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Pricing variance swaps by using two methods: replication strategy and a stochastic volatility model

Master's Thesis in Financial Mathematics

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Master's thesis in Financial Mathematics, 15 ECTS credits

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Preface

First of all I would like to thank professor Ljudmila A. Bordag for giving me the opportunity to do my master thesis at SunGard Front Arena. I would also like to thank Ph.D. Jan-Olof Johansson for advising on the topic and helping with administrative issues.

Especially, I express my gratitude to my mentors Ph.D. Jonas Persson and Gustaf Sterner, my managers Annika Mårdfalk and Håkan Norekrans and staff from SunGard Front Arena.

Last but not least, big thanks to my parents, family and friends for supporting me throughout my life. If it weren't for them, I wouldn't be here.

Abstract

In this paper we investigate pricing of variance swaps contracts. The literature is mostly dedicated to the pricing using replication with portfolio of vanilla options. In some papers the valuation with stochastic volatility models is discussed as well. Stochastic volatility is becoming more and more interesting to the investors. Therefore we decided to perform valuation with the Heston stochastic volatility model, as well as by using replication strategy.

The thesis was done at SunGard Front Arena, so for testing the replication strategy Front Arena software was used. For calibration and testing of the Heston model we used MatLab.

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Chapter 1

Introduction

Volatility is widely used as a measure of the risk and uncertainty of an asset. Powerful instruments for trading volatility are volatility and variance swaps.

The purpose of this project is to derive the theoretical fair value of a variance swap, so that it can be priced and used in practice. It is done by using two methods: replication strategy and a stochastic volatility model. Therefore, for completeness, a digression into the theory of stochastic volatility models is done.

In this work we choose the Heston stochastic volatility model, one of the most widely used models these days. When pricing with stochastic volatility a problem that arises is calibration. The formula for pricing variance swaps with Heston is much simpler than the formula for the replication strategy. However, since calibration can be quite demanding, the overall complexity is more or less the same.

This research can be extended further to pricing variance swaps with caps and floors, corridor and conditional variance swaps. Also, one can consider some other stochastic volatility models, or include dividends in pricing.

Chapter 1 is a brief introduction to the thesis. Basic properties of variance swaps are given in chapter 2. The first method, replication of variance swaps, is presented in chapter 3. Pricing with stochastic volatility is presented in chapter 4. Chapter 5 consists of calibration issues: general problem and methods. The obtained results are included in chapter 6. Conclusions are in chapter 7. Finally, detailed derivations are shown in the Appendix.

Chapter 2

Variance swaps

Variance swaps are forward contracts on future realized variance. Similarly, we have volatility swaps, which are forward contracts on future realized volatility. Both of the contracts provide pure exposure to the volatility level, so they are widely used for trading volatility. Since volatility can be regarded as a nonlinear function of variance (square root), volatility swaps are more difficult to value and hedge. In contrast to this, as will be shown later, variance swaps can be perfectly replicated with portfolio of vanilla options. This feature makes them more fundamentally significant.

Payoff

The payoff of the variance swaps is given by

$$N \cdot (\sigma_R^2 - K_{var})$$

where σ_R^2 is realized variance (quoted in annual terms) over the life of the contract, K_{var} is strike price of the swap (i.e fair strike), and N is the notional amount of the swap in dollars¹ per annualized volatility point squared.

At expiration, the holder of a variance swap receives N dollars per every point by which realized variance is higher than K_{var} . The fair value of a swap is chosen so that the swap has zero value at inception.

Convexity

As shown in Figure 2.1, the payoff of a variance swap is convex in volatility.

¹Or some other currency.

Thus, e.g. if realized volatility is 1 percent higher than the strike (considered in volatility), the payoff will be higher than if it is lower by 1 percent. This convexity is important only if the realized variance is significantly different from the strike.

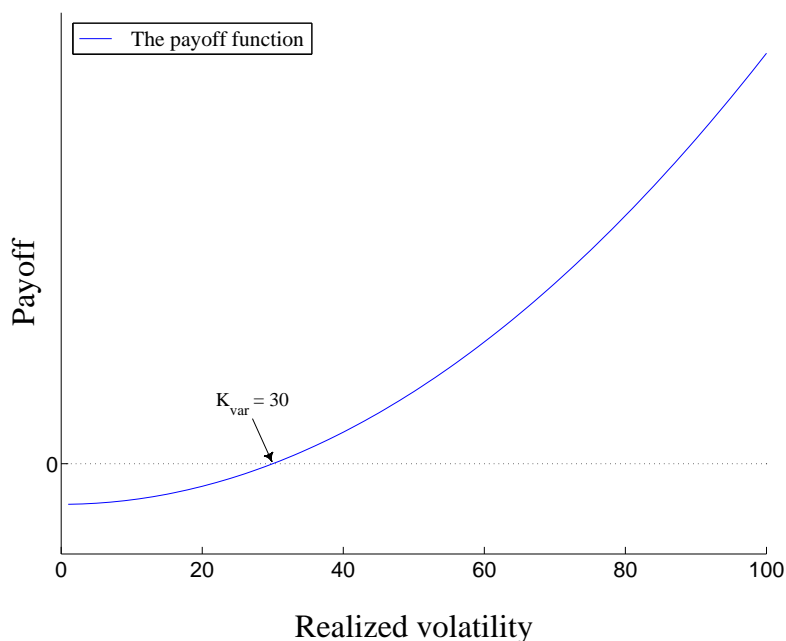


Figure 2.1: Variance swaps are convex in volatility.

The way of calculating realized variance should be defined in the contract itself. The following formula is often used²

$$\sigma_R^2 = \frac{252}{n-2} \sum_{i=1}^{n-1} \left(\ln \frac{S(t_{i+1})}{S(t_i)} \right)^2 \quad (2.1)$$

where n is the number of business days from the trading days up to (and including) the maturity day.³

²If we use daily observations.

³See *Gairat* [7]

The equation (2.1) has its continuous analogue given by

$$\sigma_R^2 = \frac{1}{T} \int_0^T \sigma_t^2 dt. \quad (2.2)$$

Here T is time to expiration in years and σ_t is volatility of the underlying.

Applications

As mentioned before, variance swaps are used for various volatility trading, namely for:

- Volatility trading: used by investors who are taking bets on volatility, both realized and implied
- Forward volatility trading: buying variance swap with one maturity, and selling variance swap with another maturity
- Trading the spread between realized and implied volatility
- Spreads on indices: capturing volatility level between two correlated indices

2.1 Variance swaps and option delta hedging

In this section we will explain replication strategy on an intuitive level. More formal derivation is presented in the next section. We assume that we are in the Black-Scholes world, and that risk free interest rate is zero.

To see how we can replicate variance swaps with vanilla options, we will start with delta hedging. As is known, the sensitivity of an option to the underlying price can be eliminated by holding a reverse position in the underlying in quantity equal to option's delta. After eliminating delta, we are left with sensitivities:

- Gamma: sensitivity to change in underlying price (in terms of the second derivative)
- Vega: sensitivity to volatility
- Theta: sensitivity to passage of time

Since the option is delta hedged, the daily $P\&L$ (profit and loss) looks like

$$\text{Daily } P\&L = \text{Gamma } P\&L + \text{Vega } P\&L + \text{Theta } P\&L + \text{Other} \quad (2.3)$$

Here 'Other' denotes impact of dividends, higher order sensitivities, etc. The previous equation can be rewritten as

$$\text{Daily } P\&L = \frac{1}{2} \Gamma \times (\Delta S)^2 + \Theta \times (\Delta t) + V \times (\Delta \sigma) + \dots \quad (2.4)$$

However, volatility is constant, and assuming that the factor 'Other' is negligible, we get the following expression for Daily $P\&L$:

$$\text{Daily } P\&L = \frac{1}{2} \Gamma \times (\Delta S)^2 + \Theta \times (\Delta t). \quad (2.5)$$

Using⁴ $\Theta \approx -\frac{1}{2}\Gamma \times (\Delta S)^2$, we obtain

$$\text{Daily } P\&L = \frac{1}{2} \Gamma S^2 \times \left[\left(\frac{\Delta S}{S} \right)^2 - \sigma^2 \Delta t \right].$$

The first term in the brackets, $\left(\frac{\Delta S}{S}\right)^2$, can be considered as realized one day variance. If we sum all daily $P\&L$ s until the maturity of the option we get final $P\&L$

$$\text{Final } P\&L = \frac{1}{2} \sum_{t=0}^n \gamma_t [r_t^2 - \sigma^2 \Delta t] \quad (2.6)$$

where t stands for time dependence, r_t for underlying's daily return, and γ_t is the so called **dollar gamma**.⁵

Paying closer look to the previous equation, we can see that it is very similar to the payoff of a variance swap. Like in the variance swap payoff, we have a sum of weighted differences between realized variance (squared daily returns) and some constant which can be interpreted as fair strike. The main difference between those two is that while in variance swap weights are constant, here they depend on the option's gamma. This dependency is known as *path-dependency* of $P\&L$.

⁴See *Bossu et al [1]*

⁵Gamma multiplied by the square of the underlying's price at time t .

We would like to have a portfolio of options with a constant total dollar gamma. *How can we create this portfolio?* Let us take a look at Figure 2.2.

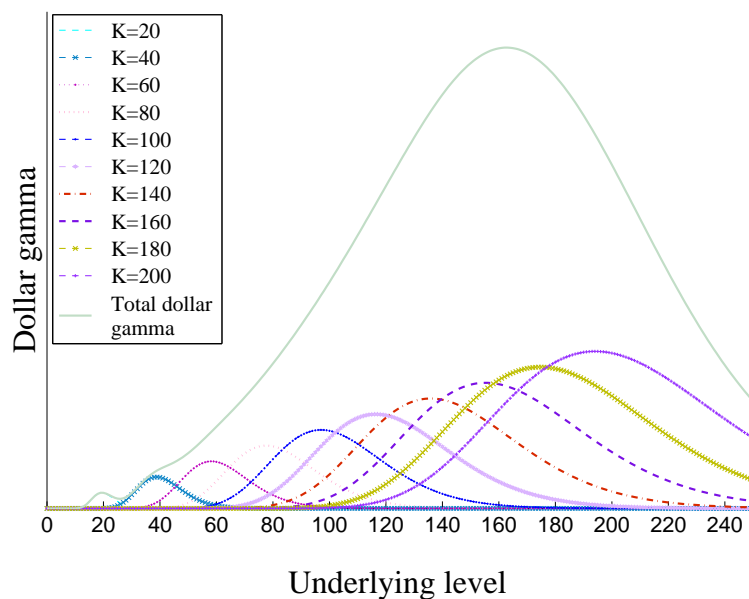


Figure 2.2: Dollar gamma for options with strikes from 20 to 200.

We can see that options with higher strikes have more influence to the aggregate dollar gamma in the contract than the options with lower strikes. By making weights inversely proportional to the strike ⁶ K , i.e. $w(K) = \frac{c}{K}$, where c is some constant, we get the result given in Figure 2.3.

The total dollar gamma is still not constant, but since we have a linear interval our position is improved. The idea now is to create new weights: $w'(K) = \frac{w(K)}{K}$. Hopefully they will provide constant gamma. The result with weights $w'(K)$ is shown in Figure 2.4.

⁶More details for this idea can be found in *Bossu et al [1]*.

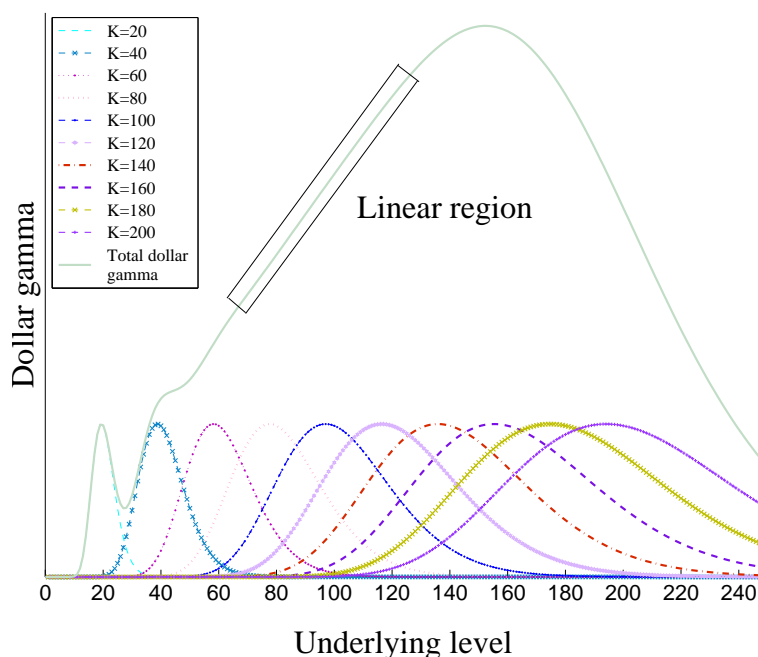


Figure 2.3: Dollar gamma for options weighted inversely proportional to the strike.

Thus, we got a constant interval for the underlying prices between 65 and 125. In order to obtain constant total dollar gamma on the whole interval, having infinite number of options with underlying struck along a continuum between 0 and ∞ weighted inversely proportional to the square of strike is necessary.

The problem of replicating variance swaps can be approached in another way as well. As was mentioned before, variance swaps depend only on volatility. Thus we need a portfolio of options whose sensitivity to variance does not depend on movement in the underlying price. It can be proved⁷ that such exposure demands portfolio of options with weights inversely proportional to the square of strikes. Hence, as was expected, the result is the same as in the previous approach.

⁷See Demeterfi et al [4]

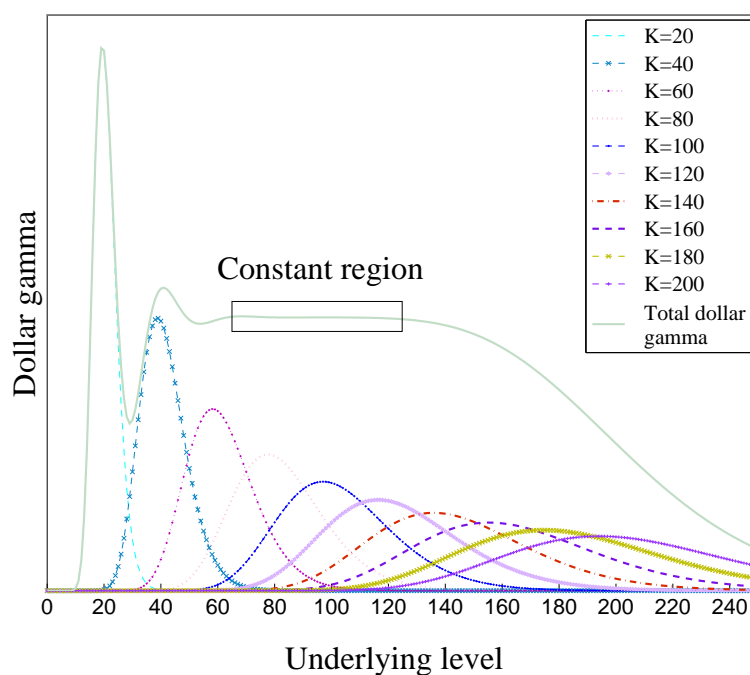


Figure 2.4: Dollar gamma for options weighted inversely proportional to the square of strike.

To make this replication on intuitive level it was assumed that we are in the Black-Scholes world, and that both dividends and interest rate are zero. The last two assumptions is not difficult to generalize. However, in the presence of the implied volatility skew, it is hard to extend replication process clearly. For those reasons, we will derive replication strategy more formally in the next chapter.

Chapter 3

Replication of variance swaps using portfolio of options

3.1 Replication in theory

In this section the only assumption that is made is that underlying's price moves continuously. Therefore, it follows

$$\frac{dS_t}{S_t} = \mu(t, \dots) dt + \sigma(t, \dots) dZ_t. \quad (3.1)$$

It is assumed that drift μ and the continuously sampled volatility are arbitrary functions of their parameters. For simplicity it is assumed that dividends are zero.

The realized variance for a given price history is the continuous integral¹

$$V = \frac{1}{T} \int_0^T \sigma^2(t, \dots) dt. \quad (3.2)$$

Since variance swaps are actually forward contracts on the realized variance, pricing them is not different from pricing any other forward contract. Let us denote by F the forward contract on future realized variance with strike K_{var} . The value of F is the expected present value of the future payoff in the risk neutral world:

$$F = E[e^{-rT}(V - K_{var})]. \quad (3.3)$$

¹This approximation of the variance of daily returns is used in most contracts.

Using that the strike of the contract is set up so that contract has zero present value we obtain

$$K_{var} = E(V). \quad (3.4)$$

Applying this to the equation (3.2) yields

$$K_{var} = \frac{1}{T} E \left[\int_0^T \sigma^2(t, \dots) dt \right]. \quad (3.5)$$

The equation (3.5) may seem quite simple, but the problem is that nobody knows for sure the value of the future variance. The used approach is good for valuing the contract. However, it does not give insight into the replication strategy; thus, we have to use a different approach. Namely, applying Itô's lemma to $\ln S_t$ yields

$$d(\ln S_t) = \left(\mu - \frac{1}{2} \sigma^2 \right) dt + \sigma dZ_t. \quad (3.6)$$

Next step is to subtract equation (3.6) from equation (3.1). This produces

$$\frac{dS_t}{S_t} - d(\ln S_t) = \frac{1}{2} \sigma^2 dt. \quad (3.7)$$

After integrating the previous equation from time 0 to T , we obtain

$$\frac{2}{T} \left[\int_0^T \frac{dS_t}{S_t} - \ln \frac{S_T}{S_0} \right] = \frac{1}{T} \int_0^T \sigma^2 dt = V. \quad (3.8)$$

Equation (3.8) describes the replication procedure. It guarantees that capturing variance does not depend on the path that underlying takes as long as it moves continuously. Now, having this new expression for variance, and using equation (3.4) we get

$$K_{var} = \frac{2}{T} E \left[\int_0^T \frac{dS_t}{S_t} - \ln \frac{S_T}{S_0} \right]. \quad (3.9)$$

In the risk neutral world, and with a constant interest rate r , the underlying actually follows the process

$$\frac{dS_t}{S_t} = r dt + \sigma(t, \dots) dZ_t. \quad (3.10)$$

By taking integral from 0 to T , and then expectation of the previous equation the following equality can be reached

$$E \left[\int_0^T \frac{dS_t}{S_t} \right] = r T. \quad (3.11)$$

Hence, we are left with

$$K_{var} = \frac{2}{T} \left(r T - E \left[\ln \frac{S_T}{S_0} \right] \right). \quad (3.12)$$

Therefore, in order to find value of the fair strike we have to find expectation of the log contract. Here we face the following problem: *there are no actively traded log contracts*. So, what we should do is to replicate the log payoff by composing its shape into a linear and nonlinear component, and then replicate each of them separately.

- The linear part can be replicated with a forward contract on the underlying with delivery time T .
- For replicating the curved part we need options with all possible strikes and the same expiration time T .

It is more convenient to do replication with liquid options, i.e.:

- using out-of-the-money calls for high underlying's values, and
- out-of-the-money puts for low underlying's values.

We will need a new parameter S_* , which represents boundary between calls and puts. The log payoff can be written as

$$\ln \frac{S_T}{S_0} = \ln \frac{S_T}{S_*} + \ln \frac{S_*}{S_0}. \quad (3.13)$$

The second term in the previous equation is constant, so we just have to replicate the first term. The term $\ln \frac{S_T}{S_*}$ is decomposed in the following way²

$$\begin{aligned}
 -\ln \frac{S_T}{S_*} &= -\frac{S_T - S_*}{S_*} && \text{(forward contract)} \\
 &+ \int_0^{S_*} \frac{1}{K^2} \text{Max}(K - S_T, 0) dK && \text{(put options)} \\
 &+ \int_{S_*}^{\infty} \frac{1}{K^2} \text{Max}(S_T - K, 0) dK && \text{(call options)}
 \end{aligned} \tag{3.14}$$

With equation (3.14) we decomposed the payoff in a short position in a log contract into a portfolio of

- a short position in $1/S_*$ forward contracts with strike S_*
- a long position in $1/K^2$ put options struck at K and with strikes from 0 to S_*
- a long position in $1/K^2$ call options struck at K and with strikes in a range from S_* to ∞

After doing the replication process, we can go back to the equation (3.12), and rewrite it as

$$\begin{aligned}
 K_{var} &= \frac{2}{T} \left[rT - \left(\frac{S_0}{S_*} e^{rT} - 1 \right) - \ln \frac{S_*}{S_0} \right. \\
 &\quad \left. + e^{rT} \int_0^{S_*} \frac{1}{K^2} P(K) dK \right. \\
 &\quad \left. + e^{rT} \int_{S_*}^{\infty} \frac{1}{K^2} C(K) dK \right]
 \end{aligned} \tag{3.15}$$

where $P(K)/C(K)$ denotes the current value of a put/call option with strike K .

²See Appendix A for complete derivation.

3.2 Replication in practice

In the previous section we derived a formula for replicating a variance swap with a portfolio of options. Then a very strong assumption that we have options with all strikes from 0 to ∞ was used. As is clear, this is quite impossible to be done in practice. Since we can only have options with strikes in a finite interval, the replication strategy will give an estimation for variance which is lower than it should be in reality.

Therefore, in this section we will derive another replication formula, which assumes that we have a finite number of strikes in our portfolio.³

Let us go back to the equation (3.9)

$$K_{var} = \frac{2}{T} E \left[\int_0^T \frac{dS_t}{S_t} - \ln \frac{S_T}{S_0} \right]. \quad (3.16)$$

As was mentioned before, $\ln \frac{S_T}{S_0}$ can be rewritten as $\ln \frac{S_T}{S_*} + \ln \frac{S_*}{S_0}$, with S_* being a so-called boundary between call and put strikes. For our calculation it is convenient to add and subtract the term $\frac{S_T - S_*}{S_*}$ from the equation (3.16).

Thus,

$$K_{var} = \frac{2}{T} E \left[\int_0^T \frac{dS_t}{S_t} - \ln \frac{S_*}{S_0} - \frac{S_T - S_*}{S_*} + \frac{S_T - S_*}{S_*} - \ln \frac{S_T}{S_0} \right]. \quad (3.17)$$

After taking expectation we have

$$K_{var} = \frac{2}{T} \left[rT - \left(\frac{S_0}{S_*} e^{rT} - 1 \right) - \ln \frac{S_*}{S_0} \right] + \frac{2}{T} E \left[\frac{S_T - S_*}{S_*} - \ln \frac{S_T}{S_*} \right]. \quad (3.18)$$

Let us denote by P the present value of the portfolio of options whose payoff at expiration is equal to

$$f(S_T) = \frac{2}{T} \left[\frac{S_T - S_*}{S_*} - \ln \frac{S_T}{S_*} \right]. \quad (3.19)$$

³See Demetrfi et al [4]

Now we have

$$K_{var} = \frac{2}{T} \left[rT - \left(\frac{S_0}{S_*} e^{rT} - 1 \right) - \ln \frac{S_*}{S_0} \right] + e^{rT} P. \quad (3.20)$$

As in the previous section, we want to replicate the payoff $f(S_T)$. Assume that we hold a portfolio of calls with strikes K_i^c , $S_* = K_0^c < K_1^c < K_2^c < K_3^c < \dots$ and puts with strikes K_j^p such that $S_* = K_0^p > K_1^p > K_2^p > K_3^p > \dots$. In order to replicate our payoff with this portfolio, we need to calculate the proper weight for each option. Providing that the weights are given with $w(K_i^c)$ for calls, and $w(K_j^p)$ for puts, the value of P is

$$P = \sum_i w(K_i^c) C(K_i^c) + \sum_j w(K_j^p) P(K_j^p). \quad (3.21)$$

where $C(K_i^c)$ denotes the present value of a call option with strike K_i^c , and similarly $P(K_j^p)$ stands for the present value of a put option with strike K_j^p . A detailed calculation of weights is presented in Appendix B. Assume having a reasonable number of options, with these weights one will always have a payoff equal to or higher than $f(S_T)$.

Chapter 4

Stochastic volatility

After the stock market crash in October 1987 it was quite clear that Black-Scholes's assumption of volatility being constant can not hold in the real world. This problem was studied in several ways¹:

- Volatility is a deterministic function of time, i.e. $\sigma \equiv \sigma(t)$
- Volatility is a function of the time and the current level of the underlying's price
- Volatility depends on some random parameter x , e.g $\sigma \equiv \sigma(x(t))$ where $x(t)$ is a random process
- Volatility follows a stochastic process, then the following SDE's are satisfied

$$dS(t) = \mu(t) S(t) dt + \sqrt{v(t)} S(t) dZ_1, \quad (4.1)$$

$$dv(t) = \alpha(S, v, t) dt + \eta \beta(S, v, t) \sqrt{v(t)} dZ_2. \quad (4.2)$$

In the previous equations the coefficients are

- $\mu(t)$ is the drift of stock price returns
- η is volatility of volatility
- dZ_1 and dZ_2 are Brownian motions with correlation ρ .

¹See Swishchuk [14]

The equation (4.1) is exactly the same as in the Black-Scholes model. In the limit $\eta \rightarrow 0$ volatility is not stochastic anymore, it depends only on time. Hence, we can move from stochastic volatility world to the Black-Scholes world.

If the Brownian motions dZ_1 and dZ_2 are not perfectly (anti) correlated the market is incomplete.² This comes from the fact that volatility is actually not a traded asset, so it can not be perfectly hedged. There are stochastic volatility models describing both complete and incomplete markets.

By choosing different functions for α and β in the equation (4.2) one can obtain different models. Some of them are:

- Hull & White (1987) $d\sigma_t = \sigma_t(\alpha dt + \gamma dW_t)$, $\rho = 0$
- Scott (1987) $d\sigma_t = \sigma_t((\alpha - \beta \sigma_t) dt + \gamma dW_t)$
- Stein & Stein (1991) $d\sigma_t = \beta(\alpha + \sigma_t) dt + \gamma dW_t$, $\rho = 0$
- Heston (1993) $dv(t) = -\kappa(v_t - \theta) dt + \eta \sqrt{v(t)} dW_t$

How to choose the best model to use? This is kind of a tricky question. Of course there is not a perfect model, for each of them it is possible to find both advantages and drawbacks. In this paper we chose to use the Heston model, considered as one of the most widely used stochastic volatility models today.

²In a complete market all claims can be perfectly hedged, and there exists a unique risk neutral measure. More about this can be found in *Potter [11]*.

4.1 Heston model

The Heston model is given by

$$dS_t = \mu(t) S_t dt + \sqrt{v_t} S_t dZ_1, \quad (4.3)$$

$$dv_t = \kappa (\theta - v_t) dt + \eta \sqrt{v_t} dZ_2. \quad (4.4)$$

Here,

- θ is long run mean average price volatility
- κ is the speed of mean reversion, and
- all other parameters are as mentioned before

The Heston model can obviously be obtained from the general one described in the previous section by choosing:

- $\alpha(S, v, t) = \kappa (\theta - v_t)$
- $\beta(S, v, t) = 1$

Advantages of the Heston model³

- (Quasi) Closed form solution for the European options
- Fits most equity market implied volatility surface reasonably well
- Volatility is mean reverting
- Correlation between asset and volatility is not fixed

Drawbacks

- Parameters are not stable over time
- Hedging all the parameters is not practical
- Not good results for a very short maturities

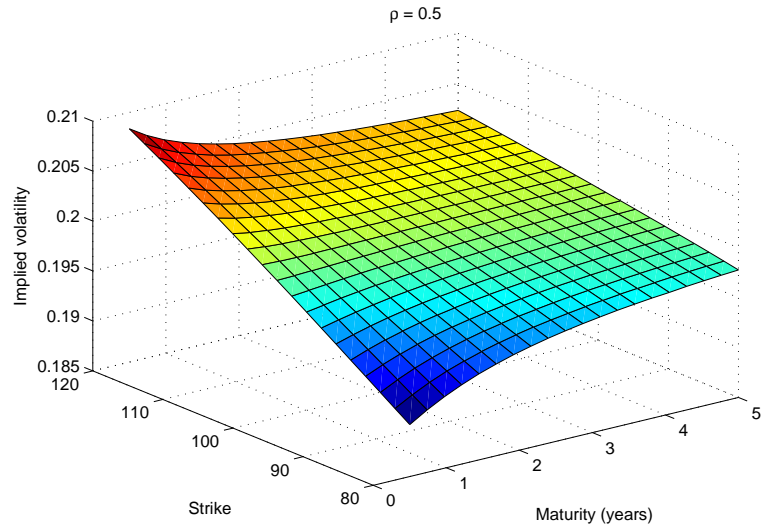


Figure 4.1: Implied volatility surface: $\rho = 0.5$, $\kappa = 2$, $\theta = 0.04$, $\eta = 0.1$, $v_0 = 0.04$, $r = 0.01$, $S_0 = 100$.

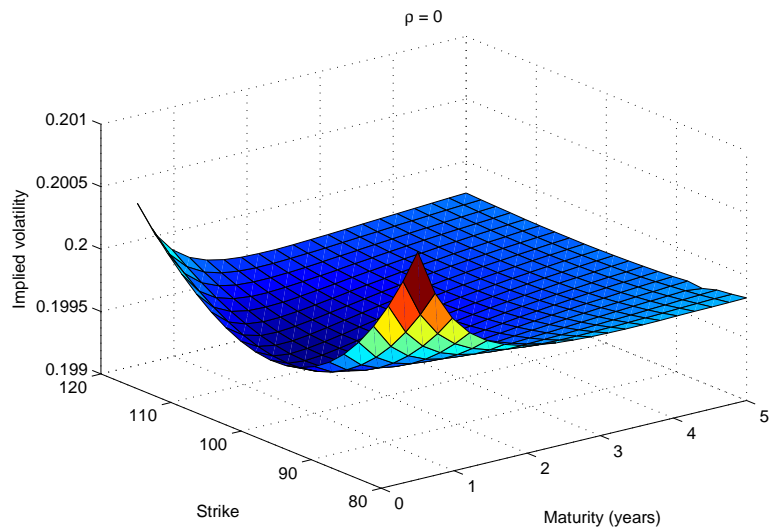


Figure 4.2: Implied volatility surface: $\rho = 0$, $\kappa = 2$, $\theta = 0.04$, $\eta = 0.1$, $v_0 = 0.04$, $r = 0.01$, $S_0 = 100$.

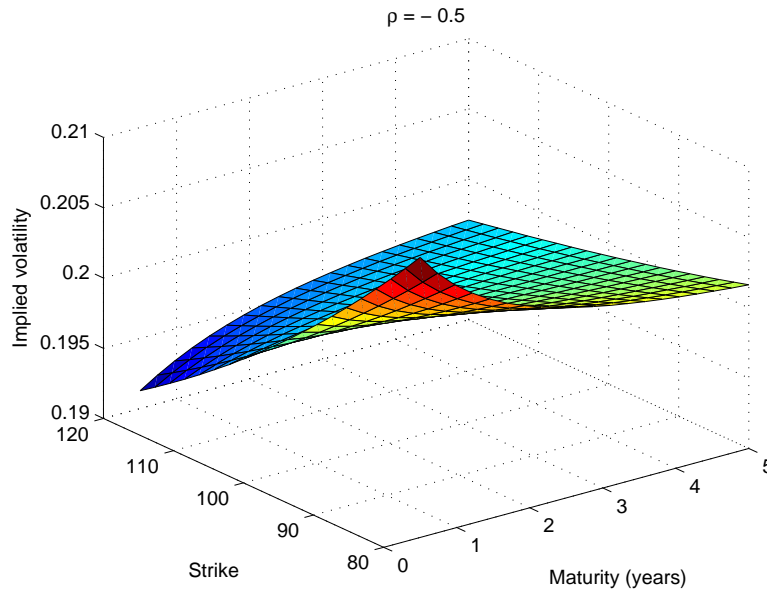


Figure 4.3: Implied volatility surface: $\rho = -0.5$, $\kappa = 2$, $\theta = 0.04$, $\eta = 0.1$, $v_0 = 0.04$, $r = 0.01$, $S_0 = 100$.

By changing the parameter ρ in the Heston model we are actually changing the skewness of the distribution. Positive ρ creates a fat right tail, and a thin left tail in a distribution of continuously compounded spot returns. The changing of skewness has an impact on the shape of implied volatility surface. We can see this in Figures⁴ 4.1, 4.2 and 4.3.

As we said η is volatility of volatility. Hence, when $\eta = 0$ volatility is deterministic and continuously compounded spot returns will have a normal distribution. If η is non-zero it increases the kurtosis of spot returns.⁵ The effect of this is raising far-in-the-money and far-out-of-the-money option prices and lowering near-the-money prices.

³See *Randjiou [12]*

⁴For simulation code see *Moodley [10]*.

⁵See *Heston [9]*

According to the Heston [9], changing ρ has also an impact to the option prices calculated with the Heston model relative to the prices calculated with the Black-Scholes model. Therefore, the prices of out-of-the money options is higher, and in-the-money is lower than prices from the Black-Scholes, for a positive correlation; and vice versa for a negative correlation.

Conclusion

When dealing with stochastic volatility models it is obviously very important how the parameters are chosen. With properly chosen ones the stochastic volatility models appear to be a very flexible and quite good description of option prices. The Black-Scholes formula produces prices almost identical to those from stochastic volatility models for at-the-money options.⁶ Moreover, options are quite often traded when they are near-the-money, hence for practical purpose the Black-Scholes model works quite well.

4.1.1 Variance swaps in the Heston model

In the section 3.1 the fair strike was evaluated, and according to the equation (3.5) we have

$$K_{var} = \frac{1}{T} E \left[\int_0^T \sigma^2(t, \dots) dt \right]. \quad (4.5)$$

As was mentioned before, in the Heston model stock price and variance satisfy the following SDEs:

$$dS_t = \mu(t) S_t dt + \sqrt{v_t} S_t dZ_1, \quad (4.6)$$

$$dv_t = \kappa(\theta - v_t) dt + \eta \sqrt{v_t} dZ_2. \quad (4.7)$$

By integrating the stochastic differential equation for variance from 0 to t , and then taking expectation⁷ we get

$$E(v_t) = (v_0 - \theta) e^{-\kappa t} + \theta. \quad (4.8)$$

Equation (3.5) can be written as

$$K_{var} = \frac{1}{T} \int_0^T E(v_t) dt. \quad (4.9)$$

⁶See Heston [9]

⁷See Appendix C for complete derivation.

Therefore, using the equation (4.8) we reach the following expression for the fair strike

$$K_{var} = \frac{1}{\kappa T} (1 - e^{-\kappa T}) (v_0 - \theta) + \theta. \quad (4.10)$$

The value of the swap at inception is equal to zero. However, if the swap is already in effect and we want to value it at some time t ($0 < t < T$), then the present value of the swap is valued using a combination of the realized variance up to time t and fair strike for the remaining life of the swap.

Recalling that the payoff of the swap is $N \cdot (\sigma_r^2 - K_{var})$, and that we denoted realized variance as $V = \frac{1}{T} \int_0^T \sigma^2(t) dt$, the present value is:

$$P_{value} = N e^{-r(T-t)} [E_{net}(V) - K_{var}], \quad (4.11)$$

where

$$E_{net}(V) = \frac{1}{T} \left(t V_{real} + \frac{1}{\kappa} (1 - e^{-\kappa(T-t)}) (v_0 - \theta) + (T - t) \theta \right). \quad (4.12)$$

V_{real} is realized variance up to the time t .

4.2 Greeks

Since we have valuation equations for both fair strike and the present value of the swap, we can now calculate some greeks. For the fair strike we have

- Theta

$$\frac{1}{365} \frac{\partial K_{var}}{\partial t} = \left[\frac{1}{\kappa^2 (T - t)} (1 - e^{-\kappa(T-t)}) - e^{-\kappa(T-t)} \right] \cdot \frac{1}{365} \frac{(v_0 - \theta)}{T - t}. \quad (4.13)$$

- Sensitivity to initial variance, v_0

$$\frac{\partial K_{var}}{\partial v_0} = \frac{1 - e^{-\kappa(T-t)}}{\kappa (T - t)}. \quad (4.14)$$

- Sensitivity to long-run variance, θ

$$\frac{\partial K_{var}}{\partial \theta} = 1 - \frac{1 - e^{-\kappa(T-t)}}{\kappa(T-t)}. \quad (4.15)$$

- Sensitivity to speed of mean reversion, κ

$$\frac{\partial K_{var}}{\partial \kappa} = \frac{(v_0 - \theta)}{\kappa} \cdot \left(\frac{1 - e^{-\kappa(T-t)}}{\kappa(T-t)} + e^{-\kappa(T-t)} \right). \quad (4.16)$$

Similarly, for the present value of a variance swap we have the following greeks:

- Theta

$$\begin{aligned} \frac{1}{365} \frac{\partial P_{value}}{\partial t} &= \frac{N e^{-r(T-t)}}{365 \cdot T} [V_{real} - (v_0 - \theta) e^{-\kappa(T-t)} - \theta] + \\ &+ \frac{r}{365} P_{value}. \end{aligned} \quad (4.17)$$

- Rho

$$\frac{1}{100} \frac{\partial P_{value}}{\partial r} = -\frac{T-t}{100} P_{value}. \quad (4.18)$$

- Sensitivity to initial variance, v_0

$$\frac{\partial P_{value}}{\partial v_0} = N e^{-r(T-t)} \frac{1 - e^{-\kappa(T-t)}}{\kappa T}. \quad (4.19)$$

- Sensitivity to long-run variance, θ

$$\frac{\partial P_{value}}{\partial \theta} = \frac{N e^{-r(T-t)}}{T} \left(T - t - \frac{1 - e^{-\kappa(T-t)}}{\kappa} \right). \quad (4.20)$$

Chapter 5

Calibration

5.1 Calibration in general

If we are pricing with stochastic volatility models one problem that appears is estimation of parameters. The procedure of estimation is called calibration of the model. The problem of calibration is not an easy problem at all. Hence it is very important which method is used.

A common way to evaluate parameters of the model is to find those parameters which produce market prices of the derivative. This is called the *inverse* problem. One of the possibilities of solving an inverse problem is to minimize the difference between model and market prices. So one needs to solve the following problem:

$$\min_{\Omega} S(\Omega) = \min_{\Omega} \sum_{i=1}^N \omega_i [C_i^{\Omega} - C_i^M]^2. \quad (5.1)$$

where Ω is set of parameters, N is the number of derivatives used for calibration, ω_i are weights, and C_i^{Ω} , C_i^M are respectively prices calculated using model and prices from the market. It is very convenient to use weights. They can be chosen in a number of ways, e.g. $\omega_i = \frac{1}{(bid_i - ask_i)^2}$, where bid_i is a bid price of the i^{th} derivative, and ask_i the corresponding ask price. Since we usually have bid and ask prices for derivatives, we can use these for weights and the middle price as a market price.

Although the minimization problem given with the equation (5.1) may seem simple, it is not simple at all. The problem that arises is that function $S(\Omega)$ is usually not convex and does not have any particular structure. Another problem is that a unique solution might not exist and that when we find a solution we can not be certain that it is a global minimum.

5.2 Calibration methods

There are a lot of methods for solving minimization problems like (5.1). In this chapter we will present some of them.

5.2.1 Excel's solver

The Solver included in Excel can be used for calibration. It uses a *Generalized Reduced Gradient* method. Therefore it is a local optimizer and as such very sensitive to the initial guess of the parameters.

Except being a local optimizer Solver is also not appropriate for calculating with very small numbers. The optimizer should only be used when we are sure that initial estimates are quite close to the optimal ones.

More information about the optimizer can be found on www.solver.com.

5.2.2 Matlab's lsqnonlin function

`lsqnonlin` is MATLAB'S function for performing the least-square non linear optimization. Like the Excel's solver, `lsqnonlin` is also a local optimizer. It uses an *interior-reflective Newton method*, and can handle bounds on parameter set which can be very useful.

The calibration is quite fast, but very sensitive to the initial values.

5.2.3 Differential evolution algorithm

The differential evolution algorithm (DE) is another tool for optimization. It was developed in the nineties by Rainer Storn and Kenneth Price¹.

The DE algorithm is claimed to be a global optimizer (still not proved). The advantage of the algorithm is that it is not dependent on initial values of the parameters. However, it has its own parameters which it is dependent on, so in case of a wrong choice of those it can miss the global minimum. Still, a lot of users suggest DE for calibration.

¹For more details see *Storn and Price [13]*

Chapter 6

Results

6.1 Simulation results

As is said before, calibration should be performed using market prices of the instrument we want to price. For some instruments a closed form solution for the model it should be priced with does not exist. In such cases common approach is to calibrate over the prices of an instrument that has a closed form solution. Although we had closed form solution for variance swaps, we faced problem with lack of market data. Hence, we decided to proceed as described in Figure 6.1. Therefore we made a set of options using the Monte Carlo simulation, and then used those options to price variance swaps with both Heston model and the replication strategy. In Front Arena software Derman model stands for pricing with the replication strategy.

In order to test the procedure described in Figure 6.1 we used following input data:

- Heston Parameters:

κ	θ	η	ρ	v_0
1.7	0.075	0.5646	-0.3521	0.0884

- *Maturities (years)* 0.126027 0.627397 1.4576
Interest rates 0.026734 0.02 0.017587
- Strikes: 60 strikes in a range 80 to 160 equally spaced
- $S_0 = 120$

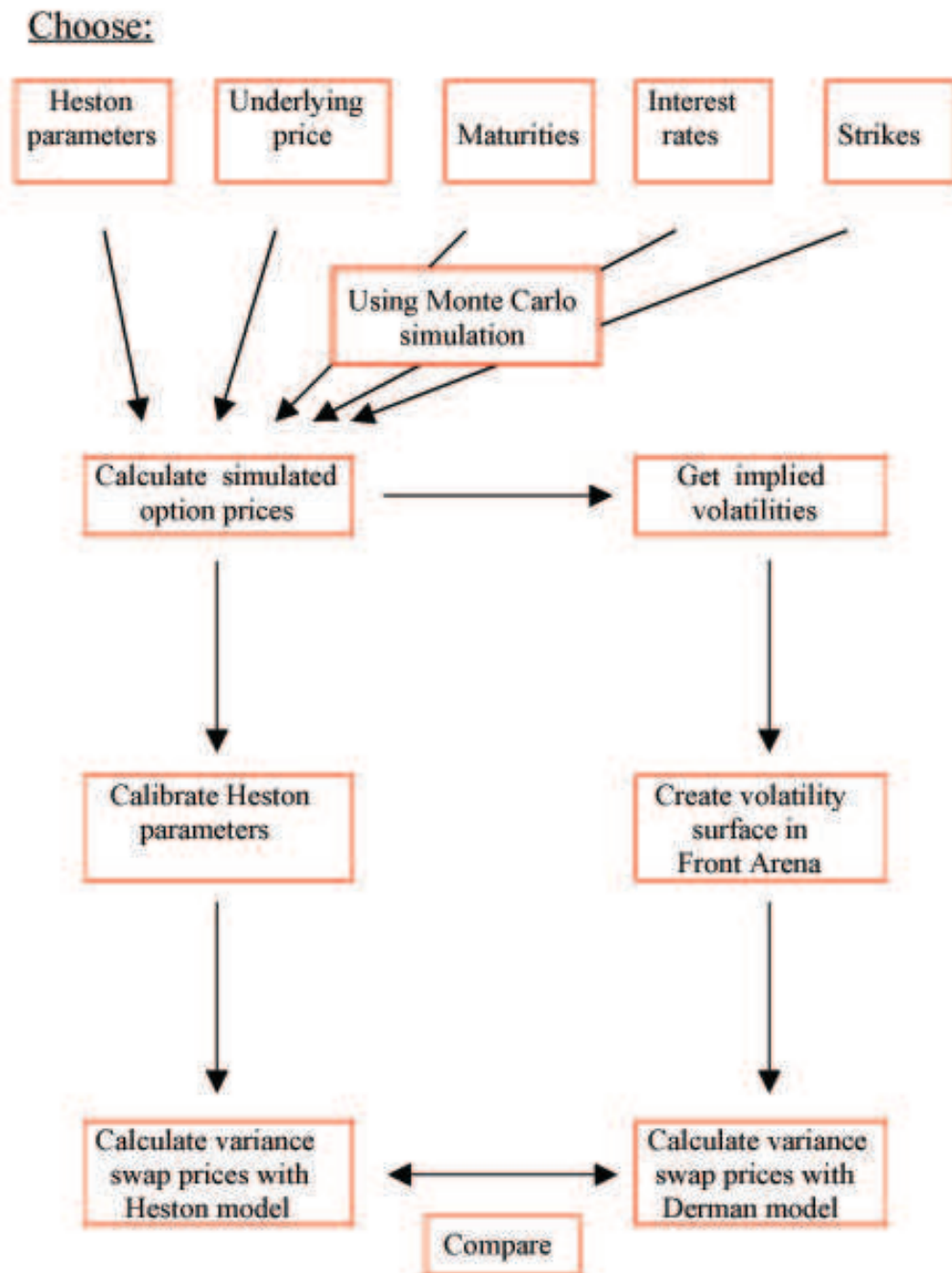


Figure 6.1: Graphical representation of Heston-Derman comparison.

After calibration we got values:

κ	θ	η	ρ	v_0
3.0098	0.0580	0.3204	-0.2640	0.0844

The result of procedure from Figure 6.1 is presented in Figure 6.2. Note that for calculating variance swaps prices with Heston we need just κ , θ and v_0 .

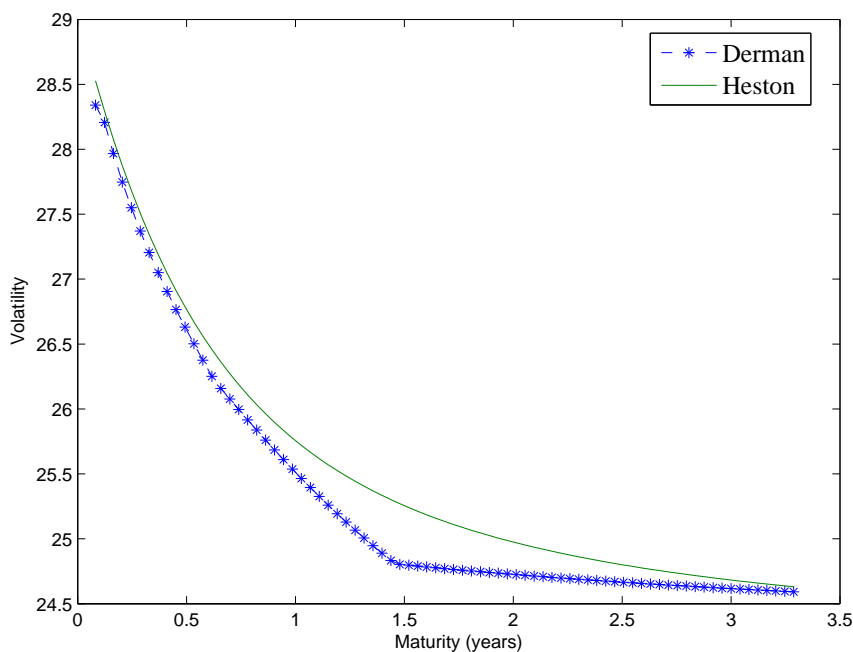


Figure 6.2: Variance swap prices calculated using both Heston and Derman models.

Here the numbers from the models are quite close. However, one has to bear in mind that we used calibrated Heston parameters, and that with different calibrated values the graphs might be fairly different. To show this, we will calculate variance swap prices with real parameters (one that we used for simulation). Plotting this prices together with prices from Figure 6.2 gives Figure 6.3.

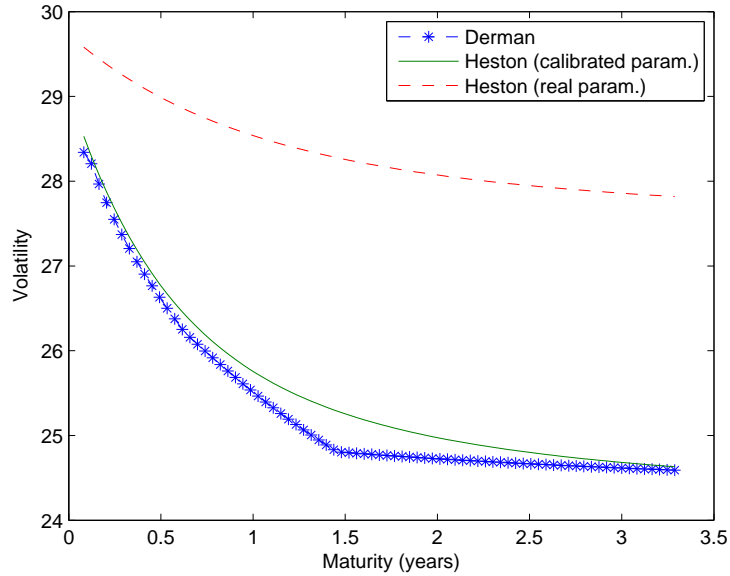


Figure 6.3: Variance swaps prices calculated with the Derman model (3.15) and two sets of parameters for the Heston model: $\kappa = 3.0098$, $v_0 = 0.0844$, $\theta = 0.0580$, and $\kappa = 1.7$, $v_0 = 0.0884$, $\theta = 0.075$.

When valuing variance swaps with the Heston model one important thing to notice is that the value is strongly dependent on the relation between v_0 and θ . The comparison is shown in Figure 6.4. We can see that for $v_0 < \theta$ prices are going up. For $v_0 > \theta$ it is vice versa.

After finishing all the valuations, a natural question that appeared was how to use variance swaps. We looked at market values of *S&P 500* index, and searched for a period where prices were both increasing and decreasing significantly. Therefore we took prices from period 24.07.2006 – 23.07.2008 and calculated corresponding realized variance. The values that we got are shown in Figure 6.5. We can see that for the most of the time when prices of the index are rising, the realized variance is going down, and vice versa. Therefore, a long position in a variance swap can be used to offset losses caused by having a long position in the index.

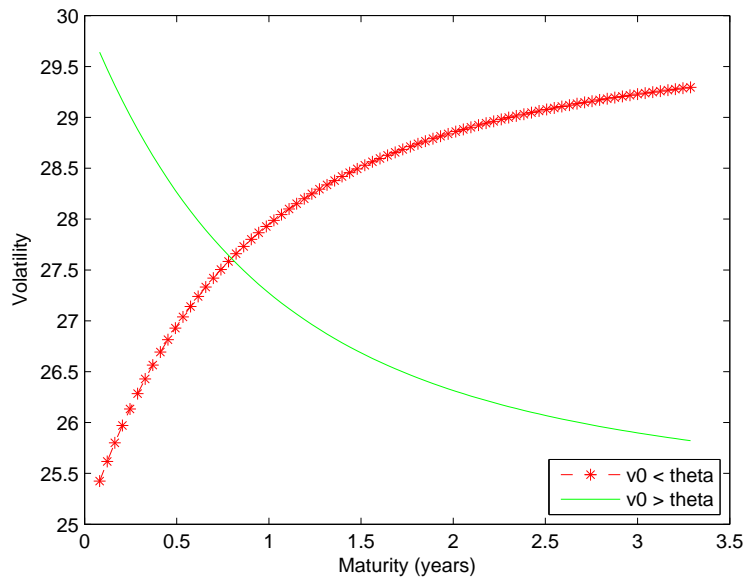


Figure 6.4: Prices of variance swaps with the Heston model (4.10) using two set of parameters: $\kappa = 2$, $v_0 = 0.0625$, $\theta = 0.09$, and $\kappa = 2$, $v_0 = 0.09$, $\theta = 0.0625$.

Assume that one is holding a portfolio consisting of long positions in the index and a variance swap contract on that index. In order to calculate the payoff of that portfolio one has to calculate the fair strike for a variance swap. Since we did not have market prices neither for *S&P 500* nor for any other options we could not get fair strike with our Heston model. So, unfortunately, we were not able to make any further conclusion about this.

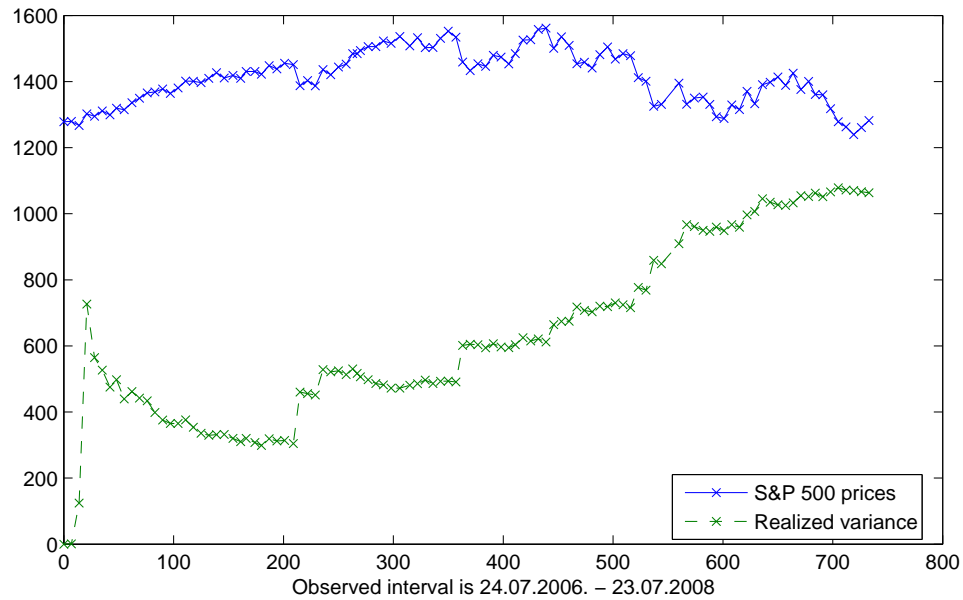


Figure 6.5: Prices and realized variance of S&P 500 index.

6.2 Results of pricing variance swaps using real data

Here we will present some results of pricing variance swaps when the Heston model is calibrated over variance swaps market data. This data is actually data from interbank offers (private communication).

As example we can look at Figures 6.6 and 6.7. On the first figure we took market values for the variance swap on DAX index from 04.06.2008. Those prices were used for calibration of the Heston model. Values that we got after calibration were used for pricing, so we got solid line as a result. Applying the same process on CAC index data yields Figure 6.7.

From the figures we can see that prices calculated with the Heston model fits quite well to the market prices. Of course this was expected since the calibration is done so that it minimizes the difference between model and market values.

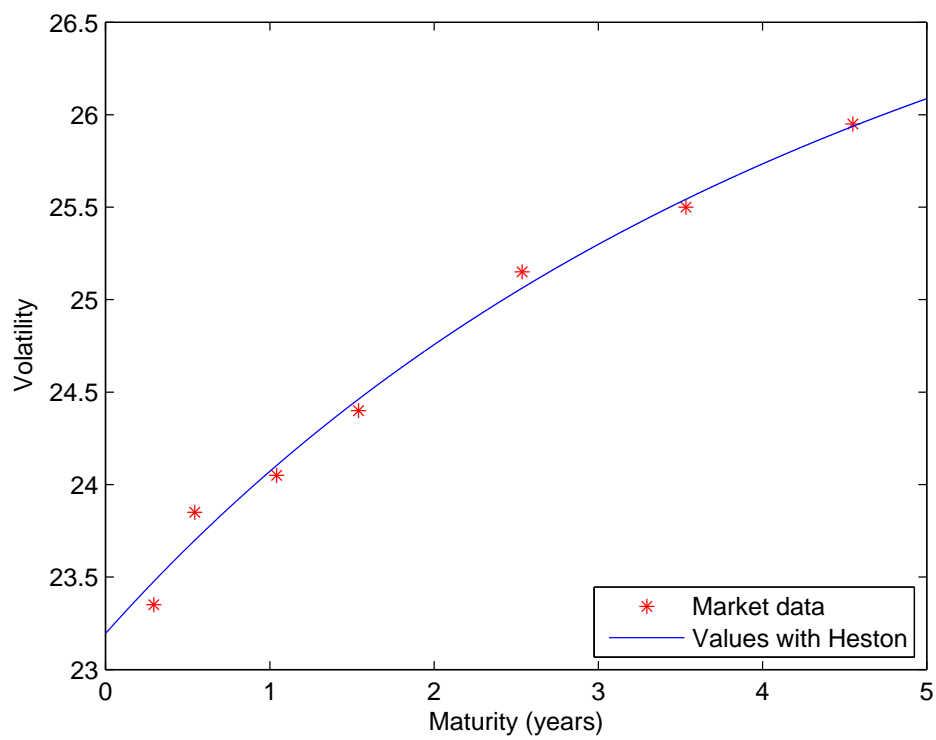


Figure 6.6: DAX variance swaps prices from 04.06.2008 and prices calculated with the Heston model with the parameters $\kappa = 0.3305$, $v_0 = 0.0538$, $\theta = 0.0817$.

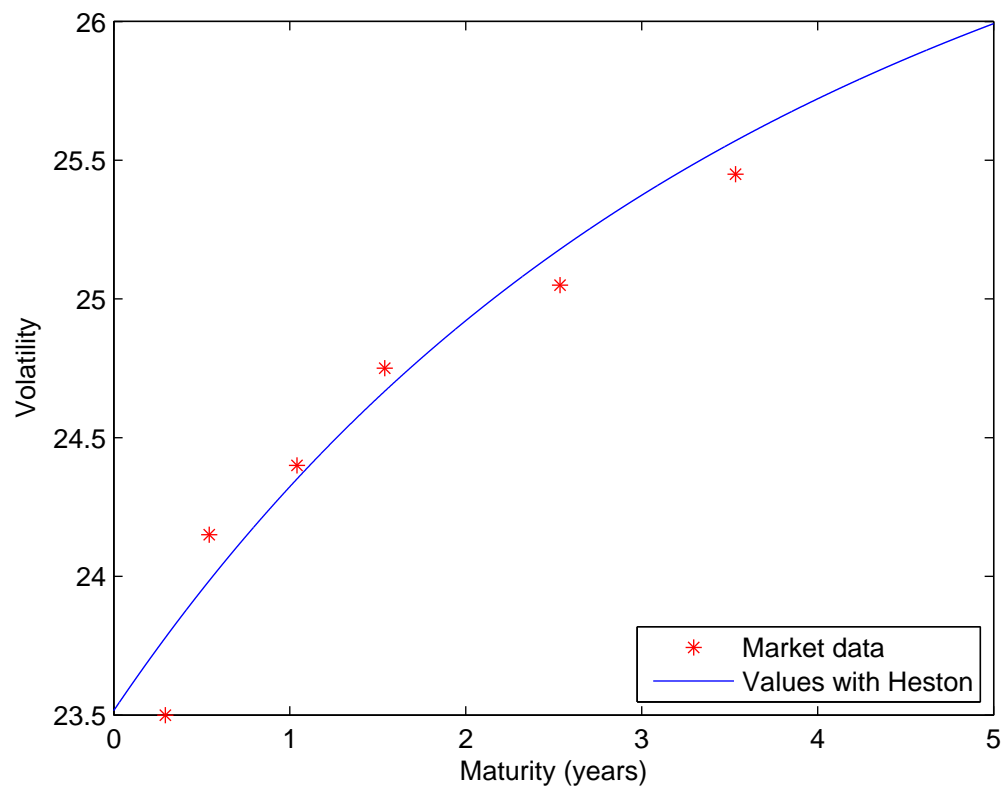


Figure 6.7: CAC variance swaps prices from 04.06.2008 and prices calculated with the Heston model with the parameters $\kappa = 0.4184$, $v_0 = 0.0553$, $\theta = 0.0764$.

Chapter 7

Conclusion

In this thesis we have derived valuation formulas for variance swaps. The replication with a portfolio of vanilla options is theoretically very good, but hard to be implemented (without any changes) in practice. In contrast to this, the formula obtained with the Heston model (4.10) is easily implemented. The accuracy, however, is strongly dependent on the results of calibration.

Like in the case of vanilla options, and some other instruments, the choice of which model to use is up to the investor.

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Appendix

APPENDIX A: *Log payoff replication*

Let f be a twice differentiable function. For any fixed κ $f(S)$ can be written

as

$$\begin{aligned}
 f(S) &= f(\kappa) + \mathbb{1}_{\{S > \kappa\}} \int_{\kappa}^S f'(u) du - \mathbb{1}_{\{S < \kappa\}} \int_S^{\kappa} f'(u) du \\
 &= f(\kappa) + \mathbb{1}_{\{S > \kappa\}} \int_{\kappa}^S \left(f'(\kappa) + \int_{\kappa}^u f''(v) dv \right) du \\
 &\quad - \mathbb{1}_{\{S < \kappa\}} \int_{\kappa}^S \left(f'(\kappa) - \int_u^{\kappa} f''(v) dv \right) du.
 \end{aligned} \tag{7.1}$$

Since $f'(\kappa)$ does not depend on u , using Fubini's theorem we get

$$\begin{aligned}
 f(S) &= f(\kappa) + f'(\kappa) (S - \kappa) + \mathbb{1}_{\{S > \kappa\}} \int_{\kappa}^S \int_v^S f''(v) du dv \\
 &\quad + \mathbb{1}_{\{S < \kappa\}} \int_S^{\kappa} \int_S^v f''(v) du dv.
 \end{aligned} \tag{7.2}$$

Now, integrating the last two terms over u produces

$$\begin{aligned}
 f(S) &= f(\kappa) + f'(\kappa) (S - \kappa) + \mathbb{1}_{\{S > \kappa\}} \int_{\kappa}^S f''(v) (S - v) dv \\
 &\quad + \mathbb{1}_{\{S < \kappa\}} \int_S^{\kappa} f''(v) (v - S) dv \\
 &= f(\kappa) + f'(\kappa) (S - \kappa) + \int_{\kappa}^{\infty} f''(v) (S - v)^+ dv \\
 &\quad + \int_0^{\kappa} f''(v) (v - S)^+ dv
 \end{aligned} \tag{7.3}$$

Using substitutions: $S = S_T$, $\kappa = S_*$, $f(y) = \ln(y)$ we finally obtain

$$\begin{aligned} \ln \frac{S_T}{S_*} &= \frac{S_T - S_*}{S_*} \\ &\quad - \int_{S_*}^{\infty} \frac{1}{v^2} (S - v)^+ dv \\ &\quad - \int_0^{S_*} \frac{1}{v^2} (v - S)^+ dv. \end{aligned} \tag{7.4}$$

which is the formula for the replication of a log payoff.

APPENDIX B: *Replication in practice: calculation of weights*

Consider the payoff

$$f(S_T) = \frac{2}{T} \left[\frac{S_T - S_*}{S_*} - \ln \frac{S_T}{S_*} \right]. \quad (7.5)$$

We are looking for a proper weights of call options with strikes $K_i^c, S_* = K_0^c < K_1^c < K_2^c < K_3^c < \dots < K_m^c$, and put options with strikes K_j^p such that $S_* = K_0^p > K_1^p > K_2^p > K_3^p > \dots > K_n^p$. Let us denote weights with $w(K_i^c)$ for calls, and $w(K_j^p)$ for puts. Our goal is to have a good approximation of $f(S_T)$. We would like to do it with a piece-wise linear function as shown in Figure 7.1.

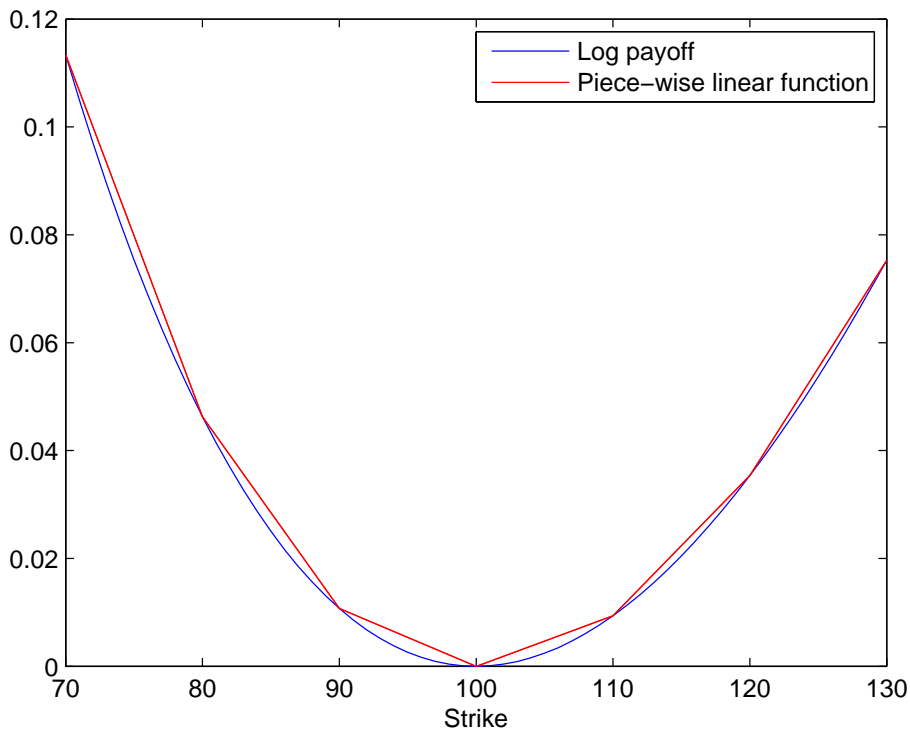


Figure 7.1: Log payoff approximated with piece-wise linear function. Here $K_0 = 100, K_1^c = 110, K_2^c = 120, K_3^c = 130; K_1^p = 90, K_2^p = 80, K_3^p = 70$.

The part from K_0^c to K_1^c is equal to the payoff of $w(K_0^c)$ call options with the strike K_0^c . Hence,

$$w(K_0^c) = \frac{f(K_1^c) - f(K_0^c)}{K_1^c - K_0^c}. \quad (7.6)$$

Similarly, the part from K_1^c to K_2^c can be viewed as a combination of calls with strikes K_0^c and K_1^c . Considering that we already hold a $w(K_0^c)$ calls with strike K_0^c , we get following system to solve for $w(K_1^c)$

$$w(K_1^c)(S - K_1^c) + w(K_0^c)(S - K_0^c) = f(S), \quad \text{for } S = K_0^c, K_1^c. \quad (7.7)$$

Solving the previous system yields

$$w(K_1^c) = \frac{f(K_2^c) - f(K_1^c)}{K_2^c - f(K_1^c)} - w(K_0^c). \quad (7.8)$$

In general, weights for calls are given by

$$w(K_n^c) = \frac{f(K_{n+1}^c) - f(K_n^c)}{K_{n+1}^c - f(K_n^c)} - \sum_{i=0}^{n-1} w(K_i^c), \quad (7.9)$$

and for puts by

$$w(K_n^p) = -\frac{f(K_{n+1}^p) - f(K_n^p)}{K_{n+1}^p - f(K_n^p)} - \sum_{i=0}^{n-1} w(K_i^p). \quad (7.10)$$

APPENDIX C: Derivation of the strike price using the Heston model

Lemma 1 (Itô lemma) Assume that we have a stochastic process X_t defined on a filtered probability space $(\Omega, \mathcal{F}, (F_t)_{t \geq 0}, Pr)$ which satisfies following SDE:

$$dX_t = a(t, \omega) dt + b(t, \omega) dW_t \quad (7.11)$$

where a and b are adapted to F_t , and satisfy following conditions

$$Pr \left(\int_0^t a(s, \omega) ds < \infty \right) = 1,$$

$$Pr \left(\int_0^t b(s, \omega) ds < \infty \right) = 1.$$

Let $F(t, x) : \mathbb{R}^+ \times \mathbb{R} \rightarrow \mathbb{R}$, $F \in C^2$. Then the process $(F(t, X_t))_{t \geq 0}$ has a stochastic differential

$$dF(t, X_t) = \left(\frac{\partial F}{\partial t} + a \frac{\partial F}{\partial X} + \frac{b^2}{2} \frac{\partial^2 F}{\partial X^2} \right) dt + b \frac{\partial F}{\partial X} dW_t. \quad (7.12)$$

As we know in the Heston model (4.3, 4.4) variance satisfies following equation

$$dv_s = \kappa (\theta - v_s) ds + \eta \sqrt{v_s} dW. \quad (7.13)$$

By integrating the previous equation from 0 to t we get

$$\int_0^t dv_s = \int_0^t \kappa (\theta - v_s) ds + \eta \int_0^t \sqrt{v_s} dW. \quad (7.14)$$

After taking the expectation of the equation (7.14) we obtain

$$E(v_t) - v_0 = \kappa \theta t - \kappa E \left(\int_0^t v_s ds \right) + E \left(\eta \int_0^t \sqrt{v_s} dW \right) \quad (7.15)$$

Using the properties of the Brownian motion: $E \left(\int_a^b g(s) dB_s \right) = 0$, and then changing the order of the integral and expectation yields

$$E(v_t) - v_0 = \kappa \theta t - \kappa \int_0^t E(v_s) ds. \quad (7.16)$$

Now, we will use substitution $\mu(t) = E(v_t)$. Therefore, we have

$$\mu(t) - v_0 = \kappa \theta t - \kappa \int_0^t \mu(s) ds. \quad (7.17)$$

Differentiating equation (7.17) in respect to t gives

$$\mu'(t) = \kappa (\theta - \mu(t)). \quad (7.18)$$

Hence, we got the first order linear differential equation which is easily solved

$$\mu(t) = \theta + e^{-\kappa t} c, \quad (7.19)$$

where c is some constant. Using that $\mu(0) = v_0$, and $\mu(t) = E(v_t)$ we finally get

$$E(v_t) = \theta + e^{-\kappa t} (v_0 - \theta).$$