Pricing of CDO Tranches by Means of Implied Expected Tranched Loss

Master's Thesis in Financial Mathematics

Anna Iakovleva
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Anna Iakovleva

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Supervisor: Ph.D. Eric Järpe
Examiner: Prof. Ljudmila A. Bordag
External referees: Prof. Vladimir Roubtsov, Prof. Ljudmila Vostrikova

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Department of Mathematics, Physics and Electrical Engineering
School of Information Science, Computer and Electrical Engineering
Halmstad University
Preface

First I would like to thank my supervisor Eric Jarpe. I appreciate greatly your support, your priceless help, well-timed and sensible advices. I enjoyed doing my thesis under your guidance.

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I am grateful to all the guest lecturers who visited Halmstad during this year with extremely interesting and useful lectures.
Abstract
In this thesis an approach to CDO tranche valuation is described. This approach allows to check market quotes for arbitrage opportunities, to obtain expected portfolio losses from the market quotes and to price CDO tranches with non-standard maturities and attachment/detachment points. A significant advantage of this approach is the possibility to avoid the necessity of construction of a correlation structure between names in the reference basket. Standard approaches to CDO valuation, based on copula functions are also considered.
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Chapter 1

Introduction

In finance, a credit derivative is a derivative with payoff, depending on the credit quality of the obligation of a third party—a so-called reference entity. A credit derivative in its simplest form is a contract between a buyer and a seller, where the seller sells a protection against some predetermined credit events, related to the reference entity. Nowadays, the most attractive credit derivatives are those, which provide protection against credit events, related to a portfolio of companies. The most popular portfolio credit derivatives are collateralized debt obligations (CDO’s) (see Hull and White [8]). The first CDO’s were issued in 1987 by bankers in now-defunct Drexel Burnham Lambert Inc. for Imperial Savings Association. 10 years later CDO’s became extremely popular. Since then this sector of securities market has been and is still growing with an enormous speed. According to the Securities Industry and Financial Markets Association, aggregate global CDO issuance totaled $157 billion in 2004, $272 billion in 2005, $552 billion in 2006 and $486 billion in 2007 [15]. The reason for it is most likely that credit derivatives allow transferring credit risks without selling the risky assets. This is in the same time good and dangerous. On the one hand, it makes possible for banks to give subprime loans. On the other hand, it extends the consequences of the default and makes the possible losses higher than the value of the defaulted loan.

CDO’s are perhaps the most complicated securities ever created. That is why it is a question of great importance to find high-quality models for valuating these derivatives. What we can see nowadays on the market is a result of a wrong estimation of risks and default dependence by commonly used models and, as a consequence, mispricing of the CDOs. For these models to be useful it must be possible to derive the correct distribution of portfolio losses. This distribution is influenced by default probabilities of separate
components of the portfolio and, what is perhaps even more important, by the dependence between them (default correlation).

In year 2000 David X. Li, an American quantitative analyst from Riskmetrics Group, presented a new approach to valuation of CDO’s. He suggested using Gaussian copulas, which allowed for fast pricing of this type of credit derivatives, see Li [10]. First, distributions of time-to-default for individual components of portfolio are obtained from market information and then the joint distribution is defined using a bivariate normal copula function. Later models based on $t$-copula, Clayton copula, Marshall Olkin copula, etc. were suggested. Some of them are more successful in fitting to market data. But all the copula models do not describe how the default environment changes with time. As an alternative to these static models, different dynamic models were developed, for example a dynamic Markov chain model (see [9]) or the dynamic model introduced by Hull and White (see [8]) and other. Another disadvantage of the copula approach is the difficulty to model the correlation structure between individual names in portfolio. This is a very non-trivial problem and its solution is usually far from reality. In this thesis an alternative approach to a risk-neutral valuation of CDO’s is considered. It suggests modelling implied portfolio loss distribution from market quotes and using this to price non-standard tranches, forward-starting CDO’s, etc. Another alternative approach was first presented by Walker (see [18, 19]). The big advantage with this model is that it can be calibrated to any set of arbitrage-free market prices avoiding the need to model correlation structure for the individual names in the basket.

In this chapter the background of my work is given. In Chapter 2 one can find a brief description of CDO valuation methods – commonly used, such as copula functions and alternative. Examples with real data are presented in Chapter 3 and discussed in Chapter 4. The data used is the market quotes for standard tranches of two most liquid CDS index families: CDX and iTraxx. The CDX indices contain 125 North American and Emerging market companies. The iTraxx indices are indices for a portfolio of 125 equally-weighted companies from the rest of the world.

The MatLab code implementing the described calculations can be found in Appendix.
Chapter 2

Methods

In this chapter a brief description of the structure of CDO’s is given. Also different approaches to CDO valuation – the copula approach and a rather new approach to risk-neutral CDO-valuation are presented. For a detailed description of the latter, see Walker [18], [19].

2.1 Collateralized Debt Obligations (CDOs)

CDO’s are very popular portfolio credit derivatives. They allow control and management of credit risks of a basket of credit-sensitive financial instruments. The whole underlying portfolio is divided into several adjacent segments – tranches. Each tranche corresponds to some specified interval of losses, the lower border of this interval is called attachment point and the upper border – detachment point. Let us consider one of the most liquid portfolio credit instruments – the iTraxx index – based on a portfolio of 125 companies. The portfolio is divided into 6 tranches: 0-3% loss, 3-6%, 6-9%, 9-12%, 12-22% and 22-100%. The first, so-called equity tranche is responsible for losses from 0 to 3% of the portfolio notional, independent of the particular names, which defaulted. The second, so-called junior tranche – for losses from 3 to 6% of the notional and so on.

In the most simple way the structure of a CDO contract can be described as following. It is a contract between two parties, where one party – protection buyer – makes regular payments, equal to predetermined percentage of the tranche notional, to the other. The payments are made until the maturity of the CDO contract or until the losses are out of the relevant range. In return, the second party – protection seller – compensates the losses in
the corresponding range to the protection buyer in case of some credit events like default. One must keep in mind, that the reference portfolio might not be in possession of one of the parties. In contrast with credit default swap, CDO securitizes a portfolio of defaultable assets. So, its payout depends on the default behavior of the whole underlying portfolio. The pricing of CDOs would not be more difficult, than CDS pricing if the defaults of single names in the reference portfolio were independent. Unfortunately, there are lots of effects which do not allow to assume the independence of the reference entities. It makes the form of dependence between the names in the reference portfolio and, consequently, their default times crucial. So, the question is how to introduce these dependences.

2.2 CDO valuation

2.2.1 Copula approach

Let us consider an underlying portfolio, consisting of \( n \) names and denote by \( \tau_i, i = 1, ..., n \) default times for these names. Obviously, \( \tau_i \) are random variables, on a common probability space \((\Omega, \mathcal{F}, P)\) with set of outcomes \( \Omega = [0, +\infty) \), sigma algebra \( \mathcal{F} \) and risk-neutral probability measure \( P \). Let us introduce the curve of cumulative default probabilities for name \( i \) as a function \( F_i(t) \) such that:

\[
F_i(t) = P[\tau_i \leq t], t \geq 0.
\] (2.1)

If we assume, that the function \( F_i(t) \) is strictly increasing with \( t \), then the existence of the inverse distribution function \( F_i^{-1}(x) \) for all \( x \in [0, 1] \) is guaranteed. And for any standard uniformly distributed \( U \), the variable \( F_i^{-1}(U) \) has the same distribution as \( \tau_i \). Using this, we are able to simulate the random default times. But for modelling and pricing portfolio credit derivatives, we also need the joint distribution of random default times.

There are many ways to specify a joint distribution of random variables knowing marginal distributions and the correlation structure. One relatively simple and useful approach is to apply copula functions, which relate univariate marginal distributions and their full multivariate distribution.

When univariate marginal distribution functions \( F_1(x_1), F_2(x_2), \ldots, F_m(x_m) \) are given, a corresponding multivariate distribution function is defined using
the copula function $C$ by:

$$C(F_1(x_1), \ldots, F_m(x_m), \rho) = F(x_1, x_2, \ldots, x_m), \quad (2.2)$$

where $\rho$ is a correlation parameter. The marginal distribution of any $X_i$ is then $C(F_1(\infty), \ldots, F_i(x_i), \ldots, F_m(\infty), \rho)$.

In 1959 perhaps the most important result in copula theory was obtained by Sklar.

**Theorem 1** (Sklar, 1959 [13]). Let $F$ be a multivariate $m$-dimensional distribution function with marginals $F_1, \ldots, F_m$. Then there exists a copula $C$ such that

$$F(x_1, x_2, \ldots, x_m) = C(F_1(x_1), \ldots, F_m(x_m)) \quad (2.3)$$

Moreover, if the marginal distributions $F_1, \ldots, F_m$ are continuous, then $C$ is unique.

The reverse statement is also true:

**Proposition 1** (Bluhm et al [2]). For any copula $C$ and (marginal) distribution functions $F_1, \ldots, F_m$, the function

$$F(x_1, \ldots, x_m) = C(F_1(x_1), \ldots, F_m(x_m)) \quad (2.4)$$

defines a multivariate distribution function with marginals $F_1, \ldots, F_m$.

So, it turns out, that one can find a unique copula representation for every multivariate distribution with continuous marginals. Moreover, it is possible to derive new multivariate distributions with specific marginals and correlation structures.

If we consider, just for simplicity, a bivariate copula function $C(x, y, \rho)$ for uniformly distributed random variables $X$ and $Y$, defined over the area $\{(x, y) | 0 < x \leq 1, 0 < y \leq 1\}$, where $\rho$ is again a correlation parameter, the following properties can be formulated [10]:

1. since $X$ and $Y$ are positive random variables

$$C(0, y, \rho) = C(x, 0, \rho) = 0 \quad (2.5)$$
2. since $X$ and $Y$ are bounded from above by 1, the marginal distributions can be obtained by:

\[
C(1, y, \rho) = y, \quad (2.6)
\]
\[
C(x, 1, \rho) = x, \quad (2.7)
\]

3. for independent random variables $X$ and $Y$:

\[
C(x, y, \rho) = xy. \quad (2.8)
\]

In 1951 Frechet [7] showed existence of upper and lower bounds of copula function. In case of bivariate copulas:

\[
\max(0, x + y - 1) \leq C(x, y) \leq \min(x, y) \quad (2.9)
\]

Further, a brief presentation of some common types of copula functions is given. For more detailed description, see [5], [6]

### 2.2.2 Elliptical copulas

The Gaussian copula was the first copula function, suggested for valuation of CDO’s (see Li [10]).

**Definition 1** [17]: Let $F$ denote the standard univariate normal distribution function and let $F_\rho$ denote the standard normal multivariate distribution with correlation matrix $\rho$. The Gaussian copula is then defined by

\[
C_{F_\rho}(x_1, x_2, \ldots, x_m, \rho) = F_\rho(F^{-1}(x_1), F^{-1}(x_2), \ldots, F^{-1}(x_m)). \quad (2.10)
\]

What should one do to sample from the Gaussian copula? Given the correlation matrix $\rho$, one must compute the Cholesky decomposition of $\rho$ (for more information on Cholesky decomposition, see e.g. [3]). Then construct a vector of $n$ independent random variables with a standard Gaussian distribution. If we transform the components of a vector $Y = AX$ into uniformly distributed random variables by $U = \Phi(Y)$, then $U \sim C_{F_\rho}$, [17]. The next type of copula functions is the Student copula.

**Definition 2** ([17]): Let $t_\nu$ denote the standard univariate Student’s $t$-distribution function with $\nu$ degrees of freedom and let $t_{\nu, \rho}$ be the multivariate Student’s $t$-distribution with correlation matrix $\rho$ and $\nu$ degrees of freedom. The Student copula is then defined by

\[
C_{t_{\nu, \rho}}(x_1, x_2, \ldots, x_m, \rho) = t_{\nu, \rho}(t^{-1}_\nu(x_1), t^{-1}_\nu(x_2), \ldots, t^{-1}_\nu(x_m)). \quad (2.11)
\]
Both these copula functions (Gaussian copula and Student copula) are elliptical copula functions. Roughly speaking, when a plot of a joint elliptical distribution is seen from above, its contours have elliptical shape.

### 2.2.3 Marshall-Olkin copulas

Another class of copula functions was first described by Marshall and Olkin in 1967 [11]. These copulas are especially useful for modelling the lifetime of objects, such that their lifetimes are related. The aim of the Marshall-Olkin copula is to build a joint distribution of exponential random variables.

Let us again consider the bivariate case. Then we have objects 1 and 2 with lifetimes $X$ and $Y$. Assume, that at any time during the lifetime of the object, one of two events can occur: the first object ”dies” or both of them ”die” together. In this case one can say, that the lifetimes follow three Poisson processes with next parameters: $\lambda_1$ shows, if the first object ”dies” (let us denote the time when this event occurs $E_1$), $\lambda_2$ shows if the second object ”dies” (corresponding time $E_2$) and $\lambda_{12}$ shows if both of them ”die” (corresponding time $E_{12}$). Then, the survival function can be defined as:

$$
\Phi(x, y) = P\{E_1 > x\} P\{E_2 > y\} P\{E_{12} > \max(x, y)\} \\
= \exp(\lambda_1 x - \lambda_2 y - \lambda_{12} \max(x, y)) \\
= \exp(- (\lambda_1 + \lambda_{12}) x - (\lambda_2 \lambda_{12}) y + \lambda_{12} \min(x, y)).
$$

Let $\alpha_1 = \lambda_{12}/(\lambda_1 + \lambda_{12})$, $\alpha_2 = \lambda_{12}/(\lambda_2 + \lambda_{12})$. This leads to the following presentation of copula functions:

$$
C_{\alpha_1, \alpha_2}(u, v) = \min(u^{1-\alpha_1} v, uv^{1-\alpha_2}). \tag{2.12}
$$

Due to the large number of variables to be sampled, this family of copulas is not as easy to sample, as elliptical copulas.

### 2.2.4 Archimedian copulas

The main advantage of presented earlier elliptical copulas is the ease of sampling from them. But they also have a disadvantage – rather complicated computations of dependence characteristics. Another class of copula functions, Archimedian copulas, partly overcomes this problem. To build a copula for dependent uniform random variables, one can employ a parametric transformation to a product copula. The parameters of this transformation can
result in any desired dependence structure. Then we have
\[ \varphi(C(x, y)) = \varphi(x) + \varphi(y). \]  
(2.13)

To proceed we need to define a generalised version of inverse function.

**Definition 3** [17]. Let \( \varphi : I \rightarrow [0, +\infty] \) be a continuous strictly decreasing function such that \( \varphi(1) = 0 \). The pseudo-inverse of \( \varphi \) is a function \( \varphi^{-1} \) with domain \([0, +\infty]\) and range \(I\) given by
\[
\varphi^{-1}(x) = \begin{cases} 
  x & \text{when } 0 \leq x \leq \varphi(x), \\
  0 & \text{when } \varphi(x) \leq x \leq +\infty.
\end{cases}
\]  
(2.14)

Solving the previous equation for \( C(x, y) \) we obtain an Archimedean copula
\[ C(x, y) = \varphi^{-1}(\varphi(x) + \varphi(y)), \]  
(2.15)

where \( \varphi \) is the generator of the Archimedean copula, \( \varphi^{-1} \) is its pseudo-inverse.

Now, the following theorem can be formulated:

**Theorem 2** [17]. Let \( \varphi \) be a continuous strictly decreasing function from \([0, 1]\) to \([0, +\infty]\), such that \( \varphi(1) = 0 \) and let \( \varphi^{-1} \) be the pseudo-inverse of \( \varphi \). Let \( C \) be a function from \([0, 1]^2\) to \([0, 1]\) given by
\[ C(x, y) = \varphi^{-1}(\varphi(x) + \varphi(y)). \]  
(2.16)

Then \( C \) is copula if and only if \( \varphi \) is convex.

For proof, (see Nelsen [12]).

### 2.3 An alternative to copula approaches

As it was mentioned in the introduction, one goal of the Walker’s approach [18] is the possibility to relax the necessary assumptions of a correlation structure between the assets in the basket and obtaining the joint basket loss and default distributions from the marginal distributions of the individuals. This is achieved by treating basket loss distribution and the expected basket loss at zero default as fundamental distributions.
Let us consider a CDO contract with maturity $T$, based on a reference basket with total notional $N$ equal to 1, divided into $N_{tr}$ tranches. Tranches are labelled with $k$. Each tranche is associated with losses from $l_{k-1}$ to $l_k$, such that $l_i \in \mathbb{R}$, $l_0 \equiv 0$ and $l_{N_{tr}} \equiv 1$. The width of the tranche $k$ at time $t = 0$ is defined as $\Delta_{k,0} = l_k - l_{k-1}$.

The total loss of the basket at time $t$ is a random variable, which magnitude depends on the number and notional of names, defaulted before time $t$ and thus can be written as:

$$L_t = \sum_{i=1}^{n} I(\tau_i < t)N_i,$$  \hspace{1cm} (2.17)

where $I(\tau_i < t)$ is 1, if name $i$ defaults before time $t$ and 0 otherwise, $N_i$ is the notional of name $i$.

If there is a recovery, i.e. that some part of the notional $N_i$ is compensated when name $i$ defaults. The compensated amount is determined by a recovery rate. Let us denote the recovery rate of the name $i$ as $R_i$, then the amount of $N_iR_i$ is compensated in case of default of the name $i$. Taking the recovery process into account, we should multiply the summand in the expression (2.17) by $(1 - R_i)$.

To value tranche $k$ it is natural to introduce such quantity as tranche-averaged loss distribution, which is the sum of losses at time $t$ in relevant range $(l_k, l_{k-1})$ divided by the tranche thickness

$$f(k, t) = \frac{1}{\Delta_{k,0}} \int_{l_{k-1}}^{l_k} F(l, t)dl,$$  \hspace{1cm} (2.18)

where $F(l, t)$ is the loss distribution.

At time $t = 0$ the present value of the notional-averaged expected loss, occurring in time interval $(0, T)$ for tranche $k$ assuming the discount factor equal to $e^{-rt}$ is

$$V_{loss}(k, T) = \int_0^T \frac{df(k, t)}{dt}e^{-rt}dt = \int_0^T e^{-rt}df(k, t).$$  \hspace{1cm} (2.19)

On the premium side, periodical payments are made at times $t_j, j = 0, \ldots, N_p$. The magnitude of the $j$-th payment (i.e. payment for the protection in time interval $\delta_j = t_j - t_{j-1}$) is

$$\omega(k, T)N_i(t_j)\delta_j,$$  \hspace{1cm} (2.20)

where $\omega(k, T)$ - is the annualized premium the for corresponding tranche $k$ and $N_i(t_j)$ is the remaining tranche notional.
Chapter 2. Methods

Typically the payments are made quarterly, so that \( \delta = 0.25 \). In the case, when the default occurs between two payment dates, an accrued payment, equal to the regular payment times \( \varphi(t) = (t_j - t_{j-1})/\delta_j \) is made.

The expected width of any tranche, except the super-senior (which corresponds to losses in the range from 22% to 100%) changes with time due to default events. Using the previously introduced notion of notional-averaged expected loss, we can write the following expression for the expected width of the tranche \( k \) at time \( t \):

\[
\Delta_{k,t} = \Delta_{k,0}(1 - f(k, t)).
\] (2.21)

The situation with the super-senior tranche is different, because in the case when default happens, there is a recovery.

Using the results, given above, Walker obtains the following expression for the expected value of premium payments

\[
V_{pr}(k, T) = u_f(k, T) + \omega(k, T) \left( \sum_{j=0}^{N_p(T)} \delta_j [1 - h(k, t_j)] e^{-rt_j} \right) + \int_0^T \delta_j \varphi(t) e^{-rt} df(k, t),
\] (2.22)

where \( h(k, t) = f(k, t) \) for all tranches except the super-senior tranche, \( u_f(k, t) \) is the upfront payment, \( N_p(T) \) is the total number of payments.

The fair value for \( \omega(k, T) \) - the annualized premium for a tranche can be found by equating the present value of expected losses to the expected present value of the premium payments.

The fair premium for the index can be calculated also by balancing the present value of expected losses and the expected present value of the premium payments. But one must be aware, that the premium payment at some time \( t \) is independent on losses and is proportional to the existing notional.

So, to price the standardized tranches and the index given the structure of the reference portfolio, we should know the notional-averaged default distribution and the expected tranche losses, \( f(k, t) \). They can be found by a calibration to all market prices for the assets in the basket. These quantities are considered as a subject to very few general constraints, what allows them to take the most universal possible form. To find out, if there exists a risk-neutral measure, that corresponds to some given set of market prices, one must simply check, if these constraints can be satisfied.

Several problems can be solved using the approach. At first, it gives an opportunity to check the market prices of CDO tranches and index for an arbitrage and determine the bounds for arbitrage-free prices. The arbitrage-free bounds for an unmarketed tranche price can be found considering a
linear-programming problem of maximizing this price (given as a function of the model parameters) over the set of all model parameters, subject to the mentioned above constraints. Also it makes possible to establish accurately interpolated term structures and tranche structures and thus value specific (unmarketed) tranches.

2.4 New method of checking CDO quotes for arbitrage

This method has almost the same purposes, as the one, previously described. It allows to check any set of market quotes for arbitrage opportunities, to price tranches with non-standard maturities and attachment/detachment points and forward-starting CDO’s. This method is a necessary addition to the procedure of pricing CDO’s by means of, for example, copula function approach, because the latter sometimes is really good in fitting the real data, but it does not detect arbitrage opportunities. The pricing non-standard CDO tranches is also an important problem on the modern fast developing credit derivative market.

Our first aim is to obtain expected portfolio losses directly from market data. As it was noticed in previous chapters, the premium leg consists of periodic premiums paid from the protection buyer to the protection seller at some defined for each tranche \( k \) rate. Let us denote this premium as \( \text{spr}_k \). Denoting the time interval between the payments as \( \delta = T_i - T_{i-1} \), the risk-free interest rate as \( r \) and the loss at time \( T_i \) for tranche \( k \) as \( L_i^k \), we can write the following expression for the net present value (NPV) of the premium leg:

\[
\text{NPV}_p = \text{spr}_k \sum_{i=1}^{N} \delta e^{-rT_i} \int_{T_{i-1}}^{T_i} (1 - L_i^k) dt. \tag{2.23}
\]

The expression for the NPV of the premium leg of equity tranches is slightly different, because they are quoted as a percentage upfront payment, assuming that payments at a rate of 500 basis points (bps) per year are made quarterly

\[
\text{NPV}_p = u_k + 0.05 \sum_{i=1}^{N} \delta e^{-rT_i} \int_{T_{i-1}}^{T_i} (1 - L_i^k) dt. \tag{2.24}
\]

On the other hand, each time, when a loss appears, the protection seller pays an amount, equal to this loss, re-scaled by the tranche thickness, to the
protection buyer. So, the NPV of the default leg can be written as

$$NPV_d = \int_0^T e^{-rt} dL_i^k.$$  \hspace{1cm} (2.25)

Let us denote the expected tranched loss (ETL) of the tranche \(k\) at time \(t\) as \(f(t, k)\). Given the market spreads one can find such ETL’s, which set the NPV of the instrument most closely to zero. To find the NPV we also need to know the ETL’s on all payment dates until maturity. At first, we interpolate between 0 (and at time 0 ETL is, of course, equal to 0) and ETL at first maturity date, getting the ETL profile for all tranches. Prices for tranches with higher maturity can be found also by an interpolation between ETL at the next maturity date and the one just found and so on. Of course, for these losses, to be realistic, certain constraints must hold. ETL must be increasing with time,

$$f(T_{i-1}, k) \leq f(T_i, k),$$  \hspace{1cm} (2.26)

decreasing with the tranche rank

$$f(T_i, k - 1) \geq f(T_i, k),$$  \hspace{1cm} (2.27)

and lie between 0 and 1.

Given ETL’s on all payment dates, one can calculate values of spreads, which will set the NPV of the instrument exactly to 0 (theoretical spreads). The differences between the market spreads and theoretical spreads allows to see, if there exist any opportunities of arbitrage. Different researchers suggest different ways of interpretation this difference. In this work the following interpretation is used: the difference is compared to the bid/ask spread and if it’s bigger, then this tranche is mispriced.

$$\Delta = \frac{spr_\text{theor} - spr_\text{mark}}{spr_\text{bid-ask}/2}.$$  \hspace{1cm} (2.28)

So, ETL’s are defined on a grid of \(t\) and \(k\), where \(t\) is the set of all payment dates and \(k\) is the set of all attachment and detachment points. If we add other attachment and detachment points to the set of \(k\) values and implement the same procedure as just described, we can obtain fair prices for tranches with non-standard attachments. Correspondingly, if we add extra values to the set \(t\), we will be able to obtain fair prices for tranches with non-standard maturities.
Chapter 3

Results

As a part of the work on this thesis project a MatLab programme, implementing the method, described in Section 2.4 was written.

Using the real data (market quotes for some CDS index tranches for a day) it is possible to obtain the expected portfolio losses and compute the theoretical premium, which allows no opportunities of arbitrage.

The results of my work can be described using the real data: market quotes for CDX index standard tranches for 28 November, 2006 from Baxter [1] showed in the Figure 3.1. The equity tranche (0 to 3%) is quoted as a percentage upfront payment, assuming that payments at a rate of 500 basis points per year are made quarterly. The other tranches are quoted in basis points per year, again assuming that payments are made quarterly.

<table>
<thead>
<tr>
<th>Tranche</th>
<th>5 Years</th>
<th>7 Years</th>
<th>10 Years</th>
</tr>
</thead>
<tbody>
<tr>
<td>1: 0-3%</td>
<td>27.5</td>
<td>48</td>
<td>58</td>
</tr>
<tr>
<td>2: 3-6%</td>
<td>78</td>
<td>184</td>
<td>520</td>
</tr>
<tr>
<td>3: 6-9%</td>
<td>24</td>
<td>46</td>
<td>98</td>
</tr>
<tr>
<td>4: 9-12%</td>
<td>10</td>
<td>26</td>
<td>42</td>
</tr>
<tr>
<td>1: 12-22%</td>
<td>5.25</td>
<td>12.25</td>
<td>22</td>
</tr>
</tbody>
</table>

Figure 3.1: Market quotes for the standardized CDX 125 S7 tranches at 28 November, 2006 [1].

To check market quotes for arbitrage opportunities one should first obtain expected tranche losses (ETL) on all payment dates. The ETL profiles for equity and upper tranches are presented on the Figures 3.2 and 3.3. Different kinds of interpolation were used (linear and cubic spline), but it shows, that the influence of the kind of interpolation is not significant.
Chapter 3. Results

Figure 3.2: The time-dependence of ETL for the equity tranche. Market quotes from Table 3.1 were used. Left plot: linear interpolation, right plot: spline interpolation.

Figure 3.3: The time-dependence of ETL for the upper tranche. Market quotes from Table 3.1 were used. Left plot: linear interpolation, right plot: spline interpolation.
The graphical representation of ETL surface through all the payment dates for all transhes is showed in Figure 3.4.

Figure 3.4: The ETL-surface, showing the distribution of ETL through time and tranches. Market quotes from Table 3.1 were used.

On the Figure 3.5 the graphical representation of the calculated theoretical premium through all the tranches and maturities is presented.
Figure 3.5: *Graphical representation of the theoretical premium surface. Market quotes from Figure 3.1 were used.*
Chapter 4

Conclusions

4.1 Discussion

The objective of this work was to implement a procedure, which makes it possible to verify the absence of arbitrage in the model and to price non-standard CDO tranches, since it is not possible in the case of copula function approaches. These procedures play an important role for valuing CDO’s nowadays, because of the fast growing diversity of the credit portfolio derivatives’ market.

The results obtained remind of earlier works by Walker, Brigo and Eberlein [18, 19, 16, 4]. A comparison with Walker’s results in checking for the arbitrage opportunities [18] shows, that the suggested method detects slightly smaller amount of mispricing, but this is absolutely natural, because looking for the theoretical premium we didn’t try to set the NPV of the contract exactly to zero, as Walker did, but wanted to find a premium, which will set the NPV as closer to 0, as possible, considering certain constraints. But in general results are similar.

Another feature, that single out this methodology from the set of CDO valuation methods is a possibility of not having to guess the right dependence structure of the reference portfolio.

4.2 Future work

The following directions of subsequent research are suggested:

- realisation of the method with recovery rate as a non-constant value;

- testing the method on larger sets of real data (Not for a single day, but for some longer period. Unfortunately, this kind of data was not
available);

- more accurate comparison to other approaches, including a numerical comparison of robustness, etc.
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function myfunc = samaya_glavnaya

global delta r a b T U spr Rec
clc;
Rec = 0.4; % recovery rate, taken as a constant
delta = 0.25; % payments are made 4 times a year
r = 0.04; % risk-free interest rate
a(1)=0; % array of attachment points
a(2)=0.03;
a(3)=0.07;
a(4)=0.1;
a(5)=0.15;
a(6)=0.3;
a(7)=1;
for i = 1:6
    b(i) = a(i+1); % array of detachment points
end;

% upfront payments for equity tranches (index - maturity)
U(2) = 0.246; U(3) = 0.406; U(4) = 0.511;
spr(2,1) = 500; spr(3,1) = 500; spr(4,1) = 500; % equity tranche
spr(2,2) = 91; spr(3,2) = 210; spr(4,2) = 426; % 3-6 tranche
spr(2,3) = 18.4; spr(3,3) = 46.8; spr(4,3) = 110; % 6-9 tranche
spr(2,4) = 6.5; spr(3,4) = 19; spr(4,4) = 51.5; % 9-12 tranche
spr(2,5) = 3.1; spr(3,5) = 6; spr(4,5) = 14.8; % 12-22 tranche
spr(2,6) = 1.4; spr(3,6) = 2.3; spr(4,6) = 3.9; % super-senior tranche
for i = 1:6
    for j = 2:4
        spr(j,i) = spr(j,i)/100;
    end;
end;
end;

% T(1) = 3; % zadаем массив maturities
T(2) = 5;
T(3) = 7;
T(4) = 10;
for mat = 2:4

% naходим ETL для equity tranche
[EL_equity] = fminbnd(@(x) npv_equity(x,mat),0,1);
f(mat,1) = EL_equity;
for k = 2:5

% naходим ETL для middle tranche
[EL_mid] = fminbnd(@(x) npv_mid(x,mat,k),0,1);
f(mat,k) = EL_mid;
end;

% naходим ETL для super-senior tranche
[EL_senior] = fminbnd(@(x) npv_senior(x,mat),0,1);
f(mat,6) = EL_senior;
end;
f(:,6)

% teper’ zaimymsya interpolation
etl_int = zeros(30,6);

% 1st index - number of payment, 2nd - tranche label
y(1) = 0;
x(1) = 0;

for k = 1:6
    clear x;
    j = 1;
    for mat = 2:4
        etl_int(1,k) = 0;
        etl_int(T(mat)*4,k) = f(mat,k);
        y(j+1) = f(mat,k);
        x(j+1) = T(mat)
        j = j+1;
    end;

    % x1 = T(mat-1)+0.25;
    x2 = T(mat)-0.25;
    xi = 0.25:.25:x2;
    yi1 = interp1(x,y,xi);
    % yi2 = interp1(x,y,’spline’,’pp’);

    etl_int(T(mat),k) = yi1;
end;

% x1 = T(mat-1)+0.25;
% x2 = T(mat)-0.25;
% xi = 0.25:.25:x2;
% yi1 = interp1(x,y,xi);
% y2 = interp1(x,y,’spline’,’pp’);
yi2 = interp1(x,y,xi,'spline');
for i = 2:(T(mat)*4-1)
    etl_int(i,k) = yi1(i-1);
end;
yi2
figure,subplot(1,2,1);plot(xi,yi1,'-o');
subplot(1,2,2);plot(xi,yi2,'-x');

% etl_int;
end;
whos etl_int
% etl_int(:,6)
figure, surf(etl_int);
%teper' nam nuzhno poschitat’ teor. spread
for mat = 2:4
    for k = 1:6
        sum_def = etl_int(1,k);
        sum_prem = 0.25*(1-etl_int(1,k)/2);
        for t = 2:T(mat)*4
            d = exp(-r*t/4);
            sum_def = sum_def + d*(etl_int(t,k)-etl_int(t-1,k));
            sum_prem = sum_prem + 0.25*d*(1-(etl_int(t,k)+etl_int(t-1,k))/2);
        end;
        spr_theory(mat,k) = sum_def/sum_prem;
    end;
end;
figure, surf(spr_theory);
whos spr_theory
spread_diff = abs(spr_theory-spr)
end

% NPV for equity tranche
function NPV1 = npv_equity(x,mat);
global r T U delta spr
    t=T(mat);
    d = exp(-r*t);
    NPV1 = abs(U(mat)+delta*spr(mat,1)*(1-x)*d-x*d);
end

% NPV for almost everything else
function NPV2 = npv_mid(x,mat,k);
global r T delta spr

25
t = T(mat);
d = exp(-r*t);
NPV2 = abs(delta*spr(mat,k)*(1-x)*d-x*d);
end

% NPV for super-senior tranche
function NPV3 = npv_senior(x,mat);
global Rec T r delta spr
    d = exp(-r*T(mat));
    NPV3 = abs(delta*spr(mat,6)*(1-x/(1-Rec))*d-x*d);
end