The feedback effects in illiquid markets, hedging strategies of large traders

Master's Thesis in Financial Mathematics

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Preface

This Masters thesis in Financial Mathematics was written at the department of Mathematics, Physics and Electrical engineering at the Halmstad University, Sweden.
I especially want to thank my supervisor of my master project, Ljudmila A. Bordag, who was always there to help and give creative tips, incentives and criticism. Without her this thesis would not be what it is.
I want to thank Dmitri Kristensson, a good Swedish friend;), who read my thesis, corrected it and discussed it with me, which was of great advantage for my thesis progress.
I even want to thank my laptop for it never let me in the lurch and with whom I spent a whole lot of hours during these last weeks.
Finally I wish You a nice reading and hope you will enjoy my work.
Halmstad, 2007.05.29
Abstract
The master thesis is devoted to an analysis of equilibrium or reaction-function models in illiquidity markets of derivatives. The main equation is a nonlinear equation which is a perturbation of Black-Scholes model. By using analytical methods we study invariant and scaling properties for the considered model.
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Chapter 1

Introduction

The financial mathematic is relatively new and developing area of science. One of the first scientists who get the most important analytical results in this area were F. Black, M. Scholes and D. Merton. They described the model defined for the ideal market of securities. They introduced a procedure that allows investors to calculate the ”fair” price of a derivative security which value depends on the value of underlying security. Now their model demands more advanced modifications because of the new kind of traders whose hedging strategies influence the market value of securities, for example, the programme traders.

Considering the influence of the feedback effect of a hedging strategy for market prices, we get one of the Black-Scholes model modifications - an equilibrium or reaction-function model - which was described by P. Sircar and G. Papanicolaou [2], P. Frey and A. Stremme [4].

The equation, which describes this model, concerns to the less investigated type of equations. The nonlinearity of this partial differential equation includes the first and the second spatial derivatives and a small parameter. This small parameter measured the influence of an program trader on an option price.

To get understanding of this equation we can use different mathematical methods. In this project we will use the method of Lie groups analysis. This method is sufficiently widespread and used in different research areas. For example, in works of L. A. Bordag [8], [9], this method was used for studying of financial models, in G. I. Barenblat[15] for equations of mathematical physics, in S. V. Spichak and V. I. Stogniy [13], V.V. Sokolov [14], M.L. Grandarias, P. Venero and J. Ramirez [16], P.A. Clarkson and E.L. Mansfield [17] for modifications of the heat equation and others.

In this paper there are six Chapters. The first Chapter is Introduction. In the second Chapter the main model and it is deducing is described. Be-
cause the model were originally introduced by Sircar and Papanicolaou [2] in
detail, in this Chapter just the brief review is stated. In the third Chapter
the Lie group analysis is described. This theory is important for the study
of equation (2.6) which is described in Chapter two. In the forth Chapter
the construction of the complete Lie algebra admitted by equation (2.6) is
made and the main results are formulated in form of a theorem (Theorem
\textbf{(4.0.3)}). The fifth Chapter is devoted to finding of invariants and to re-
duction of the ordinal partial differential equation to a non linear ordinary
differential equation. In the sixth Chapter, which is the last one, the results
are represented and discussed. The bibliography and glossary are included.
Chapter 2

A short introduction in the Sircar-Papanicolaou model

In this Chapter we describe and derive a nonlinear partial differential equation of the derivative price, and the hedging strategy in case of the equilibrium or the reaction-function model according to the article by K. Ronnie Sircar and George Papanicolaou [2].

Sircar and Papanicolaou [2] write that there are two investment opportunities on the market

- a risk-free money market account \((B)\),
- an asset \((S)\), which is a stock value,

where \(B\) is perfectly liquid investment opportunity and \(S\) is an illiquid asset. In the model the money market account is used as the \textit{numeraire}. It means that \(B_t \equiv 1\), where \(S\) is the forward price of the stock, and the interest rate can be regarded as equal to zero.

Before we get into details of the reaction-function model, it is important to define two types of traders which we will use in the following context.

The first type of traders are reference traders, who are the majority investing in the asset expecting gain. Such traders are ”price-takers”. It means that they can not influence the price of asset on the market. They trade in such a way that the equilibrium asset price follows exactly equation

\[
dS_t = \nu(S_t, t)dt + \lambda(S_t, t)dW_t,
\]

where \(\{W_t, t\}\) is a standard Brownian motion on a filtered probability space \((\Omega, F, (F_t)_{0 \leq t \leq T}, P)\).

In this equation the reference traders distribution of the wealth and the equilibrium asset price modeled by an Itô process \((S_t, t > 0)\) are independent
of each other. We can consider all reference traders as a single aggregate reference trader who represents a total action of all reference traders. Two limitations for the aggregate reference trader there exist

- An aggregate stochastic income, which is the total income of the all reference traders, can be described by an equation:

\[
dY_t = \mu(Y_t, t)dt + \eta(Y_t, t)dW_t,
\]

where \(\{W_t\}\) is a Brownian motion. Functions \(\mu(s, y)\) and \(\eta(s, y)\) will not appear in our model, so that income process need not be known in our model.

- A demand function \(\tilde{D}(S_t, Y_t, t)\) is a function of the income and of the equilibrium price process.

The second type of traders are programme or large traders, who trade the asset following a Black-Scholes type dynamic hedging strategy. The main reason for trading of the assets is to hedge own position against the risk. The programme traders are large enough to change the price of the asset corresponding to their own trading strategy.

The demand function for a large trader can be represent by following equation

\[
\phi = \varsigma \Phi,
\]

where \(\varsigma\) is the volume of options being hedged and \(\Phi\) is a smooth function, which represents the demand per security being hedged.

After we have defined the reference traders and the programme traders, we can define their total relative demand. Sircar and Papanicolaou [2] consider that the price process of the asset is determined by a market equilibrium and by the income \((Y_t)\). They make an assumption: the supply of the asset \(S_0\) is constant and \(D\) is the demand of the reference traders relative to the supply \(S_0\) (it means \(\tilde{D}(s, y, t) = S_0D(s, y, t)\)).

The relative demand function of both kinds of traders at time \(t\) has the following presentation

\[
G(s, y, t) = D(s, y, t) + \rho \Phi(s, t),
\]  

(2.1)

where \(\rho\) is equal to the ratio of the volume of options being hedged to the total supply of the asset.

By setting demand and supply to 1 at each point in time, market equilibrium will be equal to 1

\[
G(S_t, Y_t, t) \equiv 1.
\]

(2.2)
By determining the relationship between $S_t$ and $Y_t$ in the function

$$G(S_t, Y_t, t) \equiv 1,$$

We assume that the function smooth satisfies the conditions of the implicity function theorem.

In this case we obtain that $S_t = \Psi(Y_t, t)$, where $\Psi$ is a some smooth function.

This smooth function $\Psi$ defines that the process $S_t$ follows to the same Brownian motion as the process $Y_t$.

Sircar and Papanicolaou [2] shows that by using the Itô lemma, we can obtain the stochastic process

$$dS_t = \left[ \mu(Y_t, t) \frac{\partial \Psi}{\partial y} + \frac{\partial \Psi}{\partial t} + \frac{1}{2} \eta^2(Y_t, t) \frac{\partial^2 \Psi}{\partial y^2} \right] dt + \eta(Y_t, t) \frac{\partial \Psi}{\partial y} dW_t.$$ 

After some transformations the asset price process $S_t$ has the following representation

$$dS_t = \alpha(S_t, Y_t, t) dt + \nu(S_t, Y_t, t) \eta(Y_t, t) dW_t,$$

where the adjusted volatility is equal to

$$\nu(S_t, Y_t, t) = -\frac{D_y(S_t, Y_t, t)}{D_s(S_t, Y_t, t) + \rho \Phi_s(S_t, t)},$$

and the adjusted drift is given by

$$\alpha(S_t, Y_t, t) = -\left( \mu \frac{G_y}{G_s} + \frac{G_i}{G_s} + \frac{1}{2} \eta^2 \left[ \frac{G_{yy}}{G_s} - 2 \frac{G_{sy}}{G_s} + \frac{G^2_y G_{ss}}{G^3_s} \right] \right).$$

To cover the risk of the derivative, programme traders can sell or buy the asset. How much of the asset they should trade depends on the change of the derivative value in comparison to the asset value. An expression for $\Phi$ in form of the price of the derivative follows as the result of this assumption. Summing up the usual arguments, we suppose the price of the derivative is given by $u(s, t)$, where $u(S, t)$ is a some sufficient smooth function.

We have

$$\frac{\partial u}{\partial t} + \frac{1}{2} (\nu \eta)^2 \frac{\partial^2 u}{\partial s^2} + r \left( s \frac{\partial u}{\partial s} - u \right) = 0.$$ 

By using (2.3) and $\Phi(S_t, t) = u_s$, and $r = 0$ we obtain following equation on the function $u(S, t)$

$$\frac{\partial u}{\partial t} + \frac{1}{2} \eta^2 \left( \frac{D_y(S_t, Y_t, t)}{D_s(S_t, Y_t, t) + \rho \Phi_s(S_t, t)} \right)^2 \frac{\partial^2 u}{\partial s^2} = 0.$$ 

Chapter 2. A short introduction in the Sircar-Papanicolaou model

Proposition 1 If the income process \( Y_t \) is a geometric Brownian motion with the volatility parameter \( \eta \), then the feedback model reduces to the classical Black-Scholes model in the absence of programme traders if and only if the demand function is of the form

\[
D(s, t) = U \left( \frac{y\gamma}{s} \right)
\]

for some smooth increasing function \( U(\cdot) \), where \( \gamma = \frac{\sigma}{\eta} \).

The proof the Proposition 1 lead us to our main model. Thus, we have to consider the part of the proof that is interesting for us.

The geometric Brownian motion for \( Y_t \) is given by

\[
dY_t = \mu_1 Y_t dt + \eta_1 Y_t dW_t.
\]

\( \rho \Phi = 0 \) means programme traders are not on the market.

From the classical Black-Scholes equation we obtain that \( \lambda(s, t) \equiv \sigma s \).

Hence,

\[
\frac{1}{2} \eta_1^2 y^2 \left[ \frac{D_y}{D_s} \right]^2 = \frac{1}{2} \sigma^2 s^2.
\]

Here \( D \) must satisfy to the condition

\[
\frac{D_y}{D_s} = -\frac{\gamma s}{y}.
\]  \( (2.4) \)

The left hand side of (2.4) is negative under our hypothesis for the rational trading.

Thus, \( D(s, y) = U(y^\gamma/s) \) is the general solution of this partial differential equation for any differentiable function \( U(\cdot) \). Finally, if we directly differentiate the equation, we get \( D_s < 0 \) and \( D_y > 0 \) for \( s, y > 0 \) if and only if \( U'(\cdot) > 0 \).

Using definition (see Proposition 1) of a demand function, we can derive the pricing equation under the feedback effects. The diffusion coefficient is equal to

\[
\frac{1}{2} \eta_1^2 y^2 \left[ \frac{D_y}{D_s + \rho \Phi_s} \right]^2 = \frac{1}{2} \eta_1^2 y^2 \left[ \frac{\gamma s^{\gamma-1} U'(y^\gamma/s)}{y s^{\gamma}} U'(y^\gamma/s) - \rho \Phi_s \right]^2 = \frac{1}{2} \sigma^2 s^2 \left[ \frac{y^\gamma U'(y^\gamma/s)}{y s^{\gamma}} U'(y^\gamma/s) - \rho s \Phi_s \right]^2.
\]
By using (2.1) and (2.2) we obtain
\[ U \left( \frac{Y_t}{S_t} \right) = 1 - \rho \Phi(S_t, t). \]

It means that we eliminate the variable \( y \).

Let \( V(\cdot) \) be the inverse function of \( U(\cdot) \), whose existence is guaranteed by the strict monotonicity of \( V \). Then we get
\[ \frac{y^2}{s} = V(1 - \rho \Phi) \Rightarrow \frac{1}{2} \sigma^2 s^2 \left[ \frac{V(1 - \rho \Phi)U''(V(1 - \rho \Phi))}{V(1 - \rho \Phi)U'(V(1 - \rho \Phi)) - \rho s \Phi_s} \right]^2. \quad (2.5) \]

After that we made some transformations of the diffusion coefficient (2.5) and we obtain a family of nonlinear feedback pricing equations that are consistent with the classical Black-Scholes equation
\[ u_t + \frac{1}{2} \sigma^2 s^2 u_{ss} \left[ \frac{V(1 - \rho u_s)U'(V(1 - \rho u_s))}{V(1 - \rho u_s)U'(V(1 - \rho u_s)) - \rho s u_{ss}} \right]^2 = 0. \]

In our case we consider \( V \) as the linear function
\[ U(z) = \beta z, \quad \beta > 0 \]

to conclude qualitative properties of the feedback effect from the programme trading. Thus, we get the equation which describes the value of the hedging strategy for a programme trader
\[ u_t + \frac{1}{2} \sigma^2 s^2 u_{ss} \left[ \frac{1 - \rho u_s}{1 - \rho u_s - \rho s u_{ss}} \right]^2 = 0. \quad (2.6) \]
Chapter 2. A short introduction in the Sircar-Papanicolaou model
Chapter 3
The Short introduction to the Lie method

3.0.1 The basic knowledge

In this section we introduce a base knowledge of the Lie group analysis for investigation of the symmetry properties of partial differential equations.

Consider the Hausdorff space $M$, an open set $U$ in $M$ and a homomorphism $\varphi : U \rightarrow \mathbb{R}^m$ which is called map on the manifold $M$.

**Definition 3.0.1** An $m$-dimension manifold is a set $M$ together with countable collection of subsets $U_\alpha \subset M$, called coordinate charts and one-to-one functions $\varphi_\alpha : U_\alpha \rightarrow V_\alpha$. $V_\alpha$ are connected open subsets, $V_\alpha \subset \mathbb{R}^m$, called local coordinate maps which have the following properties

1. $\bigcup \alpha U_\alpha = M$,

2. for all $U_\alpha$ and $U_\beta$

   $\varphi_\beta \circ \varphi^{-1}_\alpha : \varphi_\alpha(U_\alpha \cap U_\beta) \rightarrow \varphi_\beta(U_\alpha \cap U_\beta)$

   for all $\alpha$, $\beta$, they are smooth functions,

3. If $x \in U_\alpha$ and $\bar{x} \in U_\beta$ then there exist open subset $W \subset V_\alpha$ and $\bar{W} \subset V_\beta$ with $\varphi_\alpha(x) \in W$ and $\varphi_\beta(\bar{x}) \in W$ which satisfied the following property

   $\varphi^{-1}_\alpha(W) \cap \varphi^{-1}_\beta(\bar{W}) = \emptyset$.

We introduce a definition of a Lie group:
**Definition 3.0.2** An r-parametric Lie group is a group \( G \) which also carries the structure of an r-dimensional smooth manifold in such a way that both the group operation

\[
m : G \times G \to G, \quad m(g, h) = g \circ h \quad g, h \in G
\]

and the inversion

\[
i : G \to G \quad i(g) = g^{-1}, \quad g \in G
\]

are smooth maps between manifolds.

**Definition 3.0.3** An r-parametric local Lie group consist of connected open subsets \( V_0 \subset V \subset \mathbb{R}^r \) containing coordinate origin, and smooth maps

\[
m : V \times V \to \mathbb{R}^r
\]

defining the group operation and a smooth map \( i : V_0 \to V \) - defining the group inversion with the following properties

1. If \( x, y, z \in V \) and \( m(x, y) \), \( m(y, z) \in V \) then

\[
m(x, m(y, z)) = m(m(x, y), z),
\]

2. for all \( x \in V \)

\[
m(0, x) = x = m(x, 0),
\]

3. for all \( x \in V_0 \)

\[
m(x, i(x)) = 0 = m(i(x), x).
\]

In practice we deal often with groups of transformations on some manifold \( M \). If to each group element, \( g \in G \), there is associated a map from \( M \) to itself, then a Lie group \( G \) will be represent as a group of transformations of the manifold \( M \).

**Definition 3.0.4** Let \( M \) be a smooth manifold. A local group of transformations acting on \( M \) is given by a (local) Lie group \( G \), an open subset \( U \), with \( \{ e \} \times M \subset U \subset G \times M \), definition of the group action, and a smooth map \( \Psi : U \to M \) with the following properties

- if \( (h, x) \in U, (g, \Psi(h, x)) \in U \), and also \( (g \cdot h, x) \in U \), then \( \Psi(g, \Psi(h, x)) = \Psi(g \cdot h, x) \),
- for all \( x \in M \), \( \Psi(e, x) = x \),
- if \( (g, x) \in U \), then \( (g^{-1}, \Psi(g, x)) \in U \) and \( \Psi(g^{-1}, \Psi(g, x)) = x \).
Definition 3.0.5 An orbit of a local transformation group is a minimal nonempty group invariant subset of the manifold \( M \). In other words, \( \mathcal{O} \subset M \) is an orbit provided it satisfies the conditions

- if \( x \in \mathcal{O} \), \( g \in G \) and \( g \cdot x \) is defined than \( g \cdot x \in \mathcal{O} \),
- if \( \bar{\mathcal{O}} \subset \mathcal{O} \) and \( \mathcal{O} \) satisfies first condition than \( \bar{\mathcal{O}} = \mathcal{O} \) or \( \bar{\mathcal{O}} \) is empty.

We use further notations similar that in the book P. J. Olver [19]. A curve \( C \) on a smooth manifold \( M \) is parametrized by a smooth map \( \varphi : I \to M \), where \( I \) is subinterval of \( \mathbb{R} \). In local coordinates, \( C \) is defined by \( m \) smooth functions

\[
\varphi(\epsilon) = (\varphi^1(\epsilon), \ldots, \varphi^m(\epsilon)), \quad \epsilon \in I.
\]

The curve \( C \) has a tangent vector at each point \( x = \varphi(\epsilon) \). That means

\[
\dot{\varphi}(\epsilon) = \frac{d\varphi}{d\epsilon} = (\dot{\varphi}^1(\epsilon), \ldots, \dot{\varphi}^m(\epsilon)), \text{is well defined.}
\]

By adopting the notation for tangent vectors to \( C \) at \( x = \varphi(\epsilon) \)

\[
V|_x = \dot{\varphi}(\epsilon) = \dot{\varphi}^1(\epsilon) \frac{\partial}{\partial x^1} + \ldots + \dot{\varphi}^m(\epsilon) \frac{\partial}{\partial x^m}.
\]

By \( TM|_x \) we denote the tangent space to \( M \) at \( x \). The tangent space to \( M \) at \( x \) means the set of all tangent vectors to all possible curves passing through a given point \( x \) in \( M \). \( M \) is an \( m \)-dimensional manifold, then \( TM|_x \) is an \( m \)-dimensional vector space, with \( (\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^m}) \) as a basis of \( TM|_x \) in the given local coordinates.

We define the tangent bundle of \( M \) as the collection of all tangent spaces corresponding to all points \( x \) in \( M \). We denote the tangent bundle by

\[
TM = \bigcup_{x \in M} TM|_x.
\]

According to P. J. Olver [19] a vector field \( V \) on \( M \) assigns a tangent vector \( V|_x \in TM|_x \) to each point \( x \in M \), with \( V|_x \) varying smoothly from point to point. In local coordinates \( (x^1, \ldots, x^m) \), a vector field has the form

\[
V|_x = \xi^1(x) \frac{\partial}{\partial x^1} + \ldots + \xi^m(x) \frac{\partial}{\partial x^m},
\]

where each \( \xi^i(x) \) is a smooth function of \( x \). An integral curve of a vector field \( V \) is a smooth parametrized curve \( x = \varphi(\epsilon) \) whose tangent vector at any point coincides with the value of \( V \) at the same point:

\[
\dot{\varphi}(\epsilon) = V|_{\varphi(\epsilon)}
\]
for all $\epsilon$. In local coordinates,

$$x = \varphi(\epsilon) = (\varphi^1(\epsilon), \ldots, \varphi^m(\epsilon))$$

is a solution to an autonomous system of ordinary differential equations (so called Lie equations)

$$\frac{dx^i}{d\epsilon} = \xi^i(x), \quad i = 1, \ldots, m,$$

where the $\xi^i(x)$ are the coefficients of $V$ at $x$. If the functions $\xi^i(x)$ are smooth, the standard existence and uniqueness theorems for the system of ordinary differential equations guarantee that there exists a unique solution for each set of the initial data

$$\varphi(0) = x_0.$$

This in turn implies the existence of a unique maximal integral curve $\varphi : I \to M$ passing through a given point $x_0 = \varphi(0) \in M$, where “maximal” means that it is not contained in any longer integral curve.

**Definition 3.0.6 (flow)** If $V$ is a vector field, we denote the parametrized maximal integral curve passing through $x$ in $M$ by $\Psi(\epsilon, x)$ and call $\Psi$ the flow generated by $V$. Thus for each $x \in M$, and $\epsilon$ in some interval $I_x$ containing 0, $\Psi(\epsilon, x)$ will be a point on the integral curve passing through $x$ in $M$. The flow of a vector field has the basic properties:

$$\Psi(\delta, \Psi(\epsilon, x)) = \Psi(\delta + \epsilon, x), \quad x \in M$$

for all $\delta, \epsilon \in \mathbb{R}$ such that both sides of equation are defined and

$$\Psi(0, x) = x,$$

and

$$\frac{d}{d\epsilon} \Psi(\epsilon, x) = V_{\Psi(\epsilon, x)}$$

for all $\epsilon$.

### 3.0.2 The associated Lie algebra

**Definition 3.0.7** A Lie algebra over $P(= \mathbb{R})$ is a vector (linear) space $J$ together with a bilinear operation $[\cdot, \cdot] : J \times J \to J$ called the Lie bracket (Lie multiplication) for $J$ satisfying following axioms
• bilinearity

\[ [v + \tilde{c}v, w] = c[v, w] + \tilde{c}[\tilde{v}, w], \quad [v, cw + \tilde{c}w] = c[v, w] + \tilde{c}[v, \tilde{w}] \]

\[ w, \tilde{w}, v, \tilde{v} \in J; \quad c, \tilde{c} \in P(= \mathbb{R}, \mathbb{C}), \]

• skew-symmetry

\[ [v, w] = -[w, v], \]

• Jacobi identity

\[ [v, [\nu, w]] + [w, [v, \nu]] + [\nu, [w, v]] = 0 \]

for all \( \nu, v, w \in J \).

If \( P = \mathbb{R}(\mathbb{C}) \) then \( J \) is called real (complex) Lie algebra.

In local coordinates a Lie bracket has the following form

\[ [v, w] = \sum_{i=1}^{m} \sum_{j=1}^{m} \left( \xi^i(x) \frac{\partial \eta^j}{\partial x^j} - \eta^i(x) \frac{\partial \xi^j}{\partial x^j} \right) \cdot \frac{\partial}{\partial x^i}, \]

where

\[ v = \sum_{i=1}^{m} \xi^i(x) \frac{\partial}{\partial x^i}, \quad w = \sum_{i=1}^{m} \eta^i(x) \frac{\partial}{\partial x^i}. \]

**Definition 3.0.8** A Lie algebra \( J \) is called Abelian (or commutative) algebra if \( [v, w] = 0 \) for all \( v, w \in J \).

In table form we can simple demonstrate the structure of a given Lie algebra. The commutator table for \( g \) will be the \( r \times r \) table whose \((i,j)\)-th entry expresses the Lie bracket \([v_i, v_j]\), if \( g \) is an \( r \)-dimensional Lie algebra, and \( v_1, \ldots, v_r \) form a basis for \( g \). It is important to note that the table is skew-symmetric since \([v_i, v_j] = -[v_j, v_i]\).

**Definition 3.0.9** A Lie algebra \( L_r, r < \infty \) is said to be solvable if there is a sequence (chain)

\[ L_1 \subset \ldots \subset L_{r-1} \subset L_r \]

of subalgebras of the respective dimension \( r, r - 1, \ldots, 1 \) such that \( L_k \) is an ideal in \( L_{k+1} \) where \( 1 \leq k \leq r - 1 \).

**Definition 3.0.10** An element \( V \in L \) is called a central element if \([v, \nu] = 0\) for any \( \nu \in L \). The union \( Z \) of all central elements is called the center of the Lie algebra \( L \).
3.0.3 A symmetry groups admitted by a differential equation

**Definition 3.0.11** Let \( G \) be a local group of transformations acting on a manifold \( M \). A subset \( S \in M \) is called \( G \)-invariant, and \( G \) is called a symmetry group of \( S \), if whenever \( x \in S \), and \( g \in G \) is such that \( g \cdot x \) is defined, then \( g \cdot x \in S \).

**Definition 3.0.12** Let \( G \) be a local group of transformations acting on a manifold \( M \). A function \( F : M \rightarrow N \), where \( N \) is another manifold, is called a \( G \)-invariant function if for all \( x \in M \) and all \( g \in G \) such that \( g \cdot x \) is defined,

\[
F(g \cdot x) = F(x).
\]

A real-valued \( G \)-invariant function \( \zeta : M \rightarrow \mathbb{R} \) is called an invariant of \( G \).

Consider a system \( S \) of differential equations involving \( p \) independent variables \( x = (x^1, \ldots, x^p) \), and \( q \) dependent variables \( u = (u^1, \ldots, u^q) \). We denote a solution of the system by

\[
u^\alpha = f^\alpha(x^1, \ldots, x^p), \quad \alpha = 1, \ldots, q.
\]

\( \mathbb{R}^p \) represent the space of the independent variable, and \( \mathbb{R}^q \) represent the space of the dependent variables. A symmetry group of the system \( S \) will be a local group of transformations, \( G \), acting on some open subset \( M \subseteq \mathbb{R}^p \times \mathbb{R}^q \) in such a way that “\( G \) transforms solutions of \( S \) to other solutions of \( S \)”. (The arbitrary nonlinear transformations of the both independent and dependent variables is included in this definition of symmetry).

**Definition 3.0.13** Let \( S \) be a system of differential equations. A symmetry group of the system \( S \) is a local group of transformations \( G \) acting on an open subset \( M \) of the space of independent and dependent variables for the system \( S \) with the property that whenever \( u = f(x) \) is a solution of \( S \), and whenever \( g \cdot f \) is defined for \( g \in G \), then \( u = g \cdot f(x) \) is also a solution of the system. (By solution we mean any smooth function \( u = f(x) \) defined on any sub-domain \( \Omega \subseteq X \), which satisfies the system \( S \).)

**Definition 3.0.14** Let \( J = (j_1, \ldots, j_k) \) is an unordered \( k \)-tuple multi-index of integers, with entries \( j_k \), \( 1 \leq j_k \leq p \). The order of such a multi-index which we denote by \( \#J = |J| \equiv \) is defined by \( \#J = \#j_1 + \ldots + \#j_k \)

Let us define the spaces \( U_i \) and \( U^{(n)} \). We define the spaces \( U_i \) as the spaces of the partial derivatives of the function \( u \) of the order \( i \). It means

\[
U_i = \{ \omega = \partial f u = \frac{\partial^k u}{\partial x^{j_1} \partial x^{j_2} \ldots \partial x^{j_k}}, \quad |J| = i \}.
\]

We define the space \( U^{(n)} \) as \( U^{(n)} = U_1 \times U_2 \times \ldots \times U_n \).
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Definition 3.0.15 The space $\mathbb{R}^p \times U^{(n)}$ denoted by $M^{(n)}$, whose coordinates represent the independent variables, the dependent variables and the derivatives of the dependent variables up to the order $n$ with the natural contact structure is called the $n$-th order jet bundle of the base space $M$.

Definition 3.0.16 For smooth function $u = f(x)$, with $f : \mathbb{R}^p \rightarrow \mathbb{R}^q$ and components $f^i, \ i = 1, \ldots, q$, we define the $n$-th prolongation of $f$ to be

$$\text{pr}^{(n)} f : X \rightarrow U^{(n)},$$

given by $u^{(n)} = \text{pr}^{(n)} f(x)$, $u^{(n)}_i = \partial f^a(x)$, where $J$ is a multi-index running over the space of all possible derivatives.

A system $\mathcal{S}$ of $n$-th order differential equations is given as a system of equations

$$\Delta_\nu(x, u^{(n)}) = 0, \ \nu = 1, \ldots, l.$$ 

It involves $x = (x^1, \ldots, x^p)$, $u = (u^1, \ldots, u^q)$ and the derivatives $u^{(i)}$, $i = 1, \ldots, l$ of $u$.

Similar to P. J. Olver [19] we assume that the functions

$$\Delta_\nu(x, u^{(n)}) = (\Delta_1(x, u^{(n)}), \ldots, \Delta_l(x, u^{(n)}))$$

are smooth in their arguments, so $\Delta$ is a smooth map from the jet bundle $\mathbb{R}^p \times U^{(n)}$ to some $l$-dimensional Euclidean space

$$\Delta : X \times U^{(n)} \rightarrow \mathbb{R}^l.$$ 

The equality $\Delta = 0$ determines a sub variety

$$\mathcal{J}_\Delta = \{(x, u^{(n)})|\Delta(x, u^{(n)}) = 0\} \subset M^{(n)}$$

of the jet bundle $M^{(n)}$.

Now let $M \in \mathbb{R}^p \times U$ be open. A symmetry group of $\Delta_\nu$ is a local group of transformations $G$ acting on $M$ such that when $u = f(x)$ solves $\Delta_\nu$, then $u = g \cdot f(x)$ solves $\Delta_\nu$ for all $g \in G$ where defined. Let $X$ be a vector field on $M$, and assume $X$ infinitesimally generates the symmetry group $G$ of $\Delta_\nu$. Then by projecting $X$ into $M$ via the exponential map, we may construct a local one-parameter group $\exp(\epsilon X)$. We may then define the prolongation of $X$ as

$$\text{pr}^{(n)} X = \frac{d}{d\epsilon} \text{pr}^{(n)} [\exp(\epsilon X)] (x, u^{(n)})|_{\epsilon = 0}$$

where $\exp$ is the exponential map. We also defined Jacobi matrix of $\Delta_\nu$ to be

$$J_{\Delta_\nu}(x, u^{(n)}) = \begin{pmatrix} \partial \Delta_\nu / \partial x^i & \partial \Delta_\nu / \partial u^a_j \end{pmatrix}.$$
and say $\Delta_\nu$ is maximal if the rank of $J_{\Delta_\nu}$ is $l$. The following theorem constrains the form of coefficients of the $n$-th prolongation of an infinitesimal generator for the symmetry group.

**Theorem 3.0.1 (Symmetry Group)** Let $M$ be an open subset of $X \times U$ and suppose $\Delta(x,u^{(n)})$ is an $n$-th order system of differential equations defined over $M$, with corresponding sub variety $J_\Delta \subset M^{(n)}$. Suppose $G$ is a local group of transformations acting on $M$ whose prolongation leaves $J_\Delta$ invariant, meaning that whenever $(x,u^{(n)}) \in J_\Delta$, we have $pr^{(n)} g \cdot (x,u^{(n)}) \in J_\Delta$ for all $g \in G$ where defined. Then $G$ is a symmetry group of the system of differential equations.

**Definition 3.0.17** Let

$$
\Delta_\nu(x,u^{(n)}) = 0, \quad \nu = 1, \ldots, l
$$

be a system of differential equations. The system is said to be of the maximal rank if the $l \times (p + qp^n)$ Jacobian matrix

$$
J_\Delta(x,u^{(n)}) = \left(\begin{array}{c}
\frac{\partial \Delta_\nu}{\partial x^i}, \frac{\partial \Delta_\nu}{\partial u^\alpha_J}
\end{array}\right)
$$

of $\Delta$ with respect to all the variables $(x,u^{(n)})$ is of rank $l$ whenever

$$
\Delta(x,u^{(n)}) = 0.
$$

The general prolongation formula, is given by the following definition.

**Definition 3.0.18 (The general prolongation formula)** Let

$$
X = \sum_{i=1}^p \xi^i(x,u) \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^q \varphi_\alpha(x,u) \frac{\partial}{\partial u^\alpha}
$$

be a vector field defined on an open subset $M \subset X \times U$. The $n$-th prolongation of $X$ is the vector field

$$
pr^{(n)} X = X + \sum_{\alpha=1}^q \sum_J \varphi_\alpha^J(x,u^{(n)}) \frac{\partial}{\partial u^\alpha_J}
$$

defined on the corresponding jet bundle $M^{(n)} = X \times U^{(n)}$, the second summation being over all (unordered) multi-indices $J = (j_1, \ldots, j_k)$ with $1 \leq
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The coefficient functions $\varphi^j_\alpha$ of $\text{pr}^{(n)}X$ are given by the following formula

$$
\varphi^j_\alpha(x, u^{(n)}) = D_J(\varphi_\alpha - \sum_{i=1}^{p} \xi^i u^\alpha_i) + \sum_{i=1}^{p} \xi^i u^\alpha_{J,i}
$$

where $u^\alpha_i = \partial u^\alpha / \partial x^i$ and $u^\alpha_{J,i} = \partial u^\alpha_J / \partial x^i$.

The following Theorem is called the fundamental theorem of the Lie group analysis:

**Theorem 3.0.2** Suppose

$$
\Delta_\nu(x, u^{(n)}) = 0 \quad , \quad \nu = 1, \ldots, l
$$

is a system of differential equations of maximal rank defined over $M \subset X \times U$. If $G$ is a local group of transformations acting on $M$, and

$$
\text{pr}^{(n)}X[\Delta_\nu(x, u^{(n)})] = 0 \quad , \quad \nu = 1, \ldots, l, \quad \text{whenever} \quad \Delta(x, u^{(n)}) = 0 \quad (3.1)
$$

for every infinitesimal generator $X$ of $G$, then $G$ is a symmetry group admitted by the system. The equation (3.1) is called determining equation for the group $G$. 
Chapter 3. The Short introduction to the Lie method
Chapter 4

The Lie algebraic structure of the main equation

Consider the following equation

\[ \Delta = u_t + \frac{1}{2} \sigma^2 S^2 u_{SS} \frac{(1 - \rho u_S)^2}{(1 - \rho u_S - \rho S u_{SS})^2} = 0, \quad (4.1) \]

which was introduced in the paper [2] and described in Chapter 2. Let us study the Lie symmetry admitted by this partial differential equation (PDE). The general form of the symmetry group \( G \) we obtain if we have a corresponding Lie algebra. Let

\[ X = \xi \frac{\partial}{\partial S} + \tau \frac{\partial}{\partial t} + \varphi \frac{\partial}{\partial u}. \quad (4.2) \]

be a vector field. The second prolongation \( pr^{(2)} X \) is given by

\[ pr^{(2)} X = \xi \frac{\partial}{\partial S} + \tau \frac{\partial}{\partial t} + \varphi \frac{\partial}{\partial u} + \varphi^S \frac{\partial}{\partial u_S} + \varphi^t \frac{\partial}{\partial u_t} + \varphi^{SS} \frac{\partial}{\partial u_{SS}} + \varphi^{st} \frac{\partial}{\partial u_{st}} + \varphi^{tt} \frac{\partial}{\partial u_{tt}}, \]

where coefficients \( \varphi^t, \varphi^S, \varphi^{SS} \) etc. are defined by Definition 3.0.18 (see as well in Ovsiannikov, L.V. [6]).

\[
\begin{align*}
\varphi^S & = D_S(\varphi - \xi u_S - \tau u_t) + \xi u_{SS} + \tau u_{ts}, \\
\varphi^t & = D_t(\varphi - \xi u_S - \tau u_t) + \xi u_{st} + \tau u_{tt}, \\
\varphi^{SS} & = D_S^2(\varphi - \xi u_S - \tau u_t) + \xi u_{SSS} + \tau u_{sst}, \\
\varphi^{st} & = D_S D_t(\varphi - \xi u_S - \tau u_t) + \xi u_{SSt} + \tau u_{sst}, \\
\varphi^{tt} & = D_t^2(\varphi - \xi u_S - \tau u_t) + \xi u_{stt} + \tau u_{ttt}.
\end{align*}
\]
Chapter 4. The Lie algebraic structure of the main equation

In our case, (4.1) is of the second order and if we a general vector field \( X \) in form (4.2) we have the following form for the second prolongation

\[
pr^{(2)}X = \xi \frac{\partial}{\partial S} + \varphi^S \frac{\partial}{\partial u_S} + \varphi^t \frac{\partial}{\partial u_t} + \varphi^{SS} \frac{\partial}{\partial u_{SS}},
\]

where

\[
\begin{align*}
\varphi^S &= \varphi_S + u_S(\varphi_u - \xi_S) - \tau_SU_t - \tau_uSu_t - \xi_uu_S^2, \\
\varphi^t &= \varphi_t + u_t(\varphi_u - \tau_t) - \xi_tSu_S - \xi_uSu_u - \tau_u^2, \\
\varphi^{SS} &= \varphi_{SS} + \varphi_{Su}u_S + \varphi_{uSS} + \varphi_{uu}u_S^2 - \xi_Su_S \\
&- 2\xi_Su_{SS} - 2\xi_Su_S^2 - 3\xi_uu_Su_{SS} - \xi_{uu}u_S^3 - \tau_{SS}u_t \\
&- 2\tau_{Su}Su_S - 2\tau_{Su}Su_{St} - 2\tau_{Su}Su_uSu_{St} - \tau_{uu}u_S^2u_t.
\end{align*}
\]

Now we have to find unknown functions \( \xi(t, S, u), \tau(t, S, u), \varphi(t, S, u) \) using the determining equation (3.1). If some symmetry group is admissible it has to satisfy the following condition \( mod(\Delta = 0) \). This condition means that \( pr^{(2)}X(\Delta) = 0 \) should be satisfied just on solutions of the equation (4.1). It means we should make following substitution

\[
u_t = -\frac{1}{2} \sigma^2 S^2 u_{SS} \frac{(1 - \rho u_S)^2}{(1 - \rho u_S - \rho S u_{SS})^2}
\]

(4.3)
in the determining equation

\[ pr^{(2)}X(\Delta) \mid \equiv 0, \ mod(\Delta = 0). \]

On our case it means that we replace \( u_t \) by (4.3) and obtain

\[
pr^{(2)}X(\Delta) = \frac{1}{(1 - \rho u_S - \rho S u_{SS})^3} (\xi \sigma^2 S u_{SS}(1 - \rho u_S)^3 + (\varphi_S + u_S \\
(\varphi_u - \xi_S) - \tau_SU_t - \tau_uSu_t - \xi_uu_S^2) \rho \sigma^2 S^3 u_{SS}^2(1 - \rho u_S) \\
+ (\varphi_t + u_t(\varphi_u - \tau_t) - \xi_tSu_S - \xi_uSu_u - \tau_u^2))(1 - \rho u_S \\
- \rho S u_{SS})^3 + (\varphi_{SS} + \varphi_{Su}u_u + \varphi_{uSS} + \varphi_{uu}u_S^2 - \xi_Su_S \\
- 2\xi_Su_{SS} - 2\xi_Su_S^2 - 3\xi_uu_Su_{SS} - \xi_{uu}u_S^3 - \tau_{SS}u_t - 2\tau_{Su}Su_S \\
- 2\tau_{Su}Su_{St} - 2\tau_{Su}Su_uSu_{St} - \tau_{uu}u_S^2u_t \frac{1}{2} \sigma^2 S^2 \\
(1 - \rho u_S)^2(1 - \rho u_S - \rho S u_{SS}).
\]
Correspondingly, after substitution (4.3) we obtain

\[
pr^{(2)}X(\Delta)\big|_{\Delta=0} = \frac{1}{(1 - \rho u_s - \rho S u_{SS})^3} (\xi \sigma^2 S u_{SS}(1 - \rho u_S)^3 + (\varphi_S + u_S(\varphi_u

- \xi S) - \xi_u u_S^2) \rho^2 \sigma^2 S^3 u_{SS}^2 (1 - \rho u_S) + \frac{1}{2}(\varphi_t - \xi_t u_S)(1 - \rho u_S

- \rho S u_{SS})^3 + (\varphi_{SS} + (2 \varphi_{Su} - \xi ss) u_S + (\varphi_{uu} - 2 \xi_s) u_S^2

+ (\varphi_u - 2 \xi_S) u_{SS} - 3 \xi_u u_S u_{SS} - \xi_{uu} u_S^3) \sigma^2 S^2 (1 - \rho u_S)^2

(1 - \rho u_S + \rho S u_{SS}) + \frac{1}{2}(\tau_S + \tau_u u_S) \frac{\sigma^4 S^4 \rho^2 u_{SS}^3 (1 - \rho u_S)^3}{(1 - \rho u_S - \rho S u_{SS})^2}

+ \frac{1}{4} \sigma^4 S^4 u_{SS} (1 - \rho u_S)^4 \frac{1 - \rho u_S + \rho S u_{SS}}{(1 - \rho u_S - \rho S u_{SS})^2} (\tau_S + 2 \tau_S u_S

+ \tau_u u_S + \tau_u u_S^2) - (\tau_S + \tau_u u_S) u_S \sigma^2 S^2 (1 - \rho u_S)^2 (1 - \rho u_S

+ \rho S u_{SS}) - \frac{1}{2} \sigma^2 S^2 u_{SS} (\varphi_u - \tau) - \xi_u u_S (1 - \rho u_S)^2

(1 - \rho u_S - \rho S u_{SS}) - \frac{1}{4} \tau_u \sigma^4 S^4 u_{SS}^2 (1 - \rho u_S)^4)
\]
After some transformations we obtain

\[
pr^{(2)} X(\Delta)|_{\Delta=0} = \frac{1}{(1 - \rho u_S - \rho S u_{SS})^5} (\xi \sigma^2 S(u_{SS} - 3\rho u_{SS}S + 3\rho^2 u^2_{SS} - \rho^3 u^3_{SS})(1 - 2\rho u_S + \rho^2 u^2_S - 2 \rho S u_{SS} + 2 \rho^2 S u_{SS}S)
\]

\[
+ \rho \varphi u^2_{SS} + (\varphi_S u_{SS} + (\varphi_S - \xi S - \rho \varphi)u_{SS}^2 - (\xi u)
\]

\[
+ \rho(\varphi_S - \xi S))u^2_S + \xi u \rho u^3_{SS}) + \frac{1}{2}(\varphi_S - \xi u_S)(1 - 5\rho u_S
\]

\[
+ 10\rho^2 u^2_S - 10\rho^3 u^3_S + 5\rho^4 u^4_S - \rho^5 u^5_S - 5\rho S u_{SS} + 20\rho^2 S u_{SS}S
\]

\[
- 30\rho^3 S u^2_{SS} + 20\rho^4 S u^3_{SS} - 5\rho^5 S u^4_{SS} + 10\rho^2 S^2 u_{SS}
\]

\[
- 30\rho^3 S u^2_{SS} + 30\rho^4 S u^3_{SS} - 10\rho^5 S u^4_{SS} - 10\rho^3 S^3 u^3_{SS}
\]

\[
+ 20\rho^4 S u^3_{SS} - 10\rho^5 S u^4_{SS} + 5\rho^4 S u^4_{SS} + 5\rho^5 S u^4_{SS} + 5\rho^5 S^4 u^4_{SS}
\]

\[
- \rho^5 S^5 u^5_{SS} + \sigma^2 S^2 (\varphi_S + (2\varphi_S - \xi SS)u_S + (\varphi_S - 2\xi u_S)u^2_S
\]

\[
+ (\varphi_S - 2\xi S)u_{SS} - 3\xi u S u_{SS} - \xi u u^3_S)(1 - 5 + 10\rho u^2_S
\]

\[
- 10\rho^3 u^3_S + 5\rho^4 u^4_S - \rho^5 u^5_S + 4 \rho^2 S u S S - 6 \rho^3 S u^2_{SS}
\]

\[
+ 4 \rho^3 S u^2_{SS} + (\rho^4 - 2 \rho^5 S) u^3_{SS} - \rho^5 S^2 u_{SS} - 3 \rho^3 S^3 u^3_{SS}
\]

\[
- 3 \rho^3 S^2 u^2_{SS} + \rho^5 S^2 u^3_{SS} + \rho^3 S^3 u^3_{SS} - 2 \rho^3 S^3 u^3_{SS}
\]

\[
+ \frac{1}{2} \sigma^4 S^5 \rho^2 (\tau_S u^3_{SS} + \tau u S u^3_{SS})(1 - 3 \rho u_S
\]

\[
+ 3 \rho^2 u^2_S - \rho^3 u^3_S + \frac{1}{2} \sigma^2 S^4 u_{SS} + 2 \tau_S u_S + \tau u S S + \tau u u^2_S)
\]

\[
(1 - 5 \rho u_S + 10 \rho^2 u^2_S - 10 \rho^3 u^3_S + 5 \rho^4 u^4_S - \rho^5 u^5_S + \rho S u_{SS}
\]

\[
- 4 \rho^2 S u_S u_{SS} + 6 \rho^3 S u^2_{SS} + 4 \rho^4 S u^3_{SS} + \rho^5 S u^4_{SS})
\]

\[
- \sigma^2 S^2 (\tau_S u_S + \tau u S u_S)((1 - 5 + 10 \rho^2 u^2_S - 10 \rho^3 u^3_S + 5 \rho^4 u^4_S
\]

\[
- \rho^5 u^5_S + 4 \rho^2 S u_S u_{SS} - 6 \rho^3 S u^2_{SS} + 4 \rho^4 S u^3_{SS} - \rho^5 S u^4_{SS}
\]

\[
- 3 \rho^3 S u^2_{SS} + 3 \rho^3 S u^3_{SS} - 3 \rho^4 S u^2_{SS} + \rho^5 S^2 u^3_{SS}
\]

\[
+ \rho^3 S^3 u^3_{SS} - 2 \rho^4 S^3 u^3_{SS} + \rho^5 S^3 u^3_{SS}) - \frac{1}{4} \tau_S \sigma^4 S^4 u^2_{SS}
\]

\[
(1 - 5 \rho u_S + 10 \rho^2 u^2_S - 10 \rho^3 u^3_S + 5 \rho^4 u^4_S - \rho^5 u^5_S - \rho S u_{SS}
\]

\[
+ 4 \rho^2 S u_S u_{SS} - 6 \rho^3 S u^2_{SS} + 4 \rho^4 S u^3_{SS} - \rho^5 S u^4_{SS})
\]

\[
- \frac{1}{2} \sigma^2 S^2 ((\varphi_S - \tau_S)u_{SS} - \xi u S u_{SS})(1 - 5 \rho u_S + 10 \rho^2 u^2_S
\]

\[
- 10 \rho^3 u^3_S + 5 \rho^4 u^4_S - \rho^5 u^5_S - 3 \rho S u_{SS} + 12 \rho^2 S u_S u_{SS}
\]

\[
- 18 \rho^3 S u^2_{SS} + 12 \rho^4 S u^3_{SS} - 3 \rho^5 S u^4_{SS} + 3 \rho^2 S^2 u_{SS}
\]

\[
- 9 \rho^3 S^2 u^2_{SS} + 9 \rho^4 S^2 u^3_{SS} - 3 \rho^5 S^2 u^3_{SS} - \rho^3 S^3 u^3_{SS}
\]

\[
+ 2 \rho^4 S^3 u^3_{SS} - \rho^5 S^3 u^3_{SS} - 3 \rho^4 S^3 u^3_{SS})
\])

We have to solve these system of equations \( pr^{(2)} X(\Delta)|_{\Delta=0} \equiv 0 \) by assump-
tions that the variables $S, t, u, u_s \ldots$ are independent variables in $M^{(2)}$. We should demand that all coefficients of each monomial containing derivatives $u_{SS}, u_S \ldots$ of the function $u$ must be equal to zero. It immediately gives us the following system of the differential equations on the functions $\xi, \tau, \varphi$ which is listed in Table 4.1 and Table 4.3. From the Table 4.1 we can see that $\tau$ should not depend on $u$ and $S$, i.e. $\tau(S, t, u) = \tau(t)$. Using this fact we obtain the following system of equations on the functions $\xi(s, t, u), \tau(t), \varphi(s, t, u)$ (see Table 4.3). From the Table 4.3 we get

\[
\begin{align*}
\xi_{uu} &= 0, \quad (4.4) \\
\xi_u &= 0, \quad (4.5) \\
\xi_t &= 0, \quad (4.6) \\
\varphi_t &= -2\sigma^2 S^2 \varphi_{SS}, \quad (4.7) \\
\varphi_t &= 0. \quad (4.8)
\end{align*}
\]

Hence $\xi$ does not depend on $u$ and $t$ and is a function of the variable $S$ only, $\xi(S, t, u) = \xi(S)$. Compare the equations (4.7) and (4.8), we obtain $\varphi_{SS} = 0$. Insert our results in the Table 4.3 we obtain a new system (see Table 4.4). We will insert now in the general system Table 4.4, $\tau = \tau(t), \xi = \xi(S), \varphi_{SS} = 0$ then from the Table 4.4 ((22) and (28) equations) we obtain

\[
\begin{align*}
\varphi_{uu} &= 2\xi_S, \\
2\varphi_{Su} &= \xi_{SS}.
\end{align*}
\]
We as well immediately obtain

\[ \xi + \frac{1}{2} S \varphi_u + \tau_t = 0, \quad (4.9) \]
\[-2\xi + \frac{1}{2} S \varphi_u + 2S \xi_S + S \rho \varphi_S - \frac{3}{2} S \tau_t = 0, \quad (4.10) \]
\[-4\xi - S \varphi_u - 9S \xi_S - 3S \rho \varphi_S + 6S \tau_t = 0, \quad (4.11) \]
\[-12\xi - 3S \rho \varphi_S + 6S \varphi_u + 3S \xi_s - 9S \tau_t = 0, \quad (4.12) \]
\[8\xi - 3S \varphi_u - 3S \xi_S - S \rho \varphi_S = 0, \quad (4.13) \]
\[-4\xi + S \varphi_u + 2S \xi_S - 3S \tau_t = 0, \quad (4.14) \]
\[2\xi - 4S \rho \varphi_S - 3S \varphi_u + 3S \tau_t = 0, \quad (4.15) \]
\[-6\xi - S \varphi_u + 8S \rho \varphi_S + 16S \xi_S = 0, \quad (4.16) \]
\[6\xi - S \varphi_u - 8S \xi_S - 4S \rho \varphi_S + 9S \tau_t = 0, \quad (4.17) \]
\[-2\xi - S \varphi_u + 4S \xi_S - 3S \tau_t = 0, \quad (4.18) \]
\[2\rho \varphi_S + 3\varphi_u - 4\xi_S - \tau_t = 0, \quad (4.19) \]
\[\rho \varphi_S + \xi_S - \tau_t = 0, \quad (4.20) \]
\[2\xi_S - \varphi_u + \tau_t = 0. \quad (4.21) \]

If we multiply (4.20) by two and subtract from equation (4.19) we obtain

\[3\varphi_u - 6\xi_s + \tau_t = 0. \quad (4.22) \]

Multiply (4.21) by three and subtract from (4.22) we obtain

\[4\tau_t = 0. \]

It means that \(\tau(t) = k\) where \(k\) is constant. After simplifications we obtain the following reduced system of determining equations

\[ \xi + \frac{1}{2} S \varphi_u = 0, \quad (4.23) \]
\[-2\xi + \frac{1}{2} S \varphi_u + 2S \xi_S + S \rho \varphi_S = 0, \quad (4.24) \]
\[-4\xi - S \varphi_u - 9S \xi_S - 3S \rho \varphi_S = 0, \quad (4.25) \]
\[-12\xi - 3S \rho \varphi_S + 6S \varphi_u + 3S \xi_s = 0, \quad (4.26) \]
\[8\xi - 3S \varphi_u - 3S \xi_S - S \rho \varphi_S = 0, \quad (4.27) \]
\[-4\xi + S \varphi_u + 2S \xi_S = 0, \quad (4.28) \]
\[2\xi - 4S \rho \varphi_S - 3S \varphi_u = 0, \quad (4.29) \]
\[-6\xi - S \varphi_u + 8S \rho \varphi_S + 16S \xi_S = 0, \quad (4.30) \]
\[6\xi - S \varphi_u - 8S \xi_S - 4S \rho \varphi_S = 0, \quad (4.31) \]
\[-2\xi - S \varphi_u + 4S \xi_S = 0, \quad (4.32) \]
\[2\rho \varphi_S + 3\varphi_u - 4\xi_S = 0, \quad (4.33) \]
\[\rho \varphi_S + \xi_S = 0, \quad (4.34) \]
\[2\xi_S - \varphi_u = 0. \quad (4.35) \]
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From (4.23), (4.34) and (4.35) equations we have

\[ \xi = -\frac{1}{2}S\varphi_u, \]  
\[ \xi_s = -\rho \varphi_s, \]  
\[ \xi_s = \frac{1}{2} \varphi_u. \]  

(4.36) \hspace{1cm} (4.37) \hspace{1cm} (4.38)

Let us substitute equation (4.38) to (4.36). We obtain \( \xi = S\xi_s \). Solving this equation we obtain \( \xi(S) = C S \), where \( C = \text{const.} \). Hence \( \xi_s = C \).

Substitute this result in equations (4.38) we obtain

\[ \varphi = 2Cu + g(S). \]  

(4.39)

Hence

\[ \varphi_s = g_s(S) \]

but

\[ \varphi_s = -\frac{C}{\rho}. \]

From this equations we obtain

\[ g_s(S) = -\frac{C}{\rho} \Rightarrow g(S) = -\frac{C}{\rho} S + d, \text{ where } d \equiv \text{const.} \]

Substitute the function \( g(S) \) in equations (4.39) we obtain a following structure of the function \( \varphi(S, t, u) \)

\[ \varphi(S, u) = 2Cu - \frac{C}{\rho} S + d. \]

Summarize the results of the previous calculations we obtain following values for the coefficients \( \tau, \xi, \varphi \) of the vector field \( X \) (4.39)

\[ \tau = k, \]

\[ \xi(S) = CS, \]

\[ \varphi(S, u) = 2Cu - \frac{C}{\rho} S + d, \]

where \( C, k, d \) are arbitrary constants.
Chapter 4. The Lie algebraic structure of the main equation

Theorem 4.0.3 The equation

\[ u_t + \frac{1}{2} \sigma^2 S^2 u_{SS} \frac{(1 - \rho u_S)^2}{(1 - \rho u_S - \rho u_{SS})^2} = 0 \]

admits a three dimensional Lie algebra \( L_3 \) with the following generators

\[
U_1 = \frac{\partial}{\partial t},
\]

\[
U_2 = \frac{\partial}{\partial u},
\]

\[
U_3 = S \frac{\partial}{\partial S} + \left( 2u - \frac{S}{\rho} \right) \frac{\partial}{\partial u}.
\]

The algebra \( L_3 \) has a two dimensional Abelian subalgebra \( L_2 \) spanned by infinitesimal generators \( U_1, U_2 \), i.e. \( L_2 = \langle U_1, U_2 \rangle \).

<table>
<thead>
<tr>
<th></th>
<th>( U_1 )</th>
<th>( U_2 )</th>
<th>( U_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( U_1 )</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( U_2 )</td>
<td>0</td>
<td>0</td>
<td>2( U_2 )</td>
</tr>
<tr>
<td>( U_3 )</td>
<td>0</td>
<td>-2( U_2 )</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 4.2: The commutator table for the algebra \( L_3 \) (4.40)
Table 4.3: The determining system of equations for the Lie algebra $L$. Admitted by (4.1)

<table>
<thead>
<tr>
<th>Equation</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$u_S u_{5S}^3$</td>
<td>$\frac{1}{2} \rho^5 S^5 \xi_t = 0$</td>
</tr>
<tr>
<td>$u_S^5$</td>
<td>$-\frac{1}{2} \rho^5 S^5 \varphi_t = 0$</td>
</tr>
<tr>
<td>$u_S^3 u_{5S}^4$</td>
<td>$\sigma^2 S^5 \rho^5 \xi_u - 3 \sigma^2 S^5 \rho^5 \xi_u - \frac{1}{2} \sigma^2 S^5 \rho^5 \xi_u = 0$</td>
</tr>
<tr>
<td>$u_S^4 u_{5S}^3$</td>
<td>$\sigma^2 S^5 \rho^4 \xi_u + \frac{1}{2} \sigma^2 \rho^5 S^5 (\varphi_u - \tau_t) + \sigma^2 \rho^5 S^5 (\varphi_u - 2 \xi_S) + 6 \sigma^2 \rho^4 S^5 \xi_u + \frac{5}{2} \rho^5 \xi_t - \sigma^2 S^5 \rho^4 (\xi_u + \rho(\varphi_u - \xi_S)) = 0$</td>
</tr>
<tr>
<td>$u_S^4$</td>
<td>$\sigma^2 S^5 \rho^4 \varphi_u + \frac{5}{2} \rho^4 \xi_t + \sigma^2 S^5 \rho^3 (\varphi_u - 2 \xi_S) + \frac{1}{2} \sigma^2 S^5 \rho^3 (\varphi_u - \tau_t) = 0$</td>
</tr>
<tr>
<td>$u_S^3 u_{5S}^3$</td>
<td>$\frac{3}{2} \rho^5 \xi_u - \sigma^2 S^5 \rho^5 (\varphi_u - 2 \xi_S) + 3 \sigma^2 S^4 \rho^5 \xi_u + 2 \sigma^2 S^5 \rho^4 \xi_u - 2 \sigma^2 S^4 \rho^5 \xi_u = 0$</td>
</tr>
<tr>
<td>$u_S^3 u_{5S}^3$</td>
<td>$-\sigma^2 S^3 \rho^5 \xi - 2 \sigma^2 S^4 \rho^4 \xi_u - 2 \sigma^2 S^4 \rho^4 (\xi_u + \rho(\varphi_u - \xi_S)) + 5 \sigma^2 S^3 \rho^5 \xi_t + \sigma^2 S^5 \rho^5 (2 \varphi_u - \xi_S) + \sigma^2 S^4 \rho^3 (\varphi_u - 2 \xi_S) - 2 \sigma^2 S^5 \rho^4 (\varphi_u - 2 \xi_S) + 9 \sigma^2 S^4 \rho^4 \xi_u - 2 \sigma^2 S^5 \rho^3 (\varphi_u - \tau_t) = 0$</td>
</tr>
<tr>
<td>$u_S^2 u_{5S}^3$</td>
<td>$3 \sigma^2 S^3 \rho^4 \xi + 2 \sigma^2 S^4 \rho^4 (\varphi_u - \xi_S - \rho(\varphi_u - \xi_S)) + 2 \sigma^2 S^4 \rho^3 (\xi_u - \rho(\varphi_u - \xi_S)) - 10 \sigma^3 \rho^4 \xi_t - 5 \sigma^3 \rho^5 \varphi + \sigma^2 S^5 \rho^3 \varphi SS - 2 \sigma^2 S^4 \rho^4 (2 \varphi_u - 2 \xi_S) - 3 \sigma^2 S^4 \rho^4 (\varphi_u - 2 \xi_S) + \sigma^2 S^5 \rho^3 (\varphi_u - 2 \xi_S) + 9 \sigma^2 S^4 \rho^3 \xi_u + \sigma^2 S^4 \rho^3 \xi_u - \sigma^2 S^4 \rho^3 \xi_u - \frac{9}{2} \sigma^2 S^4 \rho^3 (\varphi_u - \tau_t) = 0$</td>
</tr>
<tr>
<td>$u_S^3 u_{5S}^2$</td>
<td>$-3 \sigma^2 S^3 \rho^3 \xi _t + 5 \sigma^3 S^3 \rho^3 \xi_t + 10 \sigma^3 \rho^4 \varphi - 2 \sigma^2 S^4 \rho^4 \varphi SS + 2 \sigma^2 S^5 \rho^3 (2 \varphi_u - 2 \xi_S) - 3 \sigma^2 S^4 \rho^3 (\varphi_u - 2 \xi_S) + 3 \sigma^2 S^4 \rho^2 \xi_u - 2 \sigma^2 S^4 \rho^3 (\varphi_u - \xi_S - \rho(\varphi_u - \xi_S)) + 2 \sigma^2 S^4 \rho^4 \varphi + \frac{5}{2} \sigma^2 S^4 \rho^3 (\varphi_u - \tau_t) = 0$</td>
</tr>
<tr>
<td>$u_S^2$</td>
<td>$\sigma^2 S^3 \rho^2 \xi - 2 \sigma^2 S^4 \rho^3 \varphi_S - 5 \sigma^2 S^3 \rho^3 \varphi - 2 S \sigma^5 \rho^3 \varphi SS - \rho(\varphi_u - \tau_t) = 0$</td>
</tr>
<tr>
<td>$u_S^3 u_{5S}^2$</td>
<td>$\sigma^2 S^3 \rho^2 (\varphi_u - 2 \xi_S) - 3 \sigma^2 S^2 (\rho^4 - 2 S \rho^5) \xi_u - \sigma^2 S^4 \rho^5 \xi_u - \frac{3}{5} \sigma^2 S^3 \xi_u + \sigma^2 S^3 \rho^5 \xi_u = 0$</td>
</tr>
</tbody>
</table>
### Chapter 4. The Lie Algebraic Structure of the Main Equation

<table>
<thead>
<tr>
<th>Term</th>
<th>Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$u^3 u_{SS}$</td>
<td>$-2\sigma^2 S^2 \rho^5 \xi - \sigma^2 S^3 \rho^4 (\xi_u + \rho(\varphi_u - \xi_S)) - 2\sigma^2 S^3 \rho^4 \xi_u + 5S^2 \rho^5 \xi_t + \alpha^2 S^4 \rho^5 (2\varphi_S - 2\xi_{SS}) + \alpha^2 S^4 (\rho^4 - 2S^3 \rho^5) (\varphi_u - 2\xi_S) - 3\sigma^2 S^4 \rho^4 (\varphi_u - 2\xi_S) - 12\sigma^2 S^3 \rho^4 \xi_u - 3\sigma^2 S^3 \rho^4 \xi_u + 6S^2 \rho^4 \xi_u + \frac{3}{2} \sigma^2 S^3 \rho^5 (\varphi_u - \tau_t) = 0$</td>
</tr>
<tr>
<td>$u^3 u_{SS}^2$</td>
<td>$8\sigma^2 S^2 \rho^4 \xi + \sigma^2 S^3 \rho^4 (\varphi_u - \xi_S - \rho \varphi_S) + 2\sigma^2 S^3 \rho^3 (\xi_u + \rho(\varphi_u - \xi_S)) - 15S^2 \rho^4 \xi_t - 5S^2 \rho^5 \varphi + \alpha^2 S^4 \rho^5 (2\varphi_S - 2\xi_{SS}) + 4\sigma^2 S^3 \rho^4 (\varphi_u - 2\xi_S) - 3\sigma^2 S^3 \rho^3 (\varphi_{uu} - 2\xi_S) + 18\sigma^2 S^3 \rho^3 \xi_u + \frac{1}{3} \sigma^2 S^3 \rho^4 (\varphi_u - \tau_t) + 9\sigma^2 S^3 \rho^4 (\varphi_u - \tau_t) = 0$</td>
</tr>
<tr>
<td>$u^3 u_{SS}$</td>
<td>$-12\sigma^2 S^2 \rho^3 \xi + \sigma^2 S^3 \rho^3 (\varphi_u - \xi_S - \rho \varphi_S) - \sigma^2 S^3 \rho^4 (\xi_u + \rho(\varphi_u - \xi_S)) + 5S^2 \rho^3 \xi_t + 15S^2 \rho^4 \varphi - 3\sigma^2 S^4 \rho^4 \varphi_{SS} - 3\sigma^2 S^4 \rho^3 (2\varphi_S - \xi_{SS}) - \sigma^2 S^4 \rho^2 (\varphi_{uu} - 2\xi_S) + 6S^2 \rho^3 \xi_u + 9\sigma^2 S^3 \rho^3 (\varphi_u - \tau_t) = 0$</td>
</tr>
<tr>
<td>$u^2 u_{SS}$</td>
<td>$-2\sigma^2 S^2 \rho \xi + \sigma^2 S^3 \rho^2 (\varphi_u - \xi_S - \rho \varphi_S) - 2\sigma^2 S^3 \rho^3 \varphi - 5S^2 \rho^2 \xi_t + 30S^3 \rho^2 \varphi_{SS} - \sigma^2 S^3 \rho^2 (2\varphi_S - \xi_{SS}) + 4\sigma^2 S^3 \rho^2 (\varphi_u - 2\xi_S) + 30S^3 \rho^2 \varphi_{uu} - 6S^2 \rho^3 \xi_u - 6S^2 \rho^3 \rho^2 \xi_u - \frac{3}{2} \sigma^2 S^3 \rho \xi_u - \frac{1}{3} \sigma^2 S^3 \rho \xi_u = 0$</td>
</tr>
<tr>
<td>$u^2 u_{SS}$</td>
<td>$-2\sigma^2 S^3 \rho^2 (\varphi_u - \xi_S - \rho \varphi_S) - 2\sigma^2 S^3 \rho^3 \varphi - 5S^2 \rho^2 \xi_t + 30S^3 \rho^2 \varphi_{SS} - \sigma^2 S^3 \rho^2 (2\varphi_S - \xi_{SS}) + 4\sigma^2 S^3 \rho^2 (\varphi_u - 2\xi_S) + 30S^3 \rho^2 \varphi_{uu} - 6S^2 \rho^3 \xi_u - 6S^2 \rho^3 \rho^2 \xi_u - \frac{3}{2} \sigma^2 S^3 \rho \xi_u - \frac{1}{3} \sigma^2 S^3 \rho \xi_u = 0$</td>
</tr>
<tr>
<td>$u^2 u_{SS}$</td>
<td>$(\rho^4 - 2\rho^5 S)\sigma^2 S^2 (\varphi_u - \xi_S - \rho \varphi_S) - 4\sigma^2 S^3 \rho \xi_u - \frac{1}{2} \sigma^2 S^2 \rho \xi_u + 15\sigma^2 S^2 \rho^5 \xi_t = 0$</td>
</tr>
<tr>
<td>$u^2 u_{SS}$</td>
<td>$-\sigma^2 S^2 \rho \xi + \frac{5}{2} \rho \xi_t + (\rho^4 - 2\rho^5 S)\sigma^2 S^2 (2\varphi_S - \xi_{SS}) + \sigma^2 S^2 \rho \xi_u - 2\xi_S) + 4\sigma^2 S^3 \rho^4 (\varphi_u - 2\xi_S) - 15\sigma^2 S^2 \rho \xi_u - 6S^2 \rho^3 \xi_u + \frac{5}{2} \sigma^2 S^2 \rho \xi_u - \frac{1}{3} \sigma^2 S^3 \rho \xi_u = 0$</td>
</tr>
<tr>
<td>$u^2 u_{SS}$</td>
<td>$5\sigma^2 S^3 \rho \xi - 10\rho^4 S \xi_t + \frac{5}{2} \rho \xi_t + (\rho^4 - 2\rho^5 S)\sigma^2 S^2 \varphi_{SS} + 4\sigma^2 S^3 \rho^4 (2\varphi_S - \xi_{SS}) + 5\sigma^2 S^2 \rho^4 (\varphi_u - 2\xi_S) - 6S^2 \rho^3 (\varphi_{uu} - 2\xi_S) + 30S^3 \rho^2 \xi_u - \frac{5}{2} \sigma^2 S^2 \rho \xi_u = 0$</td>
</tr>
<tr>
<td>$u^2 u_{SS}$</td>
<td>$-10\sigma^2 S^2 \rho \xi + 15\rho \xi_t + 15S^2 \rho \xi_t + 4\sigma^2 S^3 \rho^3 \varphi_{SS} + 4\sigma^2 S^3 \rho^4 (2\varphi_S - \xi_{SS}) - 10\sigma^2 S^2 \rho \xi_u - 2\xi_S) + 4\sigma^2 S^3 \rho^2 (\varphi_{uu} - 2\xi_S) - 30S^2 \rho^2 \xi_u + \sigma^2 S^3 \rho \xi_u + 5\sigma^2 S^2 \rho \xi_u + 5\sigma^2 S^2 \rho \xi_u = 0$</td>
</tr>
<tr>
<td>$u^2 u_{SS}$</td>
<td>$10\sigma^2 S^2 \rho \xi - 15S^2 \rho \xi_t + 6S^2 \rho^2 \xi_t + 4\sigma^2 S^3 \rho^2 (2\varphi_S - \xi_{SS}) + 10\sigma^2 S^2 \rho^2 (\varphi_u - 2\xi_S) - \sigma^2 S^3 \rho (\varphi_{uu} - 2\xi_S) + 15\sigma^2 S^2 \rho \xi_u - \frac{5}{2} \sigma^2 S^2 \rho \xi_u - 5\sigma^2 S^2 \rho \xi_u - \frac{1}{3} \sigma^2 S^3 \rho \xi_u = 0$</td>
</tr>
</tbody>
</table>
The feedback effects in illiquid markets.

\[
\begin{align*}
\rho_{uu} - 5\sigma^2 S \rho \xi + \frac{5}{2} S \rho \xi_t + 10 S \rho^2 \varphi_t + 4\sigma^2 S^3 \rho^2 \varphi_{SS} - 5\sigma^2 S^2 \rho (\varphi_u - 2\xi_S) \\
-3\sigma^2 S^3 \xi_u - \frac{5}{2} \sigma^2 S^2 \rho (\varphi_u - \tau_t) - \sigma^2 S^3 \rho (2\varphi_{Su} - \xi_{SS}) + \frac{1}{2} \sigma^2 S^2 \xi_u = 0
\end{align*}
\]

\[
\begin{align*}
\rho_{SS} - \frac{5}{2} S \rho \varphi_t - \sigma^2 S^3 \rho \varphi_{SS} + \sigma^2 S^2 (\varphi_u - 2\xi_S) - \frac{1}{2} \sigma^2 S^2 (\varphi_u - \tau_t) = 0
\end{align*}
\]

\[
\begin{align*}
\rho_{u} - \frac{5}{2} S \rho \xi - \sigma^2 S^2 \rho^2 (2\varphi_{Su} - \xi_{SS}) + 5\sigma^2 S^2 \rho^4 (\varphi_{uu} - 2\xi_S) + 10\sigma^2 S^2 \rho^3 \xi_{uu} = 0
\end{align*}
\]

\[
\begin{align*}
\rho_{u} - \frac{5}{2} S \rho \xi - \frac{1}{2} \sigma^2 S^2 \rho^4 (2\varphi_{Su} - \xi_{SS}) + 5\sigma^2 S^2 \rho^4 (\varphi_{uu} - 2\xi_S) + 10\sigma^2 S^2 \rho^3 \xi_{uu} = 0
\end{align*}
\]

\[
\begin{align*}
\rho_{S} - \frac{5}{2} S \rho \xi - \frac{5}{2} \rho^4 \varphi_t + 5\sigma^2 S^2 \rho^4 \varphi_{SS} - 10\sigma^2 S^2 \rho^4 (2\varphi_{Su} - \xi_{SS}) + 10\sigma^2 S^2 \rho^3 (\varphi_{uu} - 2\xi_S) + 5\sigma^2 S^2 \rho^3 \xi_{uu} = 0
\end{align*}
\]

\[
\begin{align*}
\rho_{S} - \frac{5}{2} S \rho \xi - \frac{5}{2} \rho^4 \varphi_t - 10\sigma^2 S^2 \rho^3 \varphi_{SS} + 10\sigma^2 S^2 \rho^2 (2\varphi_{Su} - \xi_{SS}) - 5\sigma^2 S^2 \rho (\varphi_{uu} - 2\xi_S) - \sigma^2 S^2 \xi_{uu} = 0
\end{align*}
\]

\[
\begin{align*}
\rho_{S} - \frac{5}{2} S \rho \xi - \frac{5}{2} \rho^2 \varphi_t - 10\sigma^2 S^2 \rho^2 \varphi_{SS} - 5\sigma^2 S^2 \rho (2\varphi_{Su} - \xi_{SS}) + \sigma^2 S^2 (\varphi_{uu} - 2\xi_S) - \sigma^2 S^2 \xi_{uu} = 0
\end{align*}
\]

\[
\begin{align*}
\rho_{S} - \frac{5}{2} S \rho \xi + 10\sigma^2 S^2 \rho^2 \varphi_{SS} - 5\sigma^2 S^2 \rho (2\varphi_{Su} - \xi_{SS}) + \sigma^2 S^2 (\varphi_{uu} - 2\xi_S) - \sigma^2 S^2 \xi_{uu} = 0
\end{align*}
\]

\[
\begin{align*}
\rho_{S} - \frac{1}{2} \xi - \frac{5}{2} \rho \varphi_t - 5\sigma^2 S^2 \rho \varphi_{SS} + \sigma^2 S^2 (2\varphi_{Su} - \xi_{SS}) = 0
\end{align*}
\]

\[
\begin{align*}
\frac{1}{2} \varphi_t + \sigma^2 S^2 \varphi_{SS} = 0
\end{align*}
\]
Table 4.4: The reduced system of the differential equations on the functions \( \tau(t), \xi(S), \varphi(S,u) \).

<table>
<thead>
<tr>
<th></th>
<th>Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( \frac{1}{2}(\varphi_u - \tau_t) - \xi_S = 0 )</td>
</tr>
<tr>
<td>2</td>
<td>( S(\varphi_u - \xi_S - \rho \varphi_S) - 2S(\varphi_u - 2\xi_S) - (\varphi_u - \tau_t) = 0 )</td>
</tr>
<tr>
<td>3</td>
<td>( \rho \varphi_S + \frac{5}{2}(\varphi_u - 2\xi_S) + \frac{1}{2}(\varphi_u - \tau_t) = 0 )</td>
</tr>
<tr>
<td>4</td>
<td>( \varphi_{uu} - 2\xi_S = 0 )</td>
</tr>
<tr>
<td>5</td>
<td>( -\rho \xi - 2S\rho(\varphi_u - \xi_S) + S^2 \rho(2\varphi_{Su} - \xi_{SS}) + S \rho(\varphi_u - 2\xi_S) + \frac{3}{2}S \rho(\varphi_u - \tau_t) = 0 )</td>
</tr>
<tr>
<td>6</td>
<td>( 3\rho \xi + 2S\rho(\varphi_u - \xi_S - \rho \varphi_S) + 2S \rho(\varphi_u - \xi_S) - 2S^2 \rho(2\varphi_{Su} - \xi_{SS}) - 3S \rho(\varphi_u - 2\xi_S) + S^2(\varphi_{uu} - 2\xi_S) + \frac{3}{2}S \rho(\varphi_u - \tau_t) = 0 )</td>
</tr>
<tr>
<td>7</td>
<td>( -3\xi + S^2(2\varphi_{Su} - 2\xi_{SS}) - 3S(\varphi_u - 2\xi_S) - 2S(\varphi_u - \xi_S - \rho \varphi_S) + 2S \rho \varphi_S + \frac{9}{2}S(\varphi_u - \tau_t) = 0 )</td>
</tr>
<tr>
<td>8</td>
<td>( \xi - 2S \rho \varphi_S - \frac{3}{2}S(\varphi_u - \tau_t) = 0 )</td>
</tr>
<tr>
<td>9</td>
<td>( \varphi_{uu} - 2\xi_S = 0 )</td>
</tr>
<tr>
<td>10</td>
<td>( -2 \rho \xi - S \rho(\varphi_u - \xi_S) + S^2 \rho(2\varphi_{Su} - \xi_{SS}) + \frac{1}{2}S \rho(\varphi_u - 2\xi_S) - 3S^2(\varphi_{uu} - 2\xi_S) + \frac{3}{2}S \rho(\varphi_u - \tau_t) = 0 )</td>
</tr>
<tr>
<td>11</td>
<td>( 8\rho \xi + S \rho(\varphi_u - \xi_S - \rho \varphi_S) + 2S \rho(\varphi_u - \xi_S) - 3S^2 \rho(2\varphi_{Su} - \xi_{SS}) - 4S \rho(\varphi_u - 2\xi_S) - 3S^2(\varphi_{uu} - 2\xi_S) - 6S \rho(\varphi_u - \tau_t) = 0 )</td>
</tr>
<tr>
<td>12</td>
<td>( -12\rho \xi + S \rho^2 \varphi_S - 2S \rho(\varphi_u - \xi_S - \rho \varphi_S) - S \rho(\varphi_u - \xi_S) - 3S^2 \rho(2\varphi_{Su} - \xi_{SS}) - S^2(\varphi_{uu} - 2\xi_S) + 9S^2 \rho(\varphi_u - \tau_t) = 0 )</td>
</tr>
<tr>
<td>13</td>
<td>( -4\xi + S(\varphi_u - \xi_S - \rho \varphi_S) - 2S \rho \varphi_S - S^2(2\varphi_{Su} - \xi_{SS}) + 4S(\varphi_u - 2\xi_S) - 6S(\varphi_u - \tau_t) = 0 )</td>
</tr>
<tr>
<td>14</td>
<td>( -2\xi + S \rho \varphi_S - S(\varphi_u - 2\xi_S) + \frac{3}{2}S(\varphi_u - \tau_t) = 0 )</td>
</tr>
</tbody>
</table>
The feedback effects in illiquid markets.

\[
(1 - 2\rho S)(\varphi_{uu} - 2\xi_S) = 0
\]

\[
-\rho \xi + (1 - 2\rho S)S(2\varphi_{Su} - 2\xi_{SS}) - S\rho(\varphi_u - 2\xi_S) \\
+ 4S^2(\varphi_{uu} - 2\xi_S) = 0
\]

\[
5\rho \xi + 4S^2\rho(2\varphi_{Su} - \xi_{SS}) + 5S\rho(\varphi_u - 2\xi_S) - 6S(\varphi_{uu} \\
- 2\xi_S) - \frac{5}{2}S\rho(\varphi_u - \tau_t) = 0
\]

\[
-10\rho \xi + 4S^2\rho^2(2\varphi_{Su} - \xi_{SS}) - 10S\rho(\varphi_u - 2\xi_S) \\
+ 4S^2(\varphi_{uu} - 2\xi_S) + 5S\rho(\varphi_u - \tau_t) = 0
\]

\[
10\rho \xi + 4S^2\rho(2\varphi_{Su} - \xi_{SS}) + 10S\rho(\varphi_u - 2\xi_S) - S^2(\varphi_{uu} \\
- 2\xi_S) - 5S\rho(\varphi_u - \tau_t) = 0
\]

\[
\xi + S(\varphi_u - 2\xi_S) + \frac{1}{2}S(\varphi_u - \tau_t) = 0
\]

\[
\xi + S(\varphi_u - 2\xi_S) - S^2\rho(2\varphi_{Su} - \xi_{SS}) - \frac{1}{2}S(\varphi_u - \tau_t) = 0
\]

\[
\varphi_{uu} - 2\xi_S = 0
\]

\[
-2\varphi_{Su} + \xi_{SS} + 5(\varphi_{uu} - 2\xi_S) = 0
\]

\[
\rho(2\varphi_{Su} - \xi_{SS}) - 2(\varphi_{uu} - 2\xi_S) = 0
\]

\[
-\rho(2\varphi_{Su} - \xi_{SS}) + (\varphi_{uu} - 2\xi_S) = 0
\]

\[
2\rho(2\varphi_{Su} - \xi_{SS}) - \varphi_{uu} + 2\xi_S = 0
\]

\[
-5\rho(2\varphi_{Su} - \xi_{SS}) + \varphi_{uu} - 2\xi_S = 0
\]

\[
2\varphi_{Su} - \xi_{SS} = 0
\]
CHAPTER 4. THE LIE ALGEBRAIC STRUCTURE OF THE MAIN EQUATION
Chapter 5

The symmetry group admitted by the equation (4.1)

Let us now find the symmetry group admitted by the equation (4.1)

\[ u_t + \frac{1}{2} \sigma^2 S^2 u_{ss} \frac{(1 - \rho u_S)^2}{(1 - \rho u_S - \rho u_S u_S)^2} = 0. \]

According to the fundamental theorem of the Lie group analysis (see Theorem 3.0.2) to find the global representation of the group symmetry admitted by this equation we have to solve the following system of ordinary differential equations with the corresponding initial conditions

\[ \frac{dt_\epsilon}{d\epsilon} = k, \quad t_\epsilon|_{\epsilon=0} = t, \]
\[ \frac{dS_\epsilon}{d\epsilon} = CS_\epsilon, \quad S_\epsilon|_{\epsilon=0} = S, \]
\[ \frac{du_\epsilon}{d\epsilon} = 2Cu_\epsilon - \frac{CS_\epsilon}{\rho} + d., \quad u_\epsilon|_{\epsilon=0} = u. \]

(5.1)

To study different particular cases we will consider action of different subgroups. We will consider following particular cases of the system (5.1)

1. \( C \neq 0, \ k \neq 0, \)
2. \( C \neq 0, \ k = 0, \)
3. \( C = 0, \ k \neq 0, \)
4. \( C = 0, \ k = 0. \)

Now we study all listed cases and the corresponding subgroups as well as, their invariants. Using these invariants we will reduce the main equation (4.1) to ordinary differential equations.
Case 1. $C \neq 0, \ k \neq 0$ in (5.1).

It means that complete symmetry group $G_3$ admitted by the equation (4.1) is studied, i.e., all of generators of the Lie algebra $G_3$ acting on the solutions manifold $L_\Delta$ of our equation is present. We solve the first equation of the system (5.1) and obtain

$$t_\epsilon = C_1 + k \epsilon,$$

if we insert the initial condition, we obtain the value of the constant $C$

$$C_1 + k \epsilon|_{\epsilon=0} = t,$$
$$C_1 = t,$$

consequently the transformation of the variable $t$ under the action of the group $G_3$ has the following form

$$t_\epsilon = k \epsilon + t,$$  \hspace{1cm} (5.2)

where $k$ is an arbitrary constant. The second equation of the system (5.1) describe the change of the independent variable $S_\epsilon$ under the action group $G_3$ and it is general solution has a form

$$S_\epsilon = C_2 e^{b \epsilon}.$$

The initial condition defines the value of the constant $d$

$$C_2 e^{b \epsilon}|_{\epsilon=0} = S,$$
$$C_2 = S,$$

consequently the transformation of the variable $t$ under the action of the group $G_3$ has the following form

$$S_\epsilon = S e^{C_\epsilon}.$$  \hspace{1cm} (5.3)

The last equation of the system (5.1) defines the transformation of the dependent variable $u$ under the action of the group $G_3$.

To solve the linear non-homogeneous equation

$$\frac{du_\epsilon}{d\epsilon} = 2Cu_\epsilon - \frac{CS_\epsilon}{\rho} + d,$$  \hspace{1cm} (5.4)
we first solve the corresponding homogeneous equation

\[ \frac{du_\epsilon}{d\epsilon} = 2Cu_\epsilon, \]

and obtain

\[ u_\epsilon = C_3e^{2C_\epsilon}, \]
\[ S_\epsilon = Se^{C_\epsilon} \]

then using the method of the variation of the parameters we represent \( C_3 \) as a function of \( \epsilon \), i.e. \( C_3 = C_3(\epsilon) \).

We insert \( C_3(\epsilon) \) in the inhomogeneous equation (5.4) and obtain

\[ C_3'(\epsilon) = \left( -\frac{CSe^{C_\epsilon}}{\rho} + d \right) e^{-2C_\epsilon}. \]

After integration by \( \epsilon \) we obtain the expression for \( C_3(\epsilon) \)

\[ C_3(\epsilon) = \frac{S}{\rho} e^{-C_\epsilon} - \frac{d}{2C} + C_4. \]

Finally, the expression for \( u_\epsilon \) takes the form

\[ u_\epsilon = \left( \frac{S}{\rho} e^{-C_\epsilon} - \frac{d}{2C} + C_4 \right) e^{2C_\epsilon}, \]

where the constant \( C_4 \) is defined by the initial condition

\[ u_\epsilon|_{\epsilon=0} = u = \frac{S}{\rho} - \frac{d}{2C} + C_4, \]

consequently, \( C_4 \) is equal to

\[ C_4 = u - \frac{S}{\rho} + \frac{d}{2C} \]

and

\[ u_\epsilon = \frac{S}{\rho} e^{C_\epsilon} - \frac{de^{2C_\epsilon}}{2C} + \left( u - \frac{S}{\rho} + \frac{d}{2C} \right) e^{2C_\epsilon}. \] (5.5)

Now we can describe the invariants of the symmetry group \( G_3 \) (5.1), (5.2) and (5.5). From equation (5.2) we can find that

\[ \epsilon = \frac{t_\epsilon - t}{k}, \] (5.6)
by the substitution to the expression (5.3) we obtain
\[ S_\epsilon = S e^{\frac{C}{k} (u - t)} \Rightarrow S_\epsilon e^{-\frac{C}{k} t_\epsilon} = S e^{-\frac{C}{k} t}. \]
We see that this expression is unaltered under the action of the group \( G_3 \). It means that this expression is an invariant of the group \( G_3 \), i.e.
\[ \text{inv}_1 = \ln S - bt, \text{ where } b = \frac{C}{k}. \] (5.7)

**Remark 5.0.1** Any function of an invariant is an invariant, it means we represent one of the possible forms of the first invariant of \( G_3 \).

From the equation (5.3) we obtain
\[ \epsilon = \frac{1}{C} \ln \frac{S_\epsilon}{S} \]
if we insert this expression in (5.5) then
\[ u_\epsilon = \frac{S}{\rho} e^{C \frac{1}{k} \ln \frac{S_\epsilon}{S}} - \frac{d}{2C} + \left( u - \frac{S}{\rho} + \frac{d}{2C} \right) e^{2C \frac{1}{k} \ln \frac{S_\epsilon}{S}} \]
consequently we obtain
\[ u_\epsilon - \frac{S_\epsilon}{\rho} + \frac{d}{2C} = \left( u - \frac{S}{\rho} + \frac{d}{2C} \right) \left( \frac{S_\epsilon}{S} \right)^2 \]
and finally
\[ \left( u_\epsilon - \frac{S_\epsilon}{\rho} + \frac{d}{2C} \right) \frac{1}{S_\epsilon^2} = \left( u - \frac{S}{\rho} + \frac{d}{2C} \right) \frac{1}{S^2}. \]
Hence we can take as the second invariant
\[ \text{inv}_2 = \left( u - \frac{S}{\rho} + \frac{d}{2C} \right) \frac{1}{S^2}. \] (5.8)

Let us introduce new variables in equation (4.1). We denote a new independent variable by \( z \) where \( z = \text{inv}_1 = \ln S - bt \), and suppose that a new dependent variable is
\[ w(z) = \text{inv}_2 = \left( u - \frac{S}{\rho} + \frac{d}{2C} \right) \frac{1}{S^2}. \]
We reduce the number of independent variables from two to one and correspondingly obtain an ordinary differential equation whose solutions are group
The feedback effects in illiquid markets.

Let us find representation for the dependent variables using invariants \((z, w)\)

\[ u = wS^2 + \frac{S}{\rho} - \frac{d}{2C}, \]

We represent now the partial derivatives of \(u\) in the following way

\[ u_t = -bw'S^2, \]
\[ u_S = Sw' + 2Sw + \frac{1}{\rho}, \]
\[ u_{SS} = 3w' + w'' + 2w. \]

Substitute these expressions to equation (4.1) we obtain a non-linear second order ordinary differential equation

\[
\frac{bw'}{2} - \frac{1}{2}\sigma^2 (3w' + w'' + 2w) \frac{(w' + 2w)^2}{(4w' + w'' + 4w)^2} = 0
\]

**Case 2.** \(C \neq 0, k = 0\)

We take a two dimensional Lie subalgebra \(L_2\) which is spanned by generators

\[ \bar{U}_1 = \frac{\partial}{\partial u}, \]
\[ \bar{U}_2 = S \frac{\partial}{\partial S} + (2u - \frac{S}{\rho}) \frac{\partial}{\partial u}. \]

The system (5.1) describing the global group representation has now the form

\[
\frac{dt}{d\epsilon} = 0 \quad \Rightarrow \quad t_\epsilon = t,
\]
\[
\frac{dS_{\epsilon}}{d\epsilon} = CS_\epsilon, \quad S_{\epsilon |_{\epsilon = 0}} = S, \quad (5.9)
\]
\[
\frac{du_\epsilon}{d\epsilon} = 2Cu_\epsilon - \frac{CS_\epsilon}{\rho} + d, \quad u_{\epsilon |_{\epsilon = 0}} = u.
\]

As in the previous case we can see that the transformation of the variable \(S\) can be described as follows

\[ S_{\epsilon} = S e^{C\epsilon}. \quad (5.10) \]

From the equation (5.10) follows

\[ \epsilon = \frac{1}{C} \ln \frac{S_\epsilon}{S}, \]
and consequently
\[ u_ε = \frac{S}{ρ} e^{C \frac{1}{2} \ln \frac{S_ε}{S}} - \frac{d}{2C} + \left( u - \frac{S}{ρ} + \frac{d}{2C} \right) e^{2C \frac{1}{2} \ln \frac{S}{S_ε}}, \]

after transformations
\[ u_ε - \frac{S_ε}{ρ} + \frac{d}{2C} = \left( u - \frac{S}{ρ} + \frac{d}{2C} \right) \left( \frac{S_ε}{S} \right)^2, \]

and finally
\[ \left( u_ε - \frac{S_ε}{ρ} + \frac{d}{2C} \right) \frac{1}{S_ε^2} = \left( u - \frac{S}{ρ} + \frac{d}{2C} \right) \frac{1}{S^2}. \]

We can take as a first invariant the expression
\[ inv_1 = \left( u - \frac{S}{ρ} + \frac{d}{2C} \right) \frac{1}{S^2}. \]

(5.11)

Let us denote a new independent variable \( z = t \), and as a new dependent variable
\[ w(t) = inv_2 = \left( u - \frac{S}{ρ} + \frac{d}{2C} \right) \frac{1}{S^2}, \]

then the old dependent variable \( u \) is now presented by the expression
\[ u = wS^2 + \frac{S}{ρ} - \frac{d}{2C}. \]

The partial derivatives of \( u \) can be represented in the following form
\[ u_ε = S^2 w', \]
\[ u_S = 2Sw + \frac{1}{ρ}, \]
\[ u_{SS} = 2w. \]

Substitute to equation (4.1) we obtain a non-linear first order ordinary differential equation
\[ w' - \frac{1}{4} α^2 w = 0 \]

This equation describes invariant solutions to (4.1) variable \( t \) is unaltered under action of the group \( G_2 \), i.e. all solutions of the type:
\[ u(s, t) = S^2 w(t) + \frac{S}{ρ} - \frac{d}{2C}. \]
Case 3. $C = 0, \ k \neq 0$

In this case we have subalgebra of $L_3$ which is spanned by generators

\[ \tilde{U}_1 = \frac{\partial}{\partial t}, \]
\[ \tilde{U}_2 = \frac{\partial}{\partial u}. \]

The Lie equations have now the form

\[ \frac{dS}{d\epsilon} = 0, \quad \Rightarrow \quad S_\epsilon = S \]
\[ \frac{dt_\epsilon}{d\epsilon} = k, \quad t_\epsilon|_{\epsilon=0} = t, \]
\[ \frac{du_\epsilon}{d\epsilon} = d, \quad u_\epsilon|_{\epsilon=0} = u. \]  
(5.12)

We can see that the global transformations of the variables $t$ and $S$ can be represent in the following form

\[ t_\epsilon = k\epsilon + t, \quad S_\epsilon = S. \]  
(5.13)

The last equation of the system (5.12) defines transformation of on the variable $u$. To find the global form we have to solve a linear inhomogeneous equation

\[ \frac{du_\epsilon}{d\epsilon} = d, \]  
(5.14)

after transformations we obtain

\[ u_\epsilon = d\epsilon + C_6. \]
\[ u_\epsilon|_{\epsilon=0} = u = d\epsilon + C_2, \]  
(5.15)

consequently,

\[ C_2 = u \]  
(5.16)

and after the substitution (5.16) in (5.15) we obtain

\[ u_\epsilon = d\epsilon + u. \]  
(5.17)

From equation (5.13) we obtain $\epsilon = \frac{t_\epsilon - t}{k}$ and insert this result in equation (5.17). We obtain

\[ u_\epsilon = \frac{d}{k}(t_\epsilon - t) + u. \]
or in symmetrical form
\[ u_\epsilon - \frac{d}{k} t_\epsilon = u - \frac{d}{k} t. \]

Hence we can take as a first invariant the expression
\[ \text{inv}_2 = u - \frac{d}{k} t. \] (5.18)

Let us denote a new independent variable by \( z = s \), and a new dependent variable by \( \omega \)
\[ w(z) = \text{inv}_2 = u - \frac{d}{k} t. \]

In new variables
\[ u_t = w + \frac{d}{k}, \]
\[ u_s = w', \]
\[ u_{SS} = w''. \]

Substitute the results to equation (4.1) we obtain a non-linear second order ordinary differential equation
\[
w + \frac{d}{k} - \frac{1}{2} \sigma^2 S^2 w'' \left[ \frac{1 - \rho w}{1 - \rho w' - \rho S w''} \right]^2 = 0
\]

In this case the solutions depend only in trivial way on \( t \)
\[ u(S, t) = -\frac{d}{k} t + f(S), \]
where \( d \) and \( k \) are constants.

**Case 4.** \( C = 0, \quad k = 0 \)

We have one dimensional Lie algebra \( L_1 \) spanned by generator
\[ \tilde{U}_1 = \frac{\partial}{\partial u}. \]

The Lie equations has now the form
\[
\frac{d u_\epsilon}{d \epsilon} = d, \quad u_\epsilon|_{\epsilon=0} = u, \quad \Rightarrow \quad u_\epsilon = u + \epsilon d.
\]

This case is trivial. When both variables \( S, t \) to be invariants of the corresponding group \( G_1 \) we do not have any transformations.
Chapter 6

Conclusions

In this work we have considered the equilibrium or reaction-function model (2.6) by analytical methods. Because the main equation (2.6) is nonlinear, it is difficult to choose the appropriate method to find a solution.

By using the methods of Lie groups analysis, the complete symmetry algebra was found which is admitted by the following equation

\[ u_t + \frac{1}{2} \sigma^2 S^2 u_{SS} \frac{(1 - \rho u_S)^2}{(1 - \rho u_S - \rho S u_{SS})^2} = 0. \]

The main result is represented in the Theorem 4.0.5.

The structure of the model leads to the three parametrical group of symmetry \( G_3 \), given by global representations (5.2), (5.3), (5.5). By using this result we obtain a number of invariants, which were used to obtain the following reductions of the main equation

1. \[ bw' - \frac{1}{2} \sigma^2 (3w' + w'' + 2w) \frac{(w' + 2w)^2}{(4w' + w'' + 4w)^2} = 0, \]

where

\[ z = \ln S - bt, \quad w(z) = \left( u - \frac{S}{\rho} + \frac{d}{2C} \right) \frac{1}{S^2}, \]

2. \[ w' - \frac{1}{4} \sigma^2 w = 0, \]

where

\[ z = t, \quad w(z) = \left( u - \frac{S}{\rho} + \frac{d}{2C} \right) \frac{1}{S^2}, \]
3.  

\[ w + \frac{d}{k} - \frac{1}{2} \sigma^2 S^2 w'' \left[ \frac{1 - \rho w}{1 - \rho w' - \rho S w''} \right]^2 = 0, \]

where  

\[ z = S, \quad w(z) = u - \frac{d}{k} t. \]

The listed ordinary differential equations considerably more simple as than the main model (2.6). The solutions of these equations will give rise to the invariant solutions of the main equation. This task is out of the frame of the present work.
Bibliography


Glossary

**Asset.** Items of value owned by an individual. Assets that can be quickly converted into cash are considered "liquid assets." These include bank accounts, stocks, bonds, mutual funds, and so on. Other assets include real estate, personal property, and debts owed to an individual by others.

**Bond.** A debt instrument, issued by a borrower and promising a specified stream of payments to the purchaser, usually regular interest payments plus a final repayment of principal. Bonds are exchanged on open markets including, in the absence of capital controls, internationally, providing a mechanism for international capital mobility.

**Black-Scholes formula.** An equation to value a call option that uses the stock price, the exercise price, the risk-free interest rate, the time to maturity, and the standard deviation of the stock return.

\[
C_0 = S_0 N(d_1) - X e^{-rT} N(d_2),
\]

where

\[
d_1 = \frac{\ln \frac{S_0}{X} + (r - \delta + \frac{\sigma^2}{2})T}{\sigma \sqrt{T}},
\]

\[
d_2 = d_1 - \sigma \sqrt{T}
\]

and where

- \( C_0 \) - Current call option value.
- \( S_0 \) - Current stock price.
- \( N(d) \) - The probability that a random draw from a standard normal distribution will be less than d.
- \( X \) - Exercise price.
- \( r \) - Risk-free interest rate.
- \( T \) - Time to maturity of option, in years.
- \( \sigma \) - Standard deviation of the annualized continuously compounded rate of return of the stock.

**Classical Black-Scholes equation.** An Itô process for the price of the underlying is taken as given:

\[
dS_t = \nu(S_t, t)dt + \lambda(S_t, t)dW_t
\]

where \( \{W_t, t > 0\} \) is a standard Brownian motion on a probability space \((\Omega, F, P)\), and \( \nu \) and \( \lambda \) satisfy Lipschitz.

\[
\frac{\partial C}{\partial t} + \frac{1}{2} \lambda^2(s, t) \frac{\partial^2 C}{\partial s^2} + r \left( s \frac{\partial C}{\partial s} - C \right) = 0, \quad \text{for } s > 0, \ t > 0
\]
which is known as the classical Black-Scholes equation and which we used in this work.

**Common stock.** Equities, or equity securities, issued as ownership shares in a publicly held corporation. Shareholders have voting rights and may receive dividends based on their proportionate ownership.

**Demand function.** The mathematical function explaining the quantity demanded in terms of its various determinants, including income and price; thus the algebraic representation of the demand curve.

**Disequilibrium**

1. Inequality of supply and demand.

2. An untenable state of an economic system, from which it may be expected to change.

**Equilibrium.**

1. A state of a balance between offsetting forces for change, so that no change occurs.

2. In competitive markets, equality of quantity supplied and quantity demanded.

**Equilibrium price.** The market price at which the supply of an item equals the quantity demanded.

**Fair price.** In anti-dumping cases, the price to which the export price is compared, which is either the price charged in the exporter’s own domestic market or some measure of their cost, both adjusted to include any transportation cost and tariff needed to enter the importing country’s market. See dumping.

**Financial assets.** Financial assets such as stocks and bonds are claims to the income generated by real assets or claims on income from the government.

**Hedging.** Investing in an asset to reduce the overall risk of a portfolio.

**Hedging demands.** Demands for securities to hedge particular sources of consumption risk, beyond the usual mean-variance diversification motivation.

**Income.**

1. The amount of money (nominal or real) received by a person, household, or other economic unit per unit time in return for services provided or goods sold.

3. The return earned on an asset per unit time.

**Interest rate.** The amount of money charged as a fee for lending money or the price of borrowing money.

**Liquidity.** The capacity to turn assets into cash, or the amount of assets in a portfolio that have that capacity. Cash itself (i.e., money) is the most liquid asset.

**Market equilibrium.** Equality of quantity supplied and quantity demanded. See equilibrium.

**Money market.** The money market, in macroeconomics and international finance, refers to the equilibration of demand for a country’s domestic money to its money supply. Both refer to the quantity of money that people in the country hold (a stock), not to the quantity that people both in and out of the country choose to acquire during a period in the exchange market, mostly for the purpose of then using it to buy something else.

**Numeraire.** The unit in which prices are measured. This may be a currency, but in real models, such as used trade models, the numeraire is usually one of the goods, whose price is then set at one. The numeraire can also be defined implicitly by, for example, the requirement that prices sum to some constant.

**Programme trader.** Trader who trade the asset following a Black-Scholes type dynamic hedging strategy.

**Program trading.** Coordinated buy orders and sell orders of entire portfolios, usually with the aid of computers, often to achieve index arbitrage objectives.

**Reference trader.** Trader who is the majority investing in the the asset expecting gain.

**Risk-free asset.** An asset with a certain rate of return; often taken to be short-term Treasury Bills.

**Volatility.** The extent to which an economic variable, such as a price or an exchange rate, moves up and down over time.

**Volatility risk.** The risk in the value of options portfolios due to unpredictable changes in the volatility of the underlying asset.