td>990</td>
<td>42/252</td>
<td>42</td>
<td>21.1240</td>
</tr>
<tr>
<td>1008.2</td>
<td>940</td>
<td>42/252</td>
<td>78.875</td>
<td>70.2143</td>
</tr>
</tbody>
</table>

Table 4.1: Computed call option prices

If we look for the price of call option for $S_0 = 990.7$ and $K = 990.7$ then the probability $P(S_T > K)$ is equal to 0.5458 which means that according to the SVJJ model it is more possible that the price of the underlying index will rise. This fact occurred due to rising tendency of OMXS30 index in past. The price risen from 493.2 at the beginning of data to 990.72 at the end of the period for which parameters were estimated. However, this fact does not explain the lowered option price.

To compare prices we used the average price between the bid and ask prices. The spread between ask and bid can be treated as boundaries of error, which computations can be saddled. However, for considered options, spread is too small(around 3 SEK) to excuse mispricing.

The lack of transaction cost in the SVJJ model undoubtedly lowers any option price, also data used for the estimation of parameters may not reflect the actual evolution of the index. Every tick data or shorter period of time could be considered instead.

4.2 The valuation of a call option price in the frame of the Hyperbolic Model

Recall that for a hyperbolic Levy motion $X = (X_t)_t$ the hyperbolic density function for log returns is given by

$$f_{(\alpha,\beta,\delta,\mu)}(x) = \frac{\sqrt{\alpha^2 - \beta^2}}{2\alpha \delta K_1(\delta \sqrt{\alpha^2 - \beta^2})} \exp \left(-\alpha \sqrt{\delta^2 + (x - \mu)^2} + \beta (x - \mu)\right),$$

where $x$ are log returns of prices and parameters $(\alpha, \beta, \delta, \mu)$ for our data were estimated in the previous section.
Definition 4 The characteristic function of the density function $f(x)$ is defined by
\[
\varphi(u) = Ee^{iuX_t} = M(iu) = \int_{-\infty}^{\infty} e^{iux} f_{(\alpha,\beta,\delta,\mu)}(x) dx,
\]
(4.24)
where $M(u)$ is the moment generating function of the distribution and it is given by
\[
M(u) = \int_{-\infty}^{\infty} e^{ux} f(x) dx.
\]
(4.25)

In the case of the hyperbolic distribution the characteristic function $\varphi(u)$ is given by the expression
\[
\varphi(u) = e^{i\mu u} \sqrt{\frac{\alpha^2 - \beta^2}{\delta (\sqrt{\alpha^2 - \beta^2})}} \frac{K_1(\delta \sqrt{\alpha^2 - (\beta + iu)^2})}{K_1(\delta \sqrt{\alpha^2 - \beta^2})} \frac{\alpha}{\sqrt{\alpha^2 + u^2}}.
\]
(4.26)
which is valid for $|\beta + u| < \alpha$ and $u \in \mathbb{R}$.

We follow Eberlein, Keller and Prause paper [6] and consider the symmetric centered case ($\mu$ is a location parameter and $\beta$ is a skewness parameter, hence to simplify the calculation values of this parameters can be assumed to be equal to zero). In this case we obtain following equation for the characteristic function
\[
\varphi(u; \alpha, \delta) = \frac{\alpha}{K_1(\delta \alpha)} \frac{K_1(\delta \sqrt{\alpha^2 + u^2})}{\sqrt{\alpha^2 + u^2}}.
\]
(4.27)

Note that the shape parameter now is $\zeta = \alpha \delta$.

The density of $L(X_t)$ we can obtain using the characteristic function (which is given above) by the Fourier inversion formula
\[
f_t(x) = \frac{1}{\pi} \int_{0}^{\infty} \cos(ux) \left[ \varphi(u; \alpha, \delta) \right]^t du.
\]
(4.28)

Using the numerical integration the integral on the right-hand side can be computed rather efficiently.

Now we remark: in the equation (4.26) we have the following boundary $|\beta + u| < \alpha$, hence for the $\beta = 0$ (because we neglect this parameter) we have that $|u| < \alpha$ and we can modify the equation (4.28) into the form
\[
f_t(x) = \frac{1}{\pi} \int_{0}^{\alpha} \cos(ux) \left[ \varphi(u; \alpha, \delta) \right]^t du.
\]
(4.29)

The plot of the density function $f_t(x)$ is given in Figure 4.1.
Comparative analysis of the SVJJ and the hyperbolic models on the Swedish market

Figure 4.1: The density function \( f_t(x) \) for the ABB Ltd company (\( \alpha = 59.43307, \beta = 0, \delta = 0.0026037, \mu = 0 \))

Remark.\[5\] The model which produces exactly hyperbolic returns along time-intervals of length 1 is

\[
S_t = S_0 e^{X_t} \tag{4.30}
\]

and there is no unique equivalent martingale measure, that is, an equivalent measure such that the discounted process \((e^{-rt}S_t)_{t \geq 0}\) is a martingale. Here \( r \) denotes the interest rate. But for \((S_t)_{t \geq 0}\) it is easy to compute explicitly at least one equivalent martingale measure. This can be used for the valuation of derivative securities.

Therefore we can define a new density function

\[
f_t(x; \theta) = \frac{e^{\theta x} f_t(x)}{M(\theta)^t} = \frac{e^{\theta x} f_t(x)}{(\int_{-\infty}^{\infty} e^{\theta y} f_t(y) dy)^t} \tag{4.31}
\]

for some real number \( \theta \). Let \( P^\theta \) be the Esscher equivalent martingale measure such that \( dP^\theta = e^{\theta X_t - t \log M(\theta)} \) and let define \( \theta \) by \( S_0 = e^{-rT} E^\theta (S_t) \) such that \((e^{-rt}S_t)_{t \geq 0}\) is a martingale. Then we can obtain the following expression

\[
e^r = \frac{M(\theta + 1, 1)}{M(\theta, 1)}, \tag{4.32}
\]

where \( r \) is the known interest rate and \( \theta \) defines the martingale measure as the solution of (4.32).
Chapter 4. Option Pricing using the SVJJ model and the Hyperbolic model

We substitute equation (4.25) into the equation (4.32), then take the logarithm from the result and obtain

\[ r = \ln \left( \frac{K_1(\sqrt{\zeta^2 - \delta^2(\theta + 1)^2})}{K_1(\sqrt{\zeta^2 - \delta^2\theta^2})} \right) - \frac{1}{2} \ln \left( \frac{\zeta^2 - \delta^2(\theta + 1)^2}{\zeta^2 - \delta^2\theta^2} \right), \quad (4.33) \]

where \( \zeta = \alpha\delta \) and \( r \) is the interest rate. Using numerical methods it is easy to find a solution for \( \theta \) from the last equation (4.33). As the known interest rate we can take the most common of benchmark interest rate indexes the STIBOR (Stockholm interbank offered rate) and put \( r = 0.0465 \).

For a European call option the value at time 0 is given by

\[ C_0 = E^\theta [e^{-rT}(S_T - K)]_+, \quad (4.34) \]

where \( T \) is time to expiration, \( K \) is the strike price, \( H(S_T) = ((S_T - K)_+) \) is the payoff function and \( a_+ \) denotes \( \max(a, 0) \). This expectation under the equivalent martingale measure \( P^\theta \) can be presented in the following way

\[ C_t = S_0 \int_\gamma^{\infty} f_t(x; \theta + 1)dx - e^{-rt}K \int_\gamma^{\infty} f_t(x; \theta)dx, \quad (4.35) \]

for \( \gamma = \ln(K/S_0) \). The function \( f_t(x, \theta) \) is the density of the \( X_t \) distribution under the risk-neutral measure given by (4.31) which is related to the original density function \( f_t(x) \) (4.28). The value of the call option can be computed in the real time using numerical integration and the Fast Fourier transformation.

For the calculation of the call price we used the package Mathematika 5.0 (see [17]). But the direct input of equation lead us to wrong and unstable results, for instance, the value of the call option was very high. In equations for the density \( f_t(x; \theta) \) (4.31) and for the call price \( C_t \) (4.35) we have integration with respect to \( x \) (our log returns) from -Infinity to Infinity and from \( \gamma \) to Infinity. We made the assumption that potential future movements in the Swedish market should be in the same direction as movements during last 6 years (recall that the real data set was consist of daily closing prices from December 30, 2002, to April 15, 2008). Therefore from 1326 observations for log returns (that we took for the estimation of parameters for the hyperbolic and the normal density) we can find values of \( x_{\min} \) and \( x_{\max} \), in other words the minimal and the maximal value of log returns during 6 years. In the case of ABB Ltd company they were \( x_{\min} = -0.17 \) and \( x_{\max} = 0.23 \). Then we obtain from equations (4.31) and (4.35) following forms

\[ f_t(x; \theta) = \frac{e^{\theta x} f_t(x)}{(\int_{-0.23}^{0.23} e^{\theta y} f_t(y)dy)^{t}}; \quad (4.36) \]
Comparative analysis of the SVJJ and the hyperbolic models on the Swedish market

\[ C_t = S_0 \int_\gamma^{0.23} f_t(x; \theta + 1) dx - e^{-rt} K \int_\gamma^{0.23} f_t(x; \theta) dx. \]  

(4.37)

For the estimation of the call option price for the ABB Ltd company we took values of parameters \( \alpha, \delta \) for the hyperbolic model from the previous section, \( t \approx 0.3 \) (because we consider an option with the time to maturity in August 2008), the strike price \( K \) was near the stock price \( S_0 \), the parameter \( \theta = 40.98892242 \) (was found using equation (4.33) in the package Mathematika, see Appendix 5.6). For the calculation we insert equations (4.27), (4.29) and (4.36) into the equation (4.37). Obtained results for call option prices are represented in Figure 4.2 (see also Appendix 5.7).

<table>
<thead>
<tr>
<th>Date</th>
<th>Stock price (closing)</th>
<th>Strike price</th>
<th>Price using Hyperbolic model</th>
<th>Real price (Bid/Ask)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2008-04-28</td>
<td>183</td>
<td>180</td>
<td>5.88306</td>
<td>12.50/14.25</td>
</tr>
<tr>
<td></td>
<td></td>
<td>150</td>
<td>35.0780</td>
<td>33.50/37.50</td>
</tr>
<tr>
<td>2008-04-29</td>
<td>182</td>
<td>180</td>
<td>5.09892</td>
<td>11.50/14.00</td>
</tr>
<tr>
<td></td>
<td></td>
<td>150</td>
<td>34.0780</td>
<td>33.00/37.00</td>
</tr>
<tr>
<td></td>
<td></td>
<td>150</td>
<td>39.0780</td>
<td>37.50/40.75</td>
</tr>
<tr>
<td>2008-05-08</td>
<td>189</td>
<td>200</td>
<td>0.57569</td>
<td>6.25/7.50</td>
</tr>
<tr>
<td></td>
<td></td>
<td>180</td>
<td>11.4787</td>
<td>16.00/17.75</td>
</tr>
<tr>
<td></td>
<td></td>
<td>150</td>
<td>41.0780</td>
<td>38.75/42.75</td>
</tr>
</tbody>
</table>

Figure 4.2: Estimated values of call option prices for the ABB Ltd company obtained by using the Hyperbolic model with real market prices

In this Figure 4.2 we see that the value of call option-at-the-money for the company is misprized. But if the strike price is lower than stock price on 20-25 percents, then we obtain exact call option price.

For estimation of call option prices for the OMXS 30 index we took values of parameters \( \alpha, \delta \) for the hyperbolic model from the previous section, \( t \approx 0.14 \) (because we consider an option with the time to maturity in June 2008), the strike price \( K \) was near the stock price \( S_0 \), parameter \( \theta = 113.2252301 \) (was found using equation (4.33) in the package Mathematika, see Appendix 5.6). For the calculation we insert equations (4.27), (4.29) and (4.36) into the equation (4.37). Obtained results of call option prices are represented in Figure 4.3 (see also Appendix 5.7).
### Chapter 4. Option Pricing using the SVJJ model and the Hyperbolic model

<table>
<thead>
<tr>
<th>Date</th>
<th>Stock price (closing)</th>
<th>Strike price</th>
<th>Price using Hyperbolic model</th>
<th>Real price (Bid/Ask)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2008-04-28</td>
<td>1005.31</td>
<td>1000</td>
<td>9.70118</td>
<td>36.00/38.75</td>
</tr>
<tr>
<td></td>
<td>940</td>
<td></td>
<td>71.4095</td>
<td>76.50/79.00</td>
</tr>
<tr>
<td>2008-04-29</td>
<td>990.72</td>
<td>990</td>
<td>7.14397</td>
<td>32.50/34.50</td>
</tr>
<tr>
<td></td>
<td>920</td>
<td></td>
<td>76.6897</td>
<td>80.00/82.50</td>
</tr>
<tr>
<td>2008-05-02</td>
<td>1008.20</td>
<td>990</td>
<td>24.6240</td>
<td>40.75/43.25</td>
</tr>
<tr>
<td></td>
<td>940</td>
<td></td>
<td>74.2995</td>
<td>77.50/80.25</td>
</tr>
<tr>
<td>2008-05-08</td>
<td>1019.67</td>
<td>1020</td>
<td>6.28863</td>
<td>27.75/30.50</td>
</tr>
<tr>
<td></td>
<td>1000</td>
<td></td>
<td>26.1589</td>
<td>39.50/43.00</td>
</tr>
<tr>
<td></td>
<td>940</td>
<td></td>
<td>85.7695</td>
<td>84.75/87.50</td>
</tr>
</tbody>
</table>

Figure 4.3: Estimated values of call option prices of OMXS 30 index obtained by using the Hyperbolic model with real market prices

In this Figure 4.3 we see that the value of the call option-at-the-money for the index is misprized as well. But if the strike price is lower than the stock price on 8-10 percents, then we obtain the more exact call option price. We conclude that the hyperbolic model works good for the call price option in-the-money (where the stock price is greater than the strike price) and works bad for the call option price valuation at-the-money (where the stock price is equal to the strike price).
Chapter 5
Comparison of models and conclusions

In our work we try to compare real data sets for option prices with the values, that we obtained from the hyperbolic and the SVJJ models. Sometimes the explicit formula for the density function is unknown, but we can obtain characteristic function (CF) for both models and for real data values. The result of comparison of 3 characteristic functions presented in Figure 5.1 (see also Appendix 5.5).

In this Figure 5.1 characteristic functions from each model are plotted with the characteristic function from the real data for log-returns of the ABB Ltd company’ stocks. We observe that both models fit to the real data very good and it is hard to say which is better. The normal distribution which is used in the Black-Scholes model fits much worse than our improved models. We calculated an error with the formula

\[ Error = \sqrt{\frac{1}{n} \sum_{i=1}^{n} (\varphi_i - \varphi_E^i)^2}. \]  (5.1)

where \( \varphi_i \) is the considered CF and \( \varphi_E^i \) is the empirical CF. We made calculation for \( n = 10000 \) sample points and obtain following results

We see in Figure 5.2 that the SVJJ model fits slightly better than the Hyperbolic model. After calculations we obtained call option prices for the OMXS 30 index (we use index, because it is an European option). Comparable results are presented in Figure 5.3.

In Figure 5.3 we see that the value of the call option-at-the-money for the index is misprized as well. But also we can see that the Hyperbolic model works very good for the call option price which is in-the-money, but shows us mispricing for the options at-the-money and out-of-the-money (where the
Chapter 5. Comparison of models and conclusions

Figure 5.1: Plot of the empirical characteristic function (CF) (green color), the analytical CF of the SVJJ model (blue color), the analytical CF of the Hyperbolic model (red color).

<table>
<thead>
<tr>
<th>Distribution</th>
<th>Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hyperbolic</td>
<td>0.019</td>
</tr>
<tr>
<td>SVJJ</td>
<td>0.017</td>
</tr>
<tr>
<td>Normal</td>
<td>0.098</td>
</tr>
</tbody>
</table>

Figure 5.2: Approximation errors between the real and the model values

<table>
<thead>
<tr>
<th>Stock price</th>
<th>Strike price</th>
<th>Price using Hyperbolic model</th>
<th>Price using SVJJ model</th>
<th>Real price (Bid/Ask)</th>
</tr>
</thead>
<tbody>
<tr>
<td>990.72</td>
<td>1080</td>
<td>0.00</td>
<td>1.87</td>
<td>4.25/5.25</td>
</tr>
<tr>
<td>990</td>
<td>7.14</td>
<td>7.51</td>
<td>32.50/34.50</td>
<td></td>
</tr>
<tr>
<td>990</td>
<td>24.62</td>
<td>21.12</td>
<td>40.75/43.25</td>
<td></td>
</tr>
<tr>
<td>940</td>
<td>74.30</td>
<td>70.21</td>
<td>77.50/80.25</td>
<td></td>
</tr>
</tbody>
</table>

Figure 5.3: Estimated values of call option prices of the OMXS 30 index obtained by using the Hyperbolic and the SVJJ model with the real market prices
strike price is greater than the stock price). In this case the SVJJ model works slightly better, because it works normal for the call option price which is in-the-money and also works for the options at-the-money and out-of-the-money with a little mispricing. From the other side to calculate the price using the Hyperbolic model we need just a half an hour (or less), but for the calculation of the price by the SVJJ model we need about 8 hours and it is too much.

In conclusion we say that it is useful to calculate the call option price which is in-the-money by the hyperbolic model, and also it is useful to calculate each call option price by the SVJJ model, if time is not important.
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*Goodness-of-Fit Tests for the Hyperbolic Distribution.*  


*Sample Quantiles in Statistical Packages.*  

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Appendix

We present here the programming code of R software and Mathematica 5.0 for the estimation of parameters calculation.

All programs that are represented in the Appendices given for the case of the ABB Ltd stocks. For other companies and the index the program will be the same, changes must be done just in the input files with the real data and estimated parameters (Appendix 2-4).

5.1 The estimation of parameters by the MLE for the normal and the hyperbolic densities
#Firstly, we must activate libraries.
library(numDeriv)
library(HyperbolicDist)
library(gtools)
library(ghp)
library(gplots)
library(gdata)
library(MASS)

#Secondly, put the daily closing prices into the vector b.
x<-read.table("Nh\Data AllS stock.txt", dec=".")
y=x[,1]
b=c(length(y),NA)
for (i in 1:length(y))
b[i]=y[length(y)-i+1]

#Take log returns of the prices
x<-diff(log(b))
print(x)

#This function will show the parameters that estimated by MLE for the Normal distribution.
fit.dist(x,"normal")

#This function will show the parameters that estimated by MLE for the Hyperbolic distribution.
hyperbFit(x, ThetaStart = NULL, startValues = "SR", method = "Heldre-Henn")
hyperbFit(x, plot = TRUE)
hfit <- hyperbFit(x)
par(mfrow = c(1,2))
plot(hfit, which = c(1,3))
Appendix

5.2 The plot software for the theoretical normal and the theoretical hyperbolic densities with the histogram of the real data
library(numDeriv)
library(HyperbolicDist)
library(gtools)
library(qhyp)
library(gplots)
library(qdata)
library(MASS)

z<-read.table("H:\Data AEE stock.txt", dec=".")
y<-z[,1]
b=c(length(y),NA)
for (i in 1:length(y))
b[i]=y[length(y)-i+1]
x<-diff(log(b))
print(x)

##Here go initial value of the variables
n<-length(y)
err=c(n,NA)
errL=c(n,NA)
kol=c(n,NA)
a=min(x)
b=max(x)
F=c(n,NA)
Fhyp=c(n,NA)

##Here is the step for calculating.
h=1/(n-2)

##Here we input initial values for the Normal distribution.
muN=-0.014016424
sigmaN=0.0261516140

##Here we input initial values for the Hyperbolic distribution.
zetN=0.1547164
muH=-0.0007577
delta=0.0056037
piH=-0.0195275

alpha=zeta*sqrt(1+pi^2)/delta
beta=abs(pi*zeta/delta)
### The calculating of densities.

```r
m=a
i=0
err=0
while (i < m)
  (u=m+k
  i=i+1

  f[i] = 1/(sigma0*sqrt(2*pi*141553))*exp(-(((m-h/2)-muN)^2)/2*sigmaN^2))

  K=sqrt(alpha^2-betta^2)/(2*alpha*delta*besselK(zeta, 1, expn.scaled = FALSE))
  f_hyp[i] = K*exp(-alpha*sqrt(delt^2+(m-h/2)-mu)^2) + betta^2*|m-h/2|-mu|

}
print(i)
k=i
```

### The final part is for drawing the plot.

```r
h<-hist(x,breaks=150)
xhist<-x
yhist<-c(0,kol,0)

xfit<-seq(min(x),max(x),length=k)
yfit<-f_hyp

x<-seq(min(x),max(x),length=k)
y<-f

plot(xhist,yhist,type="s",ylim=c(0,max(yhist,yfit)), main="Normal pdf and histogram")
lines(xfit,yfit, col="red")
lines(x,y, col="blue")
```
Appendix

5.3 The calculation of the Errors using the area on each interval
library(numberiv)
library(HyperbolicDist)
library(gtools)
library(ghyp)
library(uplots)
library(gdata)
library(MASS)

x <- read.table("H:\Data ABE stock.txt", dec=".")
y <- x[, 1]
b <- c(length(y), NA)
for (i in 1:length(y))
  b[i] = y[length(y) - i + 1]
print(x)

nc = length(y)\nkm = c(n, NA)
a = min(x)
b = max(x)
q = c(n, NA)
qu_hp = c(n, NA)
3x = c[n, NA]
cr_n = c(n, NA)
cr_h = c(n, NA)
crr_n = c(n, NA)
crr_h = c(n, NA)
error_n = c(n, NA)
error_h = c(n, NA)

## Step-size for intervals in [a,b] from sample x.
b = i/length(x)
print(b)

mu_N = -0.0014016424
sigma_N = 0.0261516140

t = -0.1547164
mu = -0.0007277
delta = 0.0026237
pi = 0.0195275
alpha = sqrt((14 * pi^2) / delta)
betta = abs(pi * zeta / delta)
print(alpha)
print(betta)

m = a
i = 0
err_n = 0
err_h = 0
while (m > 0)
  m = m + h
  i = i + 1
  kol[i] <- length(x[(x >= m - h) & (x < m)])
  Sx[i] = kol[i] * h
  g[i] = integrate(function(x) 1 / (sigma^2 * sqrt(2 * pi * 1.414214)) * exp(-((x - mu)^2) / (2 * sigma^2)), lower = m - h, upper = m)

  # For the hyperbolic case
  g_hyp[i] = integrate(function(x) sqrt(alpha^2 - beta^2) / (2 * alpha * delta * besselI(zeta, 1, expon.scaled = FALSE) * exp(-alpha * sqrt(delta^2 + (x - mu)^2) + beta * (x - mu))), lower = m - h, upper = m)
  err_n[i] = (Sx[i] - g[i])^2
  err_h[i] = (Sx[i] - g_hyp[i])^2
  err_nh <- err_n + err_h[i]
  err_h[i] <- err_h + err_h[i]

# The final part will print out the error for the normal distribution (error n) and the error for the hyperbolic distribution (error h).
print(err_n)
print(err_h)
print(i)
error_n <- sqrt((1 / i) * err_n)
error_h <- sqrt((1 / i) * err_h)
print(error_n)
print(error_h)
Appendix

5.4 The calculation of Errors using using quantiles
library(numDeriv)
library(HyperbolicDist)
library(gtools)
library(ghyp)
library(gplots)
library(gdata)
library(MASS)

x=read.table("R:\Data ACC stock.txt", dec="."

y=x[,1]
b=c(length(y),NA)
for (i in 1:length(y))
b[i]=y[length(y)-i+1]
x<-diff(log(b))
plot(x)

n<-length(y)
er=c(n,NA)
esr=c(n,NA)
kol=c(n,NA)
a=min(x)
b=max(x)
g=c(n,NA)
g_hyp=c(n,NA)
f=c(n,NA)
f_hyp=c(n,NA)
sx=c(n,NA)
Q_x=c(n,0)
Q_n=c(n,0)
Q_hyp=c(n,0)
Q_x2=c(n,0)
Q_n2=c(n,0)
Q_hyp2=c(n,0)

h=1/(n-2)
muH=-0.001016624
sigmaH=0.001818114

zeta=0.1547554
mu=-0.0007277
delta=0.0026037
pi=0.0198275

alpha=zeta*sqrt(1+pi^2)/delta
beta=abs(pi*zeta/delta)
m=a
i=0
while(m<\n)
\n\n(m=m+h)
\ni=1+1
\n\nkol[i]<-length(x[(x >= m-h) & (x < m)])  # number of all x values between m-h and m
\nSx[i]<-kol[i]*h
\n\nf[i] = 1/(sigmaW*sqrt(2*3.141593))*exp(-((x-mu)^2)/(2*sigmaW^2))
\nX=sqrt(alpha^2-betta^2)/(2*alpha*delta*besselK[zeta, 1, expon.scaled = FALSE])
\nf_hyp[i] = X*exp(-alpha*sqrt(delta^2+(x-mu)^2) + betta*(x-mu))
\ng[i] = integrate(function (x) 1/(sigmaW*sqrt(2*3.145753))*exp(-((x-mu)^2)/2*sigmaW^2),
\nlo\ner = m-h, upper = m)#value
\ng_hyp[i] = integrate(function (x) sqrt(alpha^2-betta^2)/(2*alpha*delta*besselK[zeta, 1,
\nexpon.scaled = FALSE))*exp(-alpha*sqrt(delta^2+(x-mu)^2) + betta*(x-mu)), lower = m-h,
\nupper = m)#value
\n\n\n#Here the quantiles for real data, normal and hyperbolic densities will be calculated.
\nalpha=0
\ni=0
\nerr=0
\nerr2=0
\nwhile(alpha<=1)
\n(alpha=alpha+h)
\ni=1+1
\nS_x=0
\n\nj=0
\nwhile (S_x<=alpha/h)
\n\{\n\nj=j+1
\nS_x=S_x+kol[j]
\n\}
\nQ_x[i]=j*h+a
S_n=0
j=0
while(S_n <= alfa/h)
{ 
j=j+1
S_n = S_n + g[j]/h
}
Q_n[i]=j*h+a
S_hyp=0
j=0
while(S_hyp <=(alfa/h))
{j=j+1
S_hyp=S_hyp+g_hyp[j]/h
}
Q_hyp[i]=j*h+a
err<-(err+(Q_x[i]-Q_n[i])^2
err2<-(err2+(Q_x[i]-Q_hyp[i])^2
}

##The printing out of the error for the normal distribution (error n) and the
##error for the hyperbolic distribution(error h).
error_n<-sqrt(((1/i)*err)
print(error_n)
error_h<-sqrt(((1/i)*err2)
print(error_h)

##The final part will draw the Quantile - Quantile plot.

##For the normal density and real data.
i=0
for (i in 1:1323)
{Q_x2[i] = Q_x[i]
 Q_m2[i] = Q_n[i]
}
plot(Q_x2, Q_m2, xlab="Sample quantiles", ylab="Normal quantiles")

##For the hyperbolic density and real data.
i=0
for (i in 1:1323)
{Q_x2[i] = Q_x[i]
 Q_hyp2[i] =Q_hyp[i]
}
plot(Q_hyp2, Q_x2, xlab="Sample quantiles", ylab="Hyperbolic quantiles")
Appendix

5.5 The estimation of parameters for the SVJJ model
\(a = i u (1 - i u)\);
\(b = i \sigma \rho u - \kappa\);
\(\gamma = \sqrt{b^2 - a \sigma^2}\);
\(c = 1 - \rho 1/2 i u\);
\[\mu = \frac{\text{Exp}[\mu_1 + i(1/2) \sigma_1^2]}{1 - \rho_1 \mu_2}, -1;\]
\(\nu = 1;\)
\[\alpha[u_-, \kappa_-, \gamma_-, \tau_-, \sigma_-, \rho_-, \lambda_-, \rho_1_-, \mu_2_-, \mu_1_-, \sigma_1_-] = \]
\[\Delta u (r - \lambda \mu - \lambda) \xi - \frac{\lambda \gamma}{\sigma^2} \left( 2 \log\left[1 - \frac{\gamma + b}{2 \gamma} (1 - \text{Exp[-\gamma t])}\right] + \frac{\gamma + b}{2 \gamma} \right) - \]
\[\lambda \text{Exp}\left[\mu_1 (1 - \frac{1}{2} u^2 \sigma_1^2)\right] - \]
\[\left(\frac{2 \mu_2 a_+ \log\left[1 - \frac{(\gamma + b) c - \mu_2 a_+}{2 \gamma c} (1 - \text{Exp[-\gamma t])}\right] + \frac{(\gamma + b) \tau}{(\mu_2 c + \gamma c - b c)}\right) / (2 \gamma - (\gamma + b) (1 - \text{Exp[-\gamma t])}); (**)\]
\[\beta[u_-, \sigma_-, \rho_-, \kappa_-] = \frac{-\alpha (1 - \text{Exp[-\gamma t])}}{2 \gamma - (\gamma + b) (1 - \text{Exp[-\gamma t])}}; (**)\]

\(\phi[u_-, \kappa_-, \gamma_-, \tau_-, \sigma_-, \rho_-, \lambda_-, \rho_1_-, \mu_2_-, \mu_1_-, \sigma_1_-] = \)
\[\text{Exp}[\alpha[u, \kappa, \gamma, \tau, \sigma, \rho, \lambda, \rho_1, \mu_2, \mu_1, \sigma_1]] \left(1 - \frac{\beta[u, \sigma, \rho, \kappa]}{2 \kappa} \right) \frac{\gamma \kappa}{\sigma^2};\]

\(\text{(EMPIRICAL CHARACTERISTIC FUNCTION \*)}\)
\(\phi = \text{Import["testABB.dat", "List"]; (data importing\*)}\)
\(n = \text{Length}[\phi];\)
\(\phi_1[u_-] = \frac{1}{n} \sum_{j=1}^{n} \text{Exp}[iu \phi[j]];\)

\(\text{(MINIMIZATION \*)}\)
\(\sigma_0 = 0.85;\)
\[g[u_] = \frac{1}{\sqrt{2 \pi \sigma_0^2}} \text{Exp}\left[\frac{-u^2}{2 \sigma_0^2}\right]; (\text{weighting function \*)}\]
\[\text{fun}[\kappa_-, \gamma_-, \tau_-, \sigma_-, \rho_-, \lambda_-, \rho_1_-, \mu_2_-, \mu_1_-, \sigma_1_-] = \]
\[\text{NIntegrate}\left[\right.\]
\[\left(\text{Re}[\phi[u, \kappa, \gamma, \tau, \sigma, \rho, \lambda, \rho_1, \mu_2, \mu_1, \sigma_1]] - \text{Re}[\phi_1[u]]\right)^2 +\]
\[\left(\text{Im}[\phi[u, \kappa, \gamma, \tau, \sigma, \rho, \lambda, \rho_1, \mu_2, \mu_1, \sigma_1]] - \text{Im}[\phi_1[u]]\right)^2 g[u], [u, 0, 2.7] ;\]
\[\text{NMinimize}[\{\text{fun}[\kappa, \gamma, \tau, \sigma, \rho, \lambda, \rho_1, \mu_2, \mu_1, \sigma_1], \sigma > 0, \sigma_1 > 0, \rho > -1, \rho <= 1, \rho_1 > 0, \rho_1 <= 1, \kappa > 0, \gamma > 0, \mu_2 > 0, \lambda > 0\}, \{\kappa, \gamma, \tau, \sigma, \rho, \lambda, \rho_1, \mu_2, \mu_1, \sigma_1\}]\)
Appendix

5.6  The calculation of the Theta for the hyperbolic model

\[ \alpha = 59.43307; \]
\[ \delta = 0.0026037; \]
\[ \xi = \alpha \times \delta; \]
\[ r = 0.0465; \]

\textbf{FindRoot[}
\[ \{ r = \text{Log} \left[ \text{BesselK} \left[ 1, \sqrt{\xi^2 - \delta^2 \times (\theta + 1)^2} \right] \right] - \text{Log} \left[ \text{BesselK} \left[ 1, \sqrt{\xi^2 - \delta^2 \times \theta^2} \right] \right] - \frac{1}{2} \times \text{Log} \left[ \sqrt{\xi^2 - \delta^2 \times (\theta + 1)^2} \right] + \frac{1}{2} \times \text{Log} \left[ \sqrt{\xi^2 - \delta^2 \times \theta^2} \right] \}, \{ \theta, 0 \} \} \]
Appendix

5.7 The calculation of the European call option price for the hyperbolic and the Black-Scholes models

Calculation of call option price for the Black–Scholes model

```plaintext
r = 0.0465;
Sigma = 0.02615;
T = 0.4;
t = 0;
K = 180;
S0 = 182;
d1[r_, Sigma_, K_, S_, T_, t_] := (Log[S/K] + (r + (Sigma^2/2) * (T-t)) / (Sigma * (T-t)^1/2))
Solution = d1[r, Sigma, K, S0, T, t];
d2[r_, Sigma_, K_, S_, T_, t_] := (Log[S/K] + (r - (Sigma^2/2) * (T-t)) / (Sigma * (T-t)^1/2))
Solution = d2[r, Sigma, K, S0, T, t];
ValueCall[r_, Sigma_, K_, S_, T_, t_] :=
    S * Erf[-Infinity, d1[r, Sigma, K, S, T, t] / Sqrt[2]] / 2 -
    K * Exp[-r * (T-t)] * Erf[-Infinity, d2[r, Sigma, K, S, T, t] / Sqrt[2]] / 2;
VC = ValueCall[r, Sigma, K, S0, T, t]
```

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Calculation of call option price for the Hyperbolic model

\[ a = 59.43307; \]
\[ \beta = 0; \]
\[ \delta = 0.0026037; \]
\[ \mu = 0; \]
\[ r = 0.0465; \]
\[ S[1] = 182; \]
\[ Ka = 180; \]
\[ \theta = 40.98692242; \]
\[ y = \log[Ka/S[1]]; \]
\[ t = 0.4; \]

Calprice =

\[ S[1]* \]
\[ \text{NIntegrate}[\text{Exp}[(\theta + 1)*x]*\frac{1}{3.14}, \text{NIntegrate}[	ext{Cos}[u*x]*\left(\frac{a}{\text{BesselK}[1, \delta*x^2 + u^2]}\right)^{(t)}, \{u, 0, a\}]/\text{NIntegrate}\left[\frac{a}{\text{BesselK}[1, \delta*x^2 + u^2]}\right]^{(t)}*\frac{1}{3.14}\right] \]

\[ \text{Integrate}[\text{Cos}[u*y]*\text{Exp}[(\theta + 1)*y], \{y, -0.17, 0.23\}]*\{u, 0, a\}*\{x, y, 0.23]\right] - \text{Exp}[-r*t]*Ka* \]

\[ \text{NIntegrate}[(\theta + 1)*x]*\frac{1}{3.14}*\text{NIntegrate}[	ext{Cos}[u*x]*\left(\frac{a}{\text{BesselK}[1, \delta*x^2 + u^2]}\right)^{(t)}, \{u, 0, a\}]/\text{NIntegrate}\left[\frac{a}{\text{BesselK}[1, \delta*x^2 + u^2]}\right]^{(t)}*\frac{1}{3.14}\right] \]

\[ \text{Integrate}[\text{Cos}[u*y]*\text{Exp}[(\theta + 1)*y], \{y, -0.17, 0.23\}]*\{u, 0, a\}*\{x, y, 0.23\}\]