Analysis of An Uncertain Volatility Model in the framework of static hedging for different scenarios

Master's Thesis in Financial Mathematics

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Chapter 1

Introduction

Black and Scholes invented in 1973 a two parametric model for an option pricing, where $\sigma$ is a volatility and $\mu$ is a return of the stock. In their work these parameters are assumed to be constant. However the time series on the market have much more complicated structure and we have to deal with uncertain parameters.

In case of a plain vanilla option valuation the Black-Scholes (BS) model seems to be good enough, because the payoff function has not changing convexity. However when we deal with exotic options or some complex derivatives standard BS models can be not precise enough, because our payoffs are not always convex functions. There exists some approach, which describe more exact the behaviour of volatility. One of the classical methods is to model it stochastically. In such model we have dependence between stock price and volatility. In Oztukel and Willmot [1] authors assume that the volatility is given by a stochastic differential equation

$$d\sigma = \alpha(\sigma)dt + \beta(\sigma)dX,$$

(1.1)

where the drift $\alpha(\sigma)$ and the volatility $\beta(\sigma)$ are both functions of volatility of a stock price and they both are independent on time. This method can be applied to many financial time series. The main advantage of this method is that there exists an exact solution for a vanilla European option, moreover the stochastic process produces very good distribution function (close to real data) for market prices. An advantage of this model is that $\sigma$ and $S$ are correlated. On the other hand the model has also quite meaningful disadvantages. The model is incomplete, the parameters of the model are not stationary and we obtain skewness of implied volatility surface for the option with a short time to expiry. In general, the update of market option prices and the consequent book re-evaluation can be extremely burdensome and time consuming (see D.Brigio, F. Mercurio and F.Rapisarda [2]).
Chapter 1. Introduction

In 1995 Avellaneda, Levy and Paras [3] presented a model, where the volatility is an unknown value, but they assumed that it has to lie between two extreme values $\sigma_{\text{min}}$ and $\sigma_{\text{max}}$, known as a certainty band. This model is known as an Uncertain Volatility Model and was firstly presented in [3]. It is universal, it means that it can be applied both to exotic options and to plain vanilla options. However the main problem is that the certainty interval can be too wide and provide too large spread in calculated values of options. The most important problem in this framework is the problem of narrowing this bands to obtain more realistic results.

In our paper we prefer to concentrate more on behaviour of volatility as a function and more realistic models for the volatility, which eliminate a risk connected with behaviour of the volatility of an underlying asset. That is the reason why we will study the Uncertain Volatility Model. In Chapter 1 we will make some theoretical introduction to Uncertain Volatility Model introduced by [3] and study how it behaves in the different scenarios. In Chapter 2 we choose one of the scenarios. We also introduce BSB equation and try to make some modification to narrow the uncertainty bands using the ideas of a static hedging. In Chapter 3 we try to construct the proper portfolio for the static hedging and compare the theoretical results with real market data from the Stockholm Stock Exchange, that we collected with help of a financial program SixEdge for the periods of time between December and February 2007/08 and from March to May 2008.
Chapter 2

Models introduction

2.1 From Black-Scholes model to Black-Scholes-Barenblatt equation

In this chapter we will consider the model developed in 1995 by Avellaneda, Levy and Paras [3]. The authors avoid using term structure of volatility (it means deterministic function of time and the asset price). They also did not use a stochastic volatility model, because they chose uncertain volatility environment. The main assumption by the Uncertain Volatility Model is that volatility lies in a bounded set, but volatility is not known (undetermined). The authors of [3] assume that we study derivatives only, based on a single liquid traded stock, which pays no dividends over the duration of a contract. In our thesis we additionally assume that the interest rate r is a constant. As in framework of the Black-Scholes theory we claim that a future stock price is an Ito process

\[ dS_t = S_t \mu dt + S_t \sigma dX_t, \] (2.1)

where \( \mu \) and \( \sigma \) are non anticipative functions and \( \sigma_{\min} \leq \sigma \leq \sigma_{\max} \), \( X_t \) is a Brownian motion and \( \sigma_{\min}, \sigma_{\max} \) are constants.

According to our expectation and an uncertainty future price band we should represent upper and lower boundaries for the volatility. In [3] it is assumed that \( \sigma_{\min} \) and \( \sigma_{\max} \) are constants. Authors suggest to obtain \( \sigma_{\min} \) and \( \sigma_{\max} \) from extreme values of implied volatilities of liquid derivative instruments or from peaks in the historical volatilities. The \( \sigma \)-band can also be viewed as determining of a confidence interval for the future volatility values. However authors mention that we can modify this assumptions and suppose that \( \sigma_{\min} \) and \( \sigma_{\max} \) are functions of time and stock price, to make band as narrow as possible. In our work we would like to find the \( \sigma \)-bounds then to narrow them with help of the static hedging, using the worst-case
scenario. We verify if our portfolio is well constructed and if it is necessary how to improve it.

In this part we would like to remind the original Black-Scholes model framework. We start from [1]. Let \( V \) be a price for a derivative product, which depends on \( S \) and \( t \) \((0 \leq t \leq T)\), where \( t \) is time and \( T \) is the exercise date for a derivative product. We use Itos Lemma and obtain stochastic process

\[
dV = \left( \frac{\partial V}{\partial t} + \mu S \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt + \sigma S \frac{\partial V}{\partial S} dX. \tag{2.2}
\]

We form a portfolio of the type

\[
\Pi = V - \Delta S, \tag{2.3}
\]

with \( \Delta = \frac{\partial V}{\partial S} \), i.e., \( \Delta \)-hedging, to eliminate source of uncertainty provided by the underlying. \( \Pi \) is an instantaneously risk-free portfolio and as such it must have a return, which coincides with the risk-free return from a bank account

\[
d\Pi = dV - \Delta dS = r d\Pi dt, \tag{2.4}
\]

where \( r \) is a risk-free rate of return. We assume that \( r \) is a constant. Insert (2.1),(2.2) into (2.4) we obtain the classical BS equation, which is

\[
\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + r S \frac{\partial V}{\partial S} - rV = 0. \tag{2.5}
\]

We can solve it in closed form, when \( \sigma \) and \( r \) are constants for \( t \in [0, T] \).

To obtain a unique solution for (2.5), we should fix a terminal condition (payoff) and two boundary conditions for the function \( V(S,t) \). For instance, for a Vanilla Call we obtain that payoff is given by

\[
V(S,T) = \max(S - K, 0), \tag{2.6}
\]

and boundary conditions are

\[
V(S,T) \to 0, \text{ as } S \to 0,
\]

\[
V(S,T) \sim S - K \exp^{-r(T-t)}, \text{ as } S \to \infty.
\]

## 2.2 Uncertain parameters and certainty bands by Avellaneda, Paras and Levy

In this section we develop an uncertain parameter (\( \sigma \) and \( r \)) methodology, where parameters are uncertain, but they lie in preliminary fixed bands. Firstly
we use a portfolio of the type (2.3). The return on this risk-free investment is given by an expression

\[ d\Pi = \left( \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt. \] (2.7)

We define the certainty bands for the parameters \( \sigma \) and \( r \) in the following way

\[ \sigma_{\text{min}} \leq \sigma \leq \sigma_{\text{max}}, \] (2.8)

\[ r_{\text{min}} \leq r \leq r_{\text{max}}, \] (2.9)

where \( \sigma_{\text{min}}, \sigma_{\text{max}}, r_{\text{min}}, r_{\text{max}} \) are constants. Now we will combine parameters, within their envelopes, in order to obtain the lowest price of the portfolio. This price corresponds to a minimum change in portfolio \( \Pi \), the minimal increase and the maximal decrease, in \( d\Pi \)

\[ \min_{\sigma_{\text{min}} \leq \sigma \leq \sigma_{\text{max}}} (d\Pi) = \min_{\sigma_{\text{min}} \leq \sigma \leq \sigma_{\text{max}}} \left( \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right). \] (2.10)

We notice from above equation that the volatility \( \sigma \) is in fact the function of the second derivative of \( V \) by \( S \). If \( \frac{\partial^2 V}{\partial S^2} \leq 0 \) the minimum in \( d\Pi \) (2.5) requires \( \sigma = \sigma_{\text{max}} \), otherwise we need to take \( \sigma_{\text{min}} \) to obtain the minimum in the value of \( d\Pi \).

We can observe that \( r \) does not enter into the calculation of \( d\Pi \), so if we want to find the minimum value for \( r \), we must examine the value of the portfolio (2.3).

After such analysis we obtain equation

\[ \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + r(\Pi) S \frac{\partial V}{\partial S} - r(\Pi) V = 0, \] (2.11)

where \( \Gamma = \frac{\partial^2 V}{\partial S^2} \). We see that \( \sigma \) is a function of the second derivative of \( V \) by \( S \) and \( r \) is a function \( \Pi \) respectively.

This calculation was done for the worst case scenario by the valuating of the minimum of \( d\Pi \). We could make the same analysis for the best case scenario by valuating of the maximums \( d\Pi \) and \( \Pi \).

We choose such type of scenario, because during collecting market data, which we used in our thesis, no unexpected behaviour occured. That was the reason why a shock volatility scenario was not useful in our examples. Further explanations about scenarios are given in Chapter 3.

The equation (2.11) is called Black-Scholes-Barenblatt equation. This equation is a nonlinear, partial differential equation, which can be reduced to BS if \( \sigma \) and \( r \) are constants.
2.3 An Analytical approach

In this section we want to show how the boundaries for the option price are derived. We follow in this section the article [4]. We consider here an Up and Out European Call $V(S,t)$ with expiration $T$ and strike price $K$. The Up-and Out- Call means that we have Call option, which stops to exist when the underlying asset reaches the prespecified value (if the stock price will hit the barrier then the option is worthless). A barrier here is $S = X > K$. We know, that the price of the Up- and Out- Call is non-convex function and it is the reason of nonlinearity in the problem of an option evaluating. We assume that for $S \in (0, X)$ and $t \in (0, T]$ the Black Scholes equation

$$L(\sigma, r)V \equiv r(S,t)S \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2(S,t)S^2 \frac{\partial^2 V}{\partial S^2} + r(S,t)V - \frac{\partial V}{\partial t} = 0,$$  (2.12)

applies, where $\sigma$ and $r$ are deterministic but uncertain volatility and interest rate functions. Boundary and initial conditions for the Up- and Out- Call are

$$V(0,t) = V(X,t) = 0,$$  (2.13)

$$V(S,0) = \max(S - K, 0).$$  (2.14)

What follows

$$\lim_{S \to X} V(S,0) \neq \lim_{S \to X} V(X,t).$$  (2.15)

We see that the Up- and Out- Call will have a discontinuity in the point $(X, 0)$. We avoid that with the piecewise linear approximation for the initial condition

$$V(S,0) = \begin{cases} 
0, & \text{for } S < K, \\
S - K, & \text{for } K \leq S \leq X - \epsilon, \\
(X - \epsilon - K) \frac{X - S}{\epsilon}, & \text{for } X - \epsilon \leq S \leq X,
\end{cases}$$

where $\epsilon$ is a small parameter. It was observed in the market with the help of the SixEdge programme that the everyday fixed changing in the underlying asset price $S$ is more or equal 0.25 SEK. Such observation helps us to set boundaries in which epsilon (a small changing of $S$) will lie from economical point of view. That is the reason why the price of the option $V$ will be independent of such a small movement of its underlying asset price: $\epsilon \in [0; 0.25]$ SEK. Now we can say for sure that our results will be independent on epsilon. We assume also that problem defined by (2.12),(2.14) and (2.13) has a unique classical solution $V(S, t)$ and that such solution exists and is smooth for $t > 0$. 
If \( \sigma \) and \( r \) are unknown with certainty we suppose that we have upper and lower bounds
\[
0 < \sigma_{\text{min}}(S,t) \leq \sigma(S,t) \leq \sigma_{\text{max}}(S,t),
\]
\[
0 \leq r_{\text{min}}(S,t) \leq r(S,t) \leq r_{\text{max}}(S,t).
\]
(2.16)
(2.17)
We want to find functions \( V_{\text{min}}(S,t) \) and \( V_{\text{max}}(S,t) \) such that
\[
V_{\text{min}}(S,t) \leq V(S,t) \leq V_{\text{max}}(S,t),
\]
(2.18)
for all \( \sigma \) and \( r \) staying within the bounds, with equality holding for a specific choice of \( \sigma \) and \( r \). We assume that (2.12) has a solution for all such \( \sigma \) and \( r \), which fulfill (2.16) and (2.17).

The governing equations for \( V_{\text{min}} \) and \( V_{\text{max}} \) are derived in the literature by considering the best and the worst case scenario for the value of a portfolio as the volatility and interest rates are allowed to vary freely within their assigned ranges [3], [6]. However, the equations for \( V_{\text{min}} \) and \( V_{\text{max}} \) are already implied by the equation (2.12) and the maximum principle for parabolic equations. We recall that if a function \( V(S,t) \) satisfies
\[
L(\sigma, r)V \leq 0 \text{ on } D = (0, X) \times (0, T],
\]
where \( L(\sigma, r) \) is the operator defined by (2.12), and is continuous on \( D = [0, X] \times [0, T] \), then \( V \) cannot have a negative minimum in \( D \). If \( V \) has a minimum, which value is lower than 0 at all, then it must occur on the boundary of \( D \), i.e., at \( S = 0, S = X \) or at \( t = 0 \). Similarly, \( L(\sigma, r)V \leq 0 \) in \( D \) exclude a maximum with a value higher than zero in \( D \). Let us consider now the equation
\[
L(\sigma, r)V_{\text{min}} \equiv \frac{1}{2}S^2\left[(\sigma^2 - \sigma_{\text{min}}^2) \max \left(\frac{\partial^2 V_{\text{min}}}{\partial S^2}, 0\right) \right. \\
+ (\sigma^2 - \sigma_{\text{max}}^2) \min \left(\frac{\partial^2 V_{\text{min}}}{\partial S^2}, 0\right) \left. \right] \\
+ \left[(r - r_{\text{min}}) \max \left(S \frac{\partial V_{\text{min}}}{\partial S} - V_{\text{min}}, 0\right) \right. \\
+ (r - r_{\text{max}}) \min \left(S \frac{\partial V_{\text{min}}}{\partial S} - V_{\text{min}}, 0\right) \right]
\]
(2.19)
with two initial and boundary conditions imposed on the Call by the formula (2.12),(2.14) and (2.13). \( L(\sigma, r) \) is the Black-Scholes operator defined by equation (2.19). We see that equation (2.19) is of the form
\[
LV_{\text{min}} = F_0 \left(S, V_{\text{min}}, \frac{\partial V_{\text{min}}}{\partial S}, \frac{\partial^2 V_{\text{min}}}{\partial S^2}\right),
\]
(2.20)
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where $F_0$ stands for the right hand side of (2.12). By inspection

$$F_0 \left( S, V, \frac{\partial V}{\partial S}, \frac{\partial^2 V}{\partial S^2} \right) \geq 0. \quad (2.21)$$

We assume that $V_{min}$ is a solution of (2.19) on $(0, T] \times (0, X)$, this solution is assumed to be continuous on $[0, T] \times [0, X]$ and satisfies the initial and boundary conditions of the Up and Out Call. If we set

$$k(S, t) = V_{min}(S, t) - V(S, t), \quad (2.22)$$

then

$$L(\sigma, r)k(S, t) \geq 0 \text{ on } D,$$

and, by construction, $k(S, t) = 0$ on $S = 0$, $S = X$ and on $t = 0$. By the maximum principle the function $k(S, t)$ can not have a maximum with value higher than zero in $D$. Boundary data, which are lower or equal zero then assure that

$$k(S, t) \leq 0 \text{ in } D,$$

which implies that

$$V_{min}(S, t) \leq V(S, t),$$

so that $V_{min}$ is a lower bound to the option price for any functions $\sigma$ and $r$ between the imposed limits. If in equation (2.19) we interchange the maximum and minimum functions occurring in $F$ (i.e., $\max(\frac{\partial V}{\partial S}, 0) \rightarrow \min(\frac{\partial V}{\partial S}, 0)$ etc.) and label the new equation by

$$LV_{max} = F_1 \left( S, V_{max}, \frac{\partial V_{max}}{\partial S}, \frac{\partial^2 V_{max}}{\partial S^2} \right), \quad (2.23)$$

then

$$F_1 \left( S, V, \frac{\partial V}{\partial S}, \frac{\partial^2 V}{\partial S^2} \right) \leq 0. \quad (2.24)$$

A solution of (2.23) together with the boundary and initial conditions for the Up and Out Call satisfies

$$L(\sigma, r)(V_{max} - V) \leq 0,$$

and by the maximum principle

$$V(S, t) \leq V_{max}(S, t).$$
We observe that equation (2.19) can be rewritten in the form given in [3]. So we are left with

\[ L_0^{BSB}V_{min} = \frac{1}{2}S^2f_0(\sigma)\frac{\partial^2 V_{min}}{\partial S^2} + g_0(r) \left( S\frac{\partial V_{min}}{\partial S} - V_{min} \right) - \frac{\partial V_{min}}{\partial t} = 0, \]

(2.25)

where

\[ f_0(\sigma) = \begin{cases} \sigma_{min}(S, t), & \text{for } \frac{\partial^2 V_{min}}{\partial S^2} \geq 0, \\ \sigma_{max}(S, t), & \text{for } \frac{\partial^2 V_{min}}{\partial S^2} < 0, \end{cases} \]

and

\[ g_0(\sigma) = \begin{cases} r_{min}(S, t), & \text{for } (S\frac{\partial V_{min}}{\partial S} - V_{min}) \geq 0, \\ r_{max}(S, t), & \text{for } (S\frac{\partial V_{min}}{\partial S} - V_{min}) < 0. \end{cases} \]

We note that equation (2.25) implies that \( V_{min}(S, t) \geq 0 \) because if \( V_{min}(S, t) \) has a minimum lower than zero at some \((S^*, t^*) \in (0, X) \times (0, T]\), then

\[ \frac{\partial^2 V_{min}(S^*, t^*)}{\partial S^2} \geq 0, \quad \frac{\partial V_{min}(S^*, t^*)}{\partial S} = 0, \quad \frac{\partial V_{min}(S^*, t^*)}{\partial t} \leq 0. \]

These inequalities together with \( V_{min}(S^*, t^*) < 0 \) are inconsistent with equation (2.25), because if we transform equation (2.25) by adding to both sides of this equation a derivative of \( V \) by \( t \), we would have that left side of the equation (2.25) is larger than zero and right side of the equation (2.25) is lower than zero, what is impossible. Similarly, we obtain for the upper bound the equation

\[ L_1^{BSB}V_{max} = \frac{1}{2}S^2f_1(\sigma)\frac{\partial^2 V_{max}}{\partial S^2} + g_1(r) \left( S\frac{\partial V_{max}}{\partial S} - V_{max} \right) - \frac{\partial V_{max}}{\partial t} = 0, \]

(2.26)

where

\[ f_1(\sigma) = \begin{cases} \sigma_{min}(S, t), & \text{for } \frac{\partial^2 V_{max}}{\partial S^2} \leq 0, \\ \sigma_{max}(S, t), & \text{for } \frac{\partial^2 V_{max}}{\partial S^2} > 0, \end{cases} \]

and

\[ g_1(\sigma) = \begin{cases} r_{min}(S, t), & \text{for } (S\frac{\partial V_{max}}{\partial S} - V_{max}) \leq 0, \\ r_{max}(S, t), & \text{for } (S\frac{\partial V_{max}}{\partial S} - V_{max}) > 0. \end{cases} \]

In the literature the equations (2.25) and (2.26) are called Black Scholes Barenblatt equations (BSB) (see, e.g., [3]).
2.4 The first example of the valuating option with the BSB equation. An Electrolux Call

Let us consider just a Plain Vanilla Call for Electrolux company (we collected market data from December 2007 until February 2008). We choose the lowest and the highest observed volatility and in this way we obtained following boundaries for sigma

\[ \sigma_{min} = 0.3436, \sigma_{max} = 0.5802, \sigma = 0.41, \]

We took STIBOR (Stockholm InterBank Offered Rate) as \( r \)

\[ r_{min} = r_{max} = r = 0.0425. \]

Time to expiry is two months, \( K = 90 \) SEK and \( T = 0.1 \). We choose just the plain vanilla, because further we will try to hedge it.

The terminal condition

\[ C(S, 0) = \max(S - 90, 0), \quad (2.27) \]

and boundary conditions are

\[ C(S, T) \to 0, \text{ as } S \to 0, \]

\[ C(S, T) \sim S - 90e^{-0.0425(T-t)}, \text{ as } S \to \infty. \]

We obtain for the price of the Call option following results

\[ 4.898 \text{ SEK} \leq C \leq 8.541 \text{ SEK}. \quad (2.28) \]

We see that there is a too large spread between our boundaries. So the Black-Scholes-Barenblatt equation needs an improvement. As we know the Call option has a concave payoff function, that is why the BSB is reduced to two BS equations (for upper and lower boundaries for the price of the Call option). We will consider this example in chapter 3 and we will use a static hedging to improve the results.
2.5 The second example of the valuating option with the BSB equation. An Electrolux Double Barrier Straddle

Now let us check how the BSB equation is working for a barrier option. We choose a double barrier straddle. We have no access to Over The Counter market (OTC). Over The Counter Market is a market, where traders make a bargain by phone. Financial institutions, corporations and fund managers usually can stand as traders. That is the reason why the data are not readable from the internet or specified programs. We construct barrier by ourselves with the help of [4], where it was suggested to take the barrier less or equal to \(3 \times K\) (strike prices). We chose more realistic barrier \(S_1 = 80\) SEK, \(S_2 = 120\) SEK. The data for stock price, volatility and \(r\) was taken from the end of April 2008 and the beginning of May 2008. We considered twelve trading days for the Electrolux company and produce boundaries for the volatility of the stock price. We assumed that the option will expire in 20\(^{th}\) of June 2008 (third Friday of the month). And the interest rate is taken from the Riksbank website and is equal \(r_{\min} = r_{\max} = r = 0.0465\). We obtain following data for sigma \(\sigma_{\min} = 0.35\), \(\sigma_{\max} = 0.44\), \(\sigma = 0.45\). The strike price is equal \(K = 90\) SEK and \(T = 0.1\). The double barrier straddle is defined by following payoff

\[
V(S, T) = \max(0, S - 90) + \max(0, 90 - S),
\]

\[
V(80, t) = V(120, t) = 0.
\]

We do not take into account large price movements (shocks, jumps etc.), as we take short trading period and hence we prefer ATM (at the money) derivatives and near ATM ones and assume that stock prices will remain near the strike price of the underlying asset level. That is why we also take uncertain volatility model and suppose that \(\sigma\) remains in the intialial chosen boundaries. For such short trading period we use the constant interest rate (STIBOR). We choose implied volatility to predict the boundaries of \(\sigma\). This strategy works in the best way for option at the money and for option in the money.

2.6 A comparison the BSB results for derivative products of two companies

Now let us consider how the Black-Scholes-Barenblatt equation behaves in a strategy called Cylinder. We obtain this strategy by buying Put option.
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Figure 2.1: The price for the Double barrier straddle, envelopes $V_{\min}$ and $V_{\max}$ (dashed lines) and Black Scholes Solution (solid line).

with the strike price ($K_1$) in the money and shorting Call with the strike price ($K_2$) out of the money to finance the buying of Put. Because Put option was in the money and Call option was out of the money and we have the same underlying asset $K_1 < K_2$. So our portfolio is defined by

$$V(S,0) = \max(0, K_2 - S) - \max(0, S - K_1),$$  \hspace{1cm} (2.30)

$$\lim_{S \to \infty} \Pi(S, t) = \lim_{S \to \infty} -(S - K_2 e^{-rt}),$$  \hspace{1cm} (2.31)

$$\Pi(0, t) = K_1 e^{-rt}. $$  \hspace{1cm} (2.32)

We use data for Electrolux company from the period between the end of April 2008 and the beginning of May 2008. We choose maximum and minimum boundaries for the volatility

$$\sigma_{\min} = 0.35, \sigma_{\max} = 0.41, \sigma = 0.45,$$

As in previous examples we took the same interest rate (STIBOR) $r_{\min} = r_{\max} = r = 0.0465$ The strike price for Call is equal $K_1 = 90$ SEK and the strike price for Put is equal $K_2 = 110$ SEK), $T = 0.1$.

To compare the behaviour of the Cylinder prices for different stocks we choose
Figure 2.2: The Cylinder strategy for Electrolux derivatives, envelopes $V_{\min}$ and $V_{\max}$ (dashed lines) and Black Scholes Solution (solid line).

H&M with following data $\sigma_{\min} = 0.29$, $\sigma_{\max} = 0.32$, $\sigma = 0.305$, $K_1 = 350$ SEK (strike price for Call), $K_2 = 390$ SEK (strike price for Put), $T = 0.1$, $r_{\min} = r_{\max} = r = 0.0465$. The period, that we took data from, is the same as for Electrolux company.

We choose the strategy (2.30), because we wanted to check how the BSB equation behaves when the payoff has changing convexity. Here the Black-Scholes-Barenblatt can not be reduced to Black-Scholes equations. The minimum and maximum price of the $V(S,t)$ are narrower for H&M company than for Electrolux case, because the volatility lie in the more narrow range as in Electrolux case. It can be explained by the fact that H&M is a larger company, it takes 11% in OMX Stockholm 30 Index (30 biggest companies on the Stockholm Stock Exchange) and Electrolux takes 1.24% in OMX. This strategy is called Cylinder and it is recommended when we suppose that the price will decrease or increase, but not so far from the strike price for the Call option. The strike price for the Call needs to be at the money and the strike price for the Put needs to be in the money. We choose this compa-
2.7 The idea of the Risk diversification

In uncertain volatility models there is important to quantify the diversification of the volatility risk in portfolios of derivatives. The most obvious demonstration of the idea of the risk diversification is a consideration of two portfolios which contain two different derivatives.

Let us assume that there exist two portfolios of derivatives. The first and the second portfolio are characterized by a stream of cash flows at \( N \) and \( N' \) future dates respectively \( t_1 \leq t_2 \leq \ldots \leq t_N \) and \( t'_1 \leq t'_2 \leq \ldots \leq t'_N \)

\[
F_1(S_{t_1}), \ldots, F_N(S_{t_N}),
\]

\[
G_1(S_{t'_1}), \ldots, G_N(S_{t'_{N'}}),
\]
where \( F_j(S) \) and \( G_k(S) \) are known functions for the price of the underlying asset \( S \) of two derivative products.

Let us denote the sums of the discounted cash-flows of two derivative products by \( \Phi \) and \( \Psi \)

\[
\Phi = \sum_{j=1}^{N} \exp^{-r(t_j-t)F_j(S_{t_j})},
\]

\[
\Psi = \sum_{k=1}^{N'} \exp^{-r(t'_k-t)G_k(S_{t'_k})}.
\]  

(2.33)  

(2.34)

For the supremum and infimum of any functions we obtain following inequalities are fulfilled

\[
Sup_{PE}[\Phi + \Psi] \leq Sup_{PE}[\Phi] + Sup_{PE}[\Psi],
\]

\[
Inf_{PE}[\Phi + \Psi] \geq Inf_{PE}[\Phi] + Inf_{PE}[\Psi].
\]  

(2.35)  

(2.36)

Therefore, the optimal risk averse offer price of the portfolio \( \Phi + \Psi \) will be lower than the sum of the individual offer prices for \( \Phi \) and \( \Psi \). Analogously, the bid price for the portfolio \( \Phi + \Psi \) will be higher than the sum of the separated sell prices. Intuitively we can explain this situation in the following way a risk-averse agent, who wants delta hedging his position in the uncertain volatility environment would have to buy at the lowest volatility and to sell the items at the highest one.

In the situation where exotic options and portfolios are priced, that combine short and long positions, the situation is more complicated. Because we will have a sum of payoffs with different convexities. The BSB model can help us to price such portfolios, because it selects the volatility paths, which generate the most efficient non-arbitrageable bid/ask values. Pricing of the complete portfolio is more efficient than the adding of all prices of the individual components. We can do this pricing with the help of the BSB equation as this equation detected if the payoff function of the option is concave or convex, and uses the \( \sigma_{\min} \) and \( \sigma_{\max} \) respectively.
Chapter 3

Static hedging

3.1 Scenarios

In this chapter we will introduce a definition of a scenario, and classify possible types of scenarios. We choose one of scenarios that we prefer to use for a static hedging and the reasons why we choose this portfolio. We will mainly follow by the book of Robert Buff [6] and the paper [3].

Let us introduce a definition of the scenario [6].

**Definition 3.1 (scenario)** We call a set of (declarative) agent priorities and the (imperative) evaluation rules a **scenario**.

The definition is not strictly formal, because it needs to be formulated for a concrete scenario. In the book of Buff [6] author differs two types of scenarios

1. The worst-case volatility scenario,
2. The volatility-shock scenario.

We distinguish three types of worst-case volatility scenarios. Each of them illuminate the exposure of a portfolio to the volatility risk. All scenarios have a common feature of the portfolio

\[ C = (\sigma \sigma_{\text{min}} \leq \sigma(S_t, t) \leq \sigma_{\text{max}} \text{ and (2.1) has a solution} ) \]. (3.1)

We differ also a volatility-shock scenario. This type of scenario assume that the volatility is a constant for the most of the life of the portfolio, but it can happen some unexpected events (like mergers, announcements, devaluation, natural disasters, court rulings or others) that cannot be forecasted to happened on a specific day in the future.

Let us introduce a definition of the shock-volatility scenario [6].
Definition 3.2 (shock-volatility scenario) Let us take the volatility values in the interval \(0 < \sigma_{\min} \leq \sigma_0 \leq \sigma_{\max}\), where \(\sigma_0\) is called the prior volatility and express the subjective expectation of the agent about the true model volatility values. Here \(\sigma_{\min}\) and \(\sigma_{\max}\) are lower and upper bounds which the spot volatility can attain during periods of shock.

In [3] by Avellaneda, Paras it was presented a one-sided worst-case volatility scenario for the buy (sell) side within the specified volatility band. It is strongly connected with the worst-case price, which is the price minimizing the change of the values of the hedge portfolio (dII). If we could access the option values in the market with a price below our worst-case price then within the limits of our certainty band assumptions, we can surely obtain a profit. The main advantage of the worst-case volatility scenario is the fact that we can hedge with options: the reduction of the risk leads to super-(sub-) additive portfolio values. This type of scenarios is also studied in the literature and preferred in the most of articles about the uncertain volatility model, because upheavals occur rarely, but the volatility changes are present always in the real market (as we can observe in our data from the Swedish market).

3.2 The idea of the Static hedging

A static hedging is a strategy, which does not require readjustment of the hedge amount on the fixed time interval. In fact we use an option (or portfolio of financial derivatives) to hedge another option. It is the only way to hedge an option avoiding gamma risk and so to hedge against the volatility movements. We will show it on the simple example.

Suppose that we hold Call Option on a stock with a strike price \(K\), and a payoff \(P = \max(S - K, 0)\). We hedge this Call with a short position in other Call, but with a different strike price \(K_1\) on the same underlying asset, so a payoff \(P_1 = \max(S - K_1, 0)\). The Residual Liability (payoff) \(R = \max(S - K, 0) - \lambda \max(S - K_1, 0)\), where \(\lambda\) is the quantity of the second option (hedge amount). We have done the static hedging, because the Residual Liability for the hedged position is now much smaller than the payoff for the original contract, what makes the valuation less sensitive to the volatility movement and to the model of the underlying. The delta for a delta hedge is now a smaller amount which was mentioned before called as Residual Liability [1]. The fundamental feature of the static hedging is an optimization of the certainty band of the option value with respect to \(\lambda\). The price of the portfolio of two options is the sum of the prices of the standard option and the residual, where the price of the residual is a small part of the total sum.
In this case our spread of option values will be tighter. Heuristically we can present:

$$\text{Model Value} = \text{Min}[\text{Value of Option Hedge} + \text{Max PV(Residual Liability)}],$$

where the Residual Liability is valued under the worst-case scenario. The minimum is taken over all option portfolios. This procedure yields a position in the option for any given liability structure, which will eliminate the portfolio risk at the minimal cost (under the assumptions of the uncertain volatility model).

Let us consider the hypothetical case [3], where an agent wishes to offer the derivative security $\Phi$ to a client and hedge the risk using another derivative product $\Psi$, based on the same stock $S$, which we assume can be bought at price $G$ on the market. The agent needs to purchase $\lambda$ units of the derivative $\Psi$ to hedge the short position in $\Phi$ and provide the delta-hedging of the Residual Liability. So the additional coverage for the risk from the mismatch between $\Phi$ and $\Psi$ will be at most

$$\lambda G + \text{Sup}_P E[\Phi - \lambda \Psi].$$

The second term of (3.2) we calculate using the BSB equation.

To find an optimal number of contracts $\lambda$, we must solve the minimization problem

$$\inf_\lambda [\lambda G + \text{Sup}_PE[\Phi - \lambda \Psi]],$$

and

$$\sup_\lambda [\lambda G + \text{Inf}_PE[\Phi - \lambda \Psi]],$$

for an agent, who wants to improve the value of his portfolio and holds the derivative security with payoff $\Phi$ to hedge away some of the volatility risk of his position. In fact we need to solve BSB equation with different $\lambda$ to make the optimal hedge. This procedure can be also applied to the case with several derivatives. If amount of the derivatives that we use for hedging increases and the market is ideal (liquid, non arbitrage, frictionless and with zero drift), the range between upper and lower bounds will be progressively narrowed.

Typically by static hedging strategies one uses vanilla options as replicating instruments. It seems to be reasonable, since there is a liquid market for regular European options and the prices of these options are therefore determined by a bid and ask prices on the market. We suppose that there is no arbitrage on the market, thus the prices of the standard options are fair prices. If we derive the static hedge portfolio, we want to estimate the
price for an exotic option (a hedged option) with the help of the portfolio containing vanillas (hedging options). That is the reason why the estimated price for exotic option is also fair price. The advantage of using standard European options instead of the asset is in the more suitable payoff structure of the vanillas. The most important exotic options (barrier, asian and look back options) have a payoff similar (it means that they usually have either concave or convex payoff) to the regular European option payoff [5].

On the contrary to static hedging dynamic hedging usually uses the underlying and riskless asset (the class of instruments for the dynamic hedging is smaller than for static one). This type of hedging involves continuously readjustments of the hedge amount. Using of the dynamic hedging requires large transaction costs, what makes this strategy expensive.

Now we compare the dynamic hedging approach to the static one in more detailed way[5].

1. The main difference between the static and the dynamic hedging are the instruments, which are used for the hedging procedure. The dynamic hedging usually involves only the risk-free asset and the underlying. In contrast to that static strategies, which are mainly based on vanilla options. We notice that the underlying can be represented by a regular European Call with the strike price 0. On the other hand for large strike prices vanilla Puts behave nearly as the risk-free asset. Consequently we can use more instruments to the static hedging than to the dynamic one.

2. The other difference is a model dependence. In dynamic hedging we use the delta of the option to hedge it. Delta is the rate of the change in the price of a derivative with a change in the price of the underlying. Because of it the delta must be calculated in a model for the asset price. For instance in the BS model asset price is accepted to follow the Itoš process as we described in Chapter 2. The static hedging strategies based on plain vanilla options are completely model-independent, i.e. they work in every model. On this way for these strategies we do not have to assume a market model. The only thing we need is the absence of arbitrage on the market. However for the most of exotic options static hedging strategies are model dependent. Although the dependence is usually not that large as for dynamic hedging due to the use of vanilla options as hedging instruments.

3. The another difference between the static and the dynamic hedging is the behavior of strategies in markets with frictions. The performance of the dynamic hedging improves by adding more rebalancing
times, because it is designed for continuous trading. Since we add more transaction times the cost for the strategy can become unbounded due to the existence of transaction costs. In consequence we should limit the number of correction times (readjustments). Even if this costs are included in the model (for example in the model developed by Leland), this limitation of the transaction times leads to decrease the quality of the hedge. If there occur jumps in the market, the price approximated by dynamic hedging fail to hedge the risk. It happens when there are large movements in the market and then hedging is really needed. On the other hand the quality of static strategies does not depend on trading times.

That is the reason why the static hedging in general more appropriate for the use in the presence of essential transaction costs. However, the knowledge about the dynamic hedging is more developed that for the static one.

### 3.3 The first example of the static hedging. Hedging of an Electrolux Call with an Electrolux Put

We consider just a simple European Call on an Electrolux stock. We collected data from December 2007 until February 2008. We choose the lowest and the highest observed implied volatility and in this way we obtained following boundaries for sigma

\[ \sigma_{\text{min}} = 0.3436, \sigma_{\text{max}} = 0.5802. \]

The observed strike price is equal \( K = 90 \) SEK and \( T = 0.1 \). We took STIBOR (Stockholm InterBank Offered Rate) as interest rate in the model \( r_{\text{min}} = r_{\text{max}} = 0.0465 \). We took the same Call as in the first example in previous chapter (2.4)(we repeat here Meyer [1] calculations, but for real market data).

Now we use the idea of the static hedging to narrow our bounds, as a static hedge instrument we use European Put Option with the same datas as in Call case. We know that a Put price \( P(90,0.1)=5 \) SEK. This price corresponds to the implied volatility \( \sigma_{\text{impl}} = 0.41 \). So we construct a portfolio of the type

\[ \Pi(\lambda, S, t) = C(S, t) - \lambda P(S, t), \quad (3.5) \]

with the terminal condition

\[ \Pi(\lambda, S, 0) = \max(S - 90, 0) - \lambda \max(90 - S, 0), \quad (3.6) \]
and two boundary conditions

\[
\lim_{S \to \infty} \Pi(d, S, t) = \lim_{S \to \infty} (S - 90e^{-rt}),
\]

\[
\Pi(\lambda, 0, t) = -\lambda 90e^{-rt}.
\]

Figure 3.1: The Static hedging strategy (3.5). Upper (dashed line) and lower (solid line) bounds for pricing the strategy in example (3.5)-(3.9).

We will represent our calculations step by step. We try to find upper and lower bounds for price of Call Option. The equation below shows how to find these prices

\[
C(S(T), T) = \Pi(\lambda, S(T), T) + \lambda P(S(T), T),
\]

\[
C_{\text{min}}(\lambda, S(T), T) = \Pi_{\text{min}}(\lambda, S(T), T) + \lambda P(S(T), T),
\]

\[
C_{\text{max}}(\lambda, S(T), T) = \Pi_{\text{max}}(\lambda, S(T), T) + \lambda P(S(T), T).
\]

The maximum and minimum prices ($C_{\text{min}}$ and $C_{\text{max}}$) are dependent on \( \lambda \) in above equation, so we need to optimize them with respect to \( \lambda \)

\[
\max_{\lambda} C_{\text{min}}(\lambda, S(T), T) = \hat{C}_{\text{min}}(S(T), T),
\]
\[
\min_{\lambda} C_{max}(\lambda, S(T), T) = \hat{C}_{max}(S(T), T).
\] (3.13)

For \( \lambda \geq 1 \) the terminal condition \( \Pi(\lambda, S, 0) \) is concave and \( \Pi_{max}(\lambda, S, t) \) is the Black Scholes solution for \( \sigma_{max} = 0.5802 \). Similarly, for \( \lambda < 1 \) \( \Pi_{min}(\lambda, S, t) \) is the Black Scholes solution for \( \sigma_{min} = 0.3436 \). We use in this example Call-Put Parity
\[
C(S, t) - P(S, t) = S - Ke^{-rt},
\] (3.14)

and our formulas will look like
\[
C_{min}(\lambda, S(T), T)+S(T)-K e^{-rT} = \begin{cases} 
P_{\sigma_{min}} + \lambda (P_{\sigma_{impl}} - P_{\sigma_{min}}) & \text{if } \lambda < 1, \\
P_{\sigma_{max}} + \lambda (P_{\sigma_{impl}} - P_{\sigma_{max}}) & \text{if } \lambda \geq 1,
\end{cases}
\]

\[
C_{max}(\lambda, S(T), T)+S(T)-K e^{-rT} = \begin{cases} 
P_{\sigma_{min}} + \lambda (P_{\sigma_{impl}} - P_{\sigma_{min}}) & \text{if } \lambda \geq 1, \\
P_{\sigma_{max}} + \lambda (P_{\sigma_{impl}} - P_{\sigma_{max}}) & \text{if } \lambda < 1.
\end{cases}
\]

We see that the static hedging produces an improvement to our portfolio and now our boundaries coincide for \( \lambda = 1 \) and in this point we obtain the value for \( C(S(T), T) + P_{\sigma_{impl}} - P_{\sigma_{max}} = 5.393 \). We can easily calculate the price for Call and it is equal to 7.97 SEK. We see that the numerical method finds the optimal hedge for the upper and lower bounds, a zero spread and the value of the Call predicted by Put-Call parity. It is worth to mention that the numerical method does not use about Put-Call parity or convexity of the solution, but it solves the fully nonlinear problems in a proper way.

This value is close to real values for the Call in mentioned period of time. In market data from this period we observed value of Call 8 SEK for the strike price 90 SEK. It means the difference between theoretical result and market price for the Call is under 6 %. Thus we got a value for the option close to the real market price! This simple strategy gave us good results.

### 3.4 An example for the static hedging. A hedging barrier option with Bond and Vanilla Calls

We create a portfolio in following way: we have Up-and-Out Call and a zero-coupon Bond. We buy a portfolio of two European Calls to hedge the barrier option. The concept of this strategy can be explained in such a way

1. Buy a bond and a portfolio of European Calls.
2. If the barrier is hit, immediately sell the portfolio and remain the Bond.
3. If barrier is not hit, hold portfolio of Calls until expiration of the barrier option and hold the Bond.

We collected following data for Electrolux company from the end of April and the beginning of May 2008 (11 trading days). All options expire in the third Friday of June. In this case the volatility of the stock lies in boundaries

$$\sigma_{\text{min}} = 0.35, \sigma_{\text{max}} = 0.45.$$ 

The strike prices for Calls are $K = 95$ SEK, $K_1 = 90$ SEK, $K_2 = 100$ SEK, $T=1/8$, $r_{\text{min}} = r_{\text{max}} = 0.0465$, $X = 285$ SEK. For the static hedging we use two Call options with strike prices $K_1 = 90$ SEK, $K_2 = 110$ SEK. For the implied volatility $\sigma_{\text{impl}} = 0.41$ the Call option price is equal to $C_1(90, 1/8) = 8$ SEK and $C_2(100, 1/8) = 3$ SEK. The portfolio has a form

$$\Pi((\lambda_1, \lambda_2), S, t) = C(S, t) - \lambda_1 C_1(S, t) - \lambda_2 C_2(S, t) + B, \quad (3.15)$$

with the terminal condition

$$\Pi((\lambda_1, \lambda_2), S, 0) = \max(S - 95, 0) - \lambda_1 \max(S - 90, 0) - \lambda_2 \max(S - 100, 0) + 95, \quad (3.16)$$

and two boundary conditions

$$\Pi((\lambda_1, \lambda_2), 0, t) = 95e^{-rt}, \quad (3.17)$$

$$\Pi((\lambda_1, \lambda_2), X, t) = \max(X - 95e^{-rt}, 0) - \lambda_1 \max(X - 90e^{-rt}, 0) - \lambda_2 \max(X - 100)e^{-rt}, 0) + 95e^{-rt}. \quad (3.18)$$

We obtain following results

$$V_{\text{min}} = 99.8922 \text{ SEK for } \lambda_1 = 0.7 \text{ and } \lambda_2 = 0.6,$$

$$V_{\text{max}} = 100.2331 \text{ SEK for } \lambda_1 = 0.6 \text{ and } \lambda_2 = 0.7.$$

We see that the spread $V_{\text{max}} - V_{\text{min}}$ is equal to 0.3409. It means that our estimation is not perfect and can be improved. To make comparison with another period of the year we took the same strategy for the same company (Electrolux), but data was collected from 15th of December to 15th of January and the expiration date of the option was the third Friday of February (15th). In our data we have now bigger difference between $\sigma_{\text{min}}$ and $\sigma_{\text{max}}$. The volatility $\sigma_{\text{min}}$ is now equal to 0.3436 and $\sigma_{\text{max}}$ is now equal 0.5802. In consequence, we obtained results
Figure 3.2: The second example of the static hedging. Upper and lower bounds for pricing the strategy from example defined by (3.15)-(3.18).

\[ V_{\text{min}} = 99.7 \text{ SEK for } \lambda_1 = 0.6 \text{ and } \lambda_2 = 0.4, \]

\[ V_{\text{max}} = 100.35 \text{ SEK for } \lambda_1 = 0.6 \text{ and } \lambda_2 = 0.7. \]

The prediction give us the larger spread for the option prices, however it is a result of larger boundaries for the volatility. The larger spread occurs, because we took longer period and have more observations of the volatility. Also the market movements in December seem to be bigger, so in consequence the price spread is also larger and in this case is equal 0.8496.

The model gives us spread, which we can accept from financial point of view. We notice that for a smaller number of trading days our forecast is more exact, but there is also a high risk that real values will not lie in predicted envelopes for the option value.
3.5 New approach for choosing the volatility boundaries

To narrow the envelopes we would like to introduce new approach. We can reduce the exposure of the portfolio to inconsistence of the predicted price. Let us introduce our procedure step by step for the last example.

1. We have some boundaries derived up from extreme values of the implied volatility.

2. We want to find another method of deriving this boundaries as we are not sure if extreme values are the best values for the market, where the upheavals occur rarely.

3. We try to obtain new procedures for choosing the new envelopes for $\sigma$. We suppose that we can predict our certainty band in following way. We have data for implied volatilities for each day of 22 trading days in April- May period, for every strike price of options we are interested in (Electrolux, H & M and other companies from OMX30).

4. We suggest to make an ordered sequence from the lowest value of the implied volatility to the highest one.

5. We divide this ordered sequence for two parts (in our case each part contains 6 values of volatility).

<table>
<thead>
<tr>
<th>ImplVolat</th>
<th>OrdImplVolat</th>
</tr>
</thead>
<tbody>
<tr>
<td>40.46667</td>
<td>MAX 40.46667</td>
</tr>
<tr>
<td>39.53333</td>
<td>MIN 31.9</td>
</tr>
<tr>
<td>39.76667</td>
<td>38.53333</td>
</tr>
<tr>
<td>37.25</td>
<td>37.25</td>
</tr>
<tr>
<td>34.53333</td>
<td>36.91667</td>
</tr>
<tr>
<td>35.2</td>
<td>35.9</td>
</tr>
<tr>
<td>36.91667</td>
<td>35.75</td>
</tr>
<tr>
<td>36.9</td>
<td>36.2</td>
</tr>
<tr>
<td>36.75</td>
<td>34.53333</td>
</tr>
<tr>
<td>34.5</td>
<td>34.5</td>
</tr>
<tr>
<td>34.13333</td>
<td>34.13333</td>
</tr>
<tr>
<td>31.9</td>
<td>31.9</td>
</tr>
</tbody>
</table>

6. On the next step we make an arithmetic average of each part of the sequence.

7. This averages determine our new boundaries for sigma.
Our new boundaries for volatility are narrower. In consequence the values of option $V_{\text{min}}$ and $V_{\text{max}}$ are also narrower.

We consider strategies for options with a short time to expire and with the stock prices close to strike prices (options are ATM). We suppose that jumps in prices are rare, but the small movements are possible. As a result we can use our averaging procedure and make the envelopes tighter.

Let us check the accuracy of this method and compare with the accuracy of the previous one. We use Value at Risk measure (VaR) defined as

$$VaR = V\sigma\sqrt{\frac{1}{T11}} \times 2.33,$$

where $V$ is a market value of an option, $\sigma$ is a historical volatility of an option for last observation date, $11$ is a number of trading days and $2.33$ is a number of $\sigma$ needed for accuracy level of 99%.

Here we compare our results. The comparison is carried for options of Electrolux company.

1. VaR of the original method, which uses the highest and lowest value of the volatilities is equal: for the lowest volatility 1.43 SEK and for the highest volatility 1.84 SEK. There are the loses we can get if we would like to buy an option with a market price of 8 SEK (for this example we use values of Electrolux stock for period between December 2007 and January 2008).

2. VaR of the improved method is 1.511 SEK for lowest volatility and correspondingly 1.756 SEK for the highest one.

As we see our method gives us improvement of 40% with confidence level 99%.

We check our new predicted values in the real market. We find that our prediction is very exact, but only for 4-5 future days. Our suggestion is to update the uncertainty band every 4 days. It is similar procedure as in the semi-static hedging, where we need to make a finite number of readjustments of our portfolio.
Chapter 3. Static hedging
Chapter 4

Conclusions

In our work we considered the behaviour of the option prices for different scenarios under the uncertain volatility assumption. We used the Black-Scholes-Barenblatt equation to calculate the price. It is a nonlinear partial differential equation. We cannot obtain the price in analytical way, so we used a finite-difference schemes to calculate the prices.

In Chapter 1 we introduced different ways of dealing with the volatility. We choose the uncertain volatility model by Avellaneda, Levy and Paras [3].

In Chapter 2 we introduced and derived the Black-Scholes-Barenblatt equation (2.11), we calculated price envelopes for the strategies for Electrolux and H & M companies in sections (2.5) and (2.6) for datas collected from the Swedish Market in time period between April and May 2008. We choose firstly just a Vanilla Call (section (2.4)), then a Double Barrier Straddle (section (2.5)) and the Cylinder strategy (section (2.6)).

In Chapter 3 we introduced the concept of the static hedging under the worst-case scenario. On this way we wanted to narrow envelopes and to hedge our strategy against the volatility movements. We calculated the option prices band for the Electrolux company (section (3.4)) for datas collected from the Swedish Market in time period between April and May 2008 (calculation for 11 days) and for the longer period between December 2007 and January 2008 (calculation for 22 days). Then we checked if our predicted bands are suitable for the future values of prices in the market. We improved results using our idea presented in the Chapter 3. Our method can be applied for different types of strategies and works better for market without large frictions. It means that we obtained narrower band for the option price.
Bibliography


Glossary

Worst – case scenario - a scenario in which the parameters, within the boundaries of their bands, combined in such a way as to permanently produce the lowest price. Here the lowest price corresponds to a minimum change in $d\Pi$.

Over The Counter Market - a market where traders make a bargain by phone. Financial institutions, corporations and fund managers usually can stand as traders.

Put – Call Parity - the relationship between the price of a European Call option and the price of a European Put option with the same strike price and maturity date.

Up – and Out – Option - an option that stops to exist when the price of the underlying asset reaches the prespecified value and increases it.

Volatility Term Structure - the variation of the value of implied volatility with time to maturity.

At The Money - an option with the strike price equal to the price of the underlying asset.

Exotic option - a nonstandard European option, i.e., options with payoff, which differs from plain Vanilla Call (or Put).

Double Barrier Option - an option which payoff depends on whether the value of the underlying asset hits one of two barriers (up barrier and low barrier).
Long position - a position where the option is purchased.

Short position - a position where the option is sold.

Cylinder Strategy - a strategy obtained by buying Put option with the strike price $K_1$ in the money and shorting Call with the strike price $K_2$ out of the money to finance the buying of Put, where $K_2 > K_1$.

Double Barrier Straddle - a long position in a Double Barrier Call option and Double Barrier Put option with the same strike price.

Static Hedging - a hedging which does not change once it is initiated.

Dynamic Hedging - a procedure of hedging an option position with constant readjustments in the position held in the underlying assets.