Valuation of portfolios under uncertain volatility: Black-Scholes-Barenblatt equations and the static hedging

Master's Thesis in Financial Mathematics
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Preface

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Abstract
The famous Black-Scholes (BS) model used in the option pricing theory contains two parameters - a volatility and an interest rate. Both parameters should be determined before the price evaluation procedure starts. Usually one use the historical data to guess the value of these parameters. For short lifetime options the interest rate can be estimated in proper way, but the volatility estimation is, as well in this case, more demanding. It turns out that the volatility should be considered as a function of the asset prices and time to make the valuation self consistent. One of the approaches to this problem is the method of uncertain volatility and the static hedging. In this case the envelopes for the maximal and minimal estimated option price will be introduced. The envelopes will be described by the Black - Scholes - Barenblatt (BSB) equations. The existence of the upper and lower bounds for the option price makes it possible to develop the worse and the best cases scenario for the given portfolio. These estimations will be financially relevant if the upper and lower envelopes lie relatively narrow to each other. One of the ideas to converge envelopes to an unknown solution is the possibility to introduce an optimal static hedged portfolio.
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Chapter 1

Introduction

In option pricing literature we can found many models for the determination of the volatility. There are different methods of determination volatility value. The most simple model assumes constant volatility. It was offered by Black, Scholes and Merton in 1973. There was a famous paper, but in real world market prices of options cannot be described by constant volatility in proper way. More complicated is the implied volatility model, where volatility is constructed by the Black-Scholes equation for a currently traded contracts. A next possibility to introduce the volatility is the stochastic volatility model. In that case the volatility is assumed to follow some random process.

In our paper we will consider uncertain volatility model, numerical methods for the solving BSB equation and the static hedging. We will achieve some improvements of previous results.

In detail, Chapter 2 presents uncertain volatility model for option prices and for the sensitivity parameter delta. We will represent some examples where $\sigma_0$ and $\sigma_1$ are functions depending on time $t$.

In Chapter 3 will illustrated using the static hedging narrow the envelop. We improved spread between upper and lower bounds of the option price.

In Chapter 4 we will represent explicit and implicit finite difference methods for valuation of an option price. We study the stability problem for the explicit method. Our algorithms will be applied to different types of European and American options such as straddle, butterfly, double barrier and others.

Chapter 5 include some concluding remarks.

Finally, Chapter 6 contains most important programmes, which were used for computing.
Chapter 2

The Uncertain Volatility Method (UVM)

2.1 The uncertain volatility model

In this chapter we consider the paper of Meyer [1]. He studied a portfolio with a non-convex and a non-monotone value function. The author proved that a Black-Scholes-Barenblatt (BSB) equation deeply connected to the Black-Scholes equation. The author consider three equivalent forms of Black-Scholes-Barenblatt equation and define and analyze the idea of the static hedging.

In [1] are considered a simple Up-and-Out European Call $V(S;t)$ with an maturity date $T$, a strike price $K$ and a Up-and-Out barrier in $S = X > K$. There is a nonlinear problem because the payoff of the Up-and-Out European Call is non-convex.

For $0 < S < X$ and $0 < t \leq T$ the following differential operator is introduced

$$
L(\sigma; r)V \equiv \frac{1}{2} \sigma^2(S;t)S^2V_{SS} + r(S;t)SV_S - r(S;t)V - V_t = 0,
$$

(2.1)

where $t = T - \tau$ ($T$ is the maturity date and $\tau$ is current time). It is assumed that an uncertain volatility $\sigma$ and an interest rate $r$ are functions of time $t$ and the asset price $S$.

The boundary conditions for the Up-and-Out Call are

$$
V(0,t) = V(X,t) = 0,
$$

(2.2)

and initial the condition is

$$
V(S,0) = (S - K)^+.
$$

(2.3)

The author of [1] used in his reasoning the standard maximum principle for a partial differential equation (PDE) of the parabolic type. Let us now look at the value of the Up-and-Out Barrier option.
To use the maximum principle we shall avoid a discontinuity at \((X, 0)\). According to equations (2.2), (2.3) we can see, that

\[
\lim_{S \to X} V(S, 0) \neq \lim_{t \to 0} V(X, t).
\]

The plot of the function \(V(S, t)\) is presented in Figure 2.1

![Payoff for the Up-and-Out Call at t = 0, K = 10, S ∈ [0; 140], X = 140.](image1)

![The piecewise linear approximation for the Up-and-Out Call at t = 0, K = 10, S ∈ [0; 140], X = 140, 0 ≤ ϵ < 1.](image2)

In [1] the initial condition was replaced by a piecewise linear function, which approximate the original condition (see Figure 2.2). Let \(0 \leq ϵ < 1\),

\[
V(S, 0) = \begin{cases} 
0, & S < K \\
S - K, & K \leq S \leq (X - ϵ) \\
(X - ϵ - K) \frac{X - S}{ϵ}, & X - ϵ < S < X.
\end{cases}
\]

As laid down by [1], we assume that the functions \(σ(S, t)\) and \(r(S, t)\) are bounded above and belove

\[
σ_0(S, t) \leq σ(S, t) \leq σ_1(S, t),
\]

where \(σ_0(S, t) \geq c > 0\) and

\[
r_0(S, t) \leq r(S, t) \leq r_1(S, t),
\]

where \(r_0(S, t) \geq 0\).

The author of [1] introduced functions \(V_0(S, t)\) and \(V_1(S, t)\) such that

\[
V_0(S, t) \leq V(S, t) \leq V_1(S, t).
\]

They are the lower and upper bounds of \(V(S, t)\) for all \(S \in (0; X)\) and \(t \in (0; T]\), for all \(σ\) and \(r\) which are within the above bounds (2.4), (2.5), respectively. In [1] is assumed throughout that BS equation has a solution for all such \(σ\) and \(r\).
To find the lower and upper bounds $V_0$ and $V_1$ the author used Black-Scholes-Barenblatt equation

$$L^BSV_0 \equiv \frac{1}{2} f_0(\sigma)^2 S^2 V_{0SS} + g_0(r)(SV_{0S} - V_0) - V_0t = 0, \quad (2.7)$$

where the function $g_0(r)$ is chosen in the following way

$$g_0(r) = \begin{cases} r_0(S, t), & (SV_{0S} - V_0) \geq 0, \\ r_1(S, t), & (SV_{0S} - V_0) < 0, \end{cases} \quad (2.8)$$

and, correspondingly, the function $f_0(\sigma)$ is represented by

$$f_0(\sigma) = \begin{cases} \sigma_0(S, t), & V_{0SS} \geq 0, \\ \sigma_1(S, t), & V_{0SS} < 0. \end{cases} \quad (2.9)$$

Author of [1] to produce proofs that

$$V_0(S, t) \leq V(S, t) \leq V_1(S, t). \quad (2.10)$$

We will not present full proof in our work (see [1]). But we prove, that $V_0$ is the lower bound in case simple European Call option. Let us investigate the equation (2.7) for European Call option at time $t = 0.1$ with strike price $K = 50$ and time of expiry $T = 1$.

Figure 2.3, Figure 2.4 shows that in case of the European Call option $g_0(r) = r_0(S, t)$, i.e., $(SV_{0S} - V_0) \geq 0$ for all $S$ and $f_0(\sigma) = \sigma_0(S, t)$, i.e., $V_{0SS} \geq 0$ for all $S$.

Hence the solution $V(S, t)$ for the European Call is convex for all values $S$ and $t$ and it means that the BSB equation can be reduced to the standard Black-Scholes equation, i.e. to the equation

$$L(\sigma, r)V_0 = 0.$$

The equation (2.7) for the function $V_0(S, t)$ can be rewritten in another form, which is more convenient for further investigations. We represent $a$ as a sum $a = a^+ + a^-$, consequently $L^BSV_0$ takes the form

$$L^BSV_0 \equiv \frac{1}{2} \sigma_0^2 S^2 V_{0SS}^+ + \frac{1}{2} \sigma_1^2 S^2 V_{0SS}^- + r_0(SV_{0S} - V_0)^+ + r_1(SV_{0S} - V_0)^- - V_0t = 0. \quad (2.11)$$

Let us add and subtract $L(\sigma, r)V_0$ (see (2.1)), to the above expression (2.11). Then the right-hand expression in the previous formula take the form

$$L^BSV_0 \equiv \frac{1}{2} \sigma_0^2 S^2 V_{0SS}^+ + \frac{1}{2} \sigma_1^2 S^2 V_{0SS}^- + r_0(SV_{0S} - V_0)^+ + r_1(SV_{0S} - V_0)^- - V_0t$$

$$+ \frac{1}{2} \sigma_0^2 S^2 V_{0SS}^+ + \frac{1}{2} \sigma_1^2 S^2 V_{0SS}^- + r(SV_{0S} - V_0)^+ + r(SV_{0S} - V_0)^- - V_0t$$

$$- \frac{1}{2} \sigma_0^2 S^2 V_{0SS}^+ - \frac{1}{2} \sigma_1^2 S^2 V_{0SS}^- - r(SV_{0S} - V_0)^+ - r(SV_{0S} - V_0)^- + V_0t = 0.$$
Combine the similar terms we obtain the equation for the lower bond $V_0(S,t)$

$$
\mathcal{L}(\sigma, r)V_0 \equiv \frac{1}{2}S^2 ((\sigma^2 - \sigma_0^2)V_{0SS}^+ + (\sigma^2 - \sigma_1^2)V_{0SS}^-) \\
+ ((r - r_0)(SV_{0S} - V_0)^+ + (r - r_1)(SV_0 - V_0)^-). 
$$

(2.12)

In [1] the author give proof that the function $V_0(S,t)$ is a lower bound for the option value $V(S,t)$ for any $\sigma$ and $r$ between the imposed limits (2.4), (2.5). We will not go in detail on this fact.

Analogously, we have for the upper bound $V_1(S,t)$ the following equation

$$
\mathcal{L}^{BSB}_1V_1 \equiv \frac{1}{2}f_1(\sigma)^2S^2V_{1SS} + g_1(r)(SV_{1S} - V_1) - V_{1t} = 0,
$$

where the function $g_1(r)$ and $f_1(\sigma)$ are given by

$$
g_1(r) = \begin{cases} r_0(S,t), & (SV_{1S} - V_1) \geq 0, \\ r_1(S,t), & (SV_{1S} - V_1) < 0, \end{cases} 
$$

(2.13)

and

$$
f_1(\sigma) = \begin{cases} \sigma_0(S,t), & V_{1SS} \geq 0, \\ \sigma_1(S,t), & V_{1SS} < 0. \end{cases} 
$$

(2.14)

The equation (2.12) in this case can be rewritten in the form

$$
\mathcal{L}(\sigma, r)V_1 \equiv \frac{1}{2}S^2 ((\sigma^2 - \sigma_0^2)V_{1SS}^+ + (\sigma^2 - \sigma_1^2)V_{1SS}^-) \\
+ ((r - r_0)(SV_{1S} - V_1)^+ + (r - r_1)(SV_1 - V_1)^-). 
$$

(2.15)
We will use the following identities for obtaining a symmetric form of these equations
\[ a^+ = \frac{a + |a|}{2}, \quad a^- = \frac{a - |a|}{2}. \]

Let \( \sigma^2 \) and \( \bar{r} \) are
\[ \sigma^2 = \frac{\sigma_0^2 + \sigma_1^2}{2}, \quad \bar{r} = \frac{r_0 + r_1}{2}. \]

Insert the last expression in (2.12) and obtain
\[
\frac{1}{2} S^2 \left( \frac{(\sigma_0^2 + \sigma_1^2)}{2} - \sigma_0^2 \right) V_0^+ + \left( \frac{(\bar{r} + r_1)}{2} - r_0 \right) (SV_0 - V_0)^+ + \left( \frac{\sigma_0^2}{2} - \sigma_1^2 \right) V_0^- \left( \frac{SV_0}{2} - V_0 \right)^-.
\]

Let us collect the similar terms
\[
\frac{1}{2} S^2 \sigma_0^2 + \frac{1}{2} \sigma_1^2 + \frac{1}{2} (V_0^+ - V_0^-) + \frac{r_1}{2} - \frac{r_0}{2} \left( (SV_0 - V_0)^+ - (SV_0 - V_0)^- \right),
\]
so that introducing the function
\[
F(S, V, V_0, |V_{SS}|) = \frac{1}{2} \sigma_0^2 - \frac{1}{2} \sigma_1^2 S^2 |V_{SS}| + \frac{r_1}{2} - \frac{r_0}{2} |SV_0 - V|,
\]
equations (2.12), (2.13) can be rewrite in another forms
\[
\mathcal{L}(\sigma, r)V_0 = F(S, V, V_0, |V_{SS}|), \tag{2.17}
\]
\[
\mathcal{L}(\sigma, r)V_1 = -F(S, V, V_1, |V_{SS}|). \tag{2.18}
\]

The initial and boundary conditions for equations (2.17), (2.18) are
\[
V_0(S, 0) = V_1(S, 0) = V(S, 0) = 0, \quad V_0(0, t) = V_1(0, t) = V(0, t) = 0, \quad V_0(X, t) = V_1(X, t) = V(X, t) = 0.
\]

The equations (2.12), (2.7), (2.17) and (2.15), (2.13), (2.18) are equivalent to each other. They are be called Black-Scholes-Barenblatt equations.

**Example 2.1.1** We repeat calculations given in [1]. At first step we construct the envelopes of a Double-Barrier-European-Straddle \( V(S, t) \) for \( t = T = 0.1 \) which is defined by
\[
\mathcal{L}(\sigma, r)V(S, t) = 0, \quad V(S, 0) = (S - 100)^+ + (100 - S)^+, \quad V(80, t) = V(120, t) = 0, \tag{2.19}
\]
where
\[
0.1 < \sigma < 0.2, \quad 0.05 < r < 0.06.
\]

The results of calculations we are given in Figure 2.5.
Chapter 2. The Uncertain Volatility Method (UVM)

Figure 2.5: Envelopes \( V_0(S,t) \) and \( V_1(S,t) \) (dashed lines) and Black-Scholes solution \( V(S,t) \) (solid line) for the European-Double-Barrier-Straddle (2.19), with \( \sigma = 0.1, \ r = 0.55, \ K = 100, \ S \in [80,120], t = T = 0.1, \sigma_0 = 0.1, r_0 = 0.05, \sigma_1 = 0.2, r_1 = 0.06.\)

Figure 2.6: Envelopes \( V_0(S,t) \) and \( V_1(S,t) \) (dashed lines) and Black-Scholes solution \( V(S,t) \) (solid line) for the Double-Barrier-European-Butterfly-Spread (2.20), with \( \sigma = 0.2, \ r = 0.1, K_1 = 90, K_2 = 110, S \in [80,120], t = T = 0.25, \sigma_0 = 0.15, r_0 = 0.1, \sigma_1 = 0.25, r_1 = 0.1.\)

Example 2.1.2 Now we represent Double-Barrier-European-Butterfly-Spread \( V(S,t) \) at \( T = 0.1 \) with \( K_1 = 90, K_2 = 110. \) It was considered in the paper of D.M.Pooey, P.A.Forsyth and K.R.Vetzal (2003) [8].

Option price is described by:

\[
V(S,0) = (S - 90)^+ - 2(S - (110 + 90)/2)^+ + (S - 110)^+, \\
V(80,t) = V(120,t) = 0, \\
0.15 < \sigma < 0.25, \\
r = 0.1. \\
\]

(2.20)

In Figure 2.6 we represent results of calculations. It is similar to the example which was given in [8].

Example 2.1.3 The BSB equations can be also applied to an American option. We again repeat calculations given in [1]. From the beginning we consider Straddle similar to (2.19) with early exercise boundary \( s(t) \) for \( S < K. \) The option price \( V(S,t) \) at \( t = T = 0.1 \) is described by

\[
\mathcal{L}(\sigma, r)V(S,t) = 0, \\
V(S,0) = (S - 100)^+ + (100 - S)^+, \\
V(120,t) = 0, \\
V(s(t),t) = 100 - s(t), \\
V_S(s(t),t) = -1. \\
\]

(2.21)
where

\[ 0.1 < \sigma < 0.4, \]
\[ 0.04 < r < 0.06. \]

The results of calculations are represented in Figure 2.7. The curve between the envelopes is the solution of the Straddle at time \( t = T \) for the fixed values \( \sigma = 0.2 \) and \( r = 0.05 \).

Figure 2.7: Envelopes \( V_0(S,t) \) and \( V_1(S,t) \) (dashed lines) and Black-Scholes solution \( V(S,t) \) (solid line) for the American Straddle (2.21), with \( \sigma = 0.2, r = 0.05, K = 100, S \in [70,120], t = T = 0.1, \sigma_0 = 0.1, r_0 = 0.04, \sigma_1 = 0.4, r_1 = 0.06. \)

Further we consider the Straddle similar to (2.21) with early exercise boundary \( s(t) \) for all \( S \)

\[
\mathcal{L}(\sigma,r)V(S,t) = 0, \\
V(S,0) = (S - 100)^+ + (100 - S)^+, \\
V(120,t) = 0, \\
V(s(t),t) = 100 - s(t), \\
V_S(s(t),t) = \pm 1,
\]

(2.22)

where

\[ 0.1 < \sigma < 0.4, \]
\[ 0.04 < r < 0.06. \]

The results of calculations we present in Figure 2.8.
We can now see that envelopes $V_0$ and $V_1$ locally approach a constant volatility solution of the Black-Scholes equation and they have jumps when the convexity of the option value changes.

We used implicit and explicit methods for solving this problems. They give the similar to each other results.

### 2.2 Upper and Lower bonds estimations for the sensitivity parameter delta $V_S(S,t)$ for a Call option

We introduce the sensitivity parameter delta ($\delta$) as usual like first derivative option price $V$ with respect to the asset price $S$.

To simplify the reasoning in [1] author assumed that $\sigma(S,t)$ and $r(S,t)$ are constant or function of $t$ only. We will repeat Meyer’s argumentation in this section and consider new more advanced examples.

After differentiating the Black-Scholes equation (2.1) with respect to $S$ we obtain

$$\frac{1}{2}\sigma^2 S^2 V_{SSS} + \sigma^2 S V_{SS} + r S V_S + r V_S - r V_t - V_S^t = 0,$$

$$\frac{1}{2}\sigma^2 S^2 V_{SSS} + (\sigma^2 + r) V_{SS} S - V_S^t = 0.$$

And now we can write down the following differential equation for the delta $V_S \equiv \delta(S,t)$.

$$M(\sigma, r) \delta \equiv \frac{1}{2} S^2 \delta_{SS} + (\sigma^2 + r) S \delta_S - \delta_t = 0. \quad (2.23)$$

We study the Up-and-Out Call. In this case we have following initial conditions

$$\delta(S,0) = \begin{cases} 0, & S < K, \\ 1, & S > K. \end{cases}$$

It is known that for $\sigma$ and $r$ in (2.4), (2.5) all solutions of the Black-Scholes equation have to lie between the envelopes $V_0$ and $V_1$. As long as $V_0(0,t) = V_1(0,t)$ and $V_0(X,t) = V_1(X,t)$ we can write down following inequalities for $\delta(0,t)$ and $\delta(X,t)$

$$V_0S(0,t) \leq \delta(0,t) \leq V_1S(0,t),$$

$$V_1S(X,t) \leq \delta(X,t) \leq V_0S(X,t). \quad (2.24)$$
Similar to previous results for the option price $V(S, t)$ we transform of BSB model (2.23) to equation on a lower bond for $\delta(S, t)$

$$M(\sigma, r)\delta_0 = \frac{1}{2} S^2 (\sigma^2 - \sigma_0^2) \delta_{0SS}^+ + (\sigma^2 - \sigma_1^2) \delta_{0SS}^- + (\sigma^2 - \sigma_0^2) S \delta_{0S}^+ + (\sigma^2 - \sigma_1^2) S \delta_{0S}^- + (r - r_0) S \delta_{0S}^+ + (r - r_1) S \delta_{0S}^-,$$

(2.25)

Suppose that

$$\delta_{0S}^- + \delta_{0S}^+ = \delta_{0S},$$

then we can rewrite previous expression in the following form

$$-\frac{1}{2} \sigma_0^2 S^2 \delta_{0SS}^+ - \frac{1}{2} \sigma_1^2 S^2 \delta_{0SS}^- - \sigma_0^2 S \delta_{0S}^+ - \sigma_1^2 S \delta_{0S}^- - r_0 S \delta_{0S}^+ - r_1 S \delta_{0S}^- + \delta_t = 0,$$

or

$$\frac{1}{2} \sigma_0^2 S^2 \delta_{0SS}^+ + \frac{1}{2} \sigma_1^2 S^2 \delta_{0SS}^- + (\sigma_0^2 + r_0) S \delta_{0S}^+ + (\sigma_1^2 + r_1) S \delta_{0S}^- - \delta_t = 0.$$

We obtain equation for the lower bond on $\delta(S, t)$ in the following form

$$\frac{1}{2} \hat{f}_0(\sigma)^2 S^2 \delta_{0SS} + \left( \hat{k}_0(\sigma)^2 + \hat{g}_0(r) \right) S \delta_{0S} - \delta_t = 0.$$

(2.26)

Here we denoted by $\hat{f}_0(\sigma), \hat{k}_0(\sigma), \hat{g}_0(\sigma)$ the following functions

$$\hat{f}_0(\sigma) = \begin{cases} \sigma_0, & \delta_{0SS} \geq 0, \\ \sigma_1, & \delta_{0SS} < 0, \end{cases}$$

(2.27)

$$\hat{k}_0(\sigma) = \begin{cases} \sigma_0, & \delta_{0S} \geq 0, \\ \sigma_1, & \delta_{0S} < 0, \end{cases}$$

(2.28)

$$\hat{g}_0(\sigma) = \begin{cases} r_0, & \delta_{0S} \geq 0, \\ r_1, & \delta_{0S} < 0. \end{cases}$$

(2.29)

A symmetric form of these equations we obtain if we use the identities

$$\delta_{0SS}^+ = \frac{\delta_{0SS} + |\delta_{0SS}|}{2},$$

$$\delta_{0SS}^- = \frac{\delta_{0SS} - |\delta_{0SS}|}{2},$$

$$\bar{\sigma}^2 = \frac{\sigma_0^2 + \sigma_1^2}{2},$$

$$\bar{\tau} = \frac{r_0 + r_1}{2}.$$
Insert this identities in (2.23) we obtain the representation

\[ M(\sigma, r)\delta_0 = \frac{1}{2}\sigma^2 S^2 \delta_{SS} + (\sigma^2 + r)S\delta_S - \delta_0 = \]

\[ + \frac{1}{2}(\sigma^2 - \sigma_0^2)S\delta_0 + |\delta_0| + \frac{1}{2}(\sigma^2 - \sigma_1^2)S\delta_0 - |\delta_0| \]

\[ + (r - r_0)S\delta_0 + |\delta_0| + (r - r_1)S\delta_0 - |\delta_0| = \]

\[ \frac{1}{2}S^2 \left( \sigma^2 \delta_{SS} + \frac{\delta_{SS}}{2}(\sigma_1^2 - \sigma_0^2) \right) + S \left( \sigma^2 \delta_{SS} - \frac{\delta_{SS}}{2}(\sigma_0^2 + \sigma_1^2) + \frac{\sigma_1^2 - \sigma_0^2}{2} \right) \]

\[ + S \left( r\delta_0 - \frac{r_0 + r_1}{2}\delta_0 + \frac{r_1 - r_0}{2}|\delta_0| \right) = \]

\[ \frac{1}{2}S^2 \frac{\sigma_1^2 - \sigma_0^2}{2}|\delta_{SS}| + \frac{\sigma_1^2 - \sigma_0^2}{2}S|\delta_S| + S\frac{r_1 - r_0}{2}|\delta_0| \]

Finally the symmetric form of the BSB equation for the lower bond of \( \delta(S, t) \) is

\[ M(\sigma, r)\delta_0 = \frac{1}{2}S^2 \frac{\sigma_1^2 - \sigma_0^2}{2}|\delta_{SS}| + S \left( \frac{\sigma_1^2 - \sigma_0^2}{2} + \frac{r_1 - r_0}{2} \right)|\delta_0|. \quad (2.30) \]

**Example 2.2.1** The valuation of \( \delta \) for the Up-and-Out Call (2.31) at \( t = T = 0.1 \) is represented in Figure 2.9.

\[ \mathcal{L}(\sigma, r)V(S, t) = 0, \quad (2.31) \]

\[ V(S, 0) = (S - 100)^+, \]

\[ V(0, t) = V(120, t) = 0. \]

For \( \sigma \) and \( r \) we have constraints

\[ 0.2 = \sigma_0 \leq \sigma \leq \sigma_1 = 0.4, \]

\[ 0.04 = r_0 \leq r \leq r_1 = 0.06, \]

where for the Black-Scholes (2.1) solution we use \( \sigma = 0.3 \) and \( r = 0.05 \).

In Figure 2.9 you can see two different envelopes for the delta function. The results of calculation with the explicit method is represented by dash-dot curves and with the implicit method by dashed curves. The bounding dash-dot curves are not envelopes. It repeats the results of [1]. In our paper we have more precise results (dashed curves) obtained because we used the implicit finite difference method and apply new IT-decision.
It many papers about the uncertain volatility were stated that the boundary functions $\sigma_0$ and $\sigma_1$ dependent on $S$ and $t$. But in all examples the boundary functions $\sigma_0$ and $\sigma_1$ were constant. We will represent an example where $\sigma_0$ and $\sigma_1$ are functions dependent on time $t$. We will use the implicit method to calculate value of (2.31) in this example.

**Example 2.2.2** Now we consider quite the same Up-and-Out Call, but for a range of $\sigma$ we take a linear function with respect to time $t$. This case was not considered before in our knowledge. We have

\[
\mathcal{L}(\sigma, r)V(S, t) = 0, \\
V(S, 0) = (S - 100)^+, \\
V(0, t) = V(120, t) = 0.
\]

$\sigma$ and $r$ lie in the following boundaries

\[
0.2 + t = \sigma_0 \leq \sigma \leq \sigma_1 = 0.4 - t, \\
0.04 = r_0 \leq r \leq r_1 = 0.06,
\]

where for a Black-Scholes solution we use $\sigma = 0.3$ and $r = 0.05$.

In Figure 2.10 two different envelopes for delta are represented. Solid lines represents results of previous Example 2.2.1. Dashed lines represents Example 2.2.2. Functions $\sigma_0$ and $\sigma_1$ constrict the boundary of the uncertain volatility with increasing $t$. For Call options it leads to narrowing of the envelopes.
Chapter 3

The Static Hedging

It is known, that hedging is a method of an insurance a real transaction from a market risk. It is realized usually with help of special option portfolio. Hedgers minimize the risk by buying or selling other options.

The modern hedging was possible after three important events only. All this events happened around 1973.

- The Breton-Wood agreement was reversed, as a result of it appear a possibility of the market risk initiation.

- The Chicago Board Options Exchange (CBOE) was opened. There is one of the biggest places where options are traded. The option trading give us a real possibilities to manage the market risk.

- The famous paper of Black, Scholes & Merton about the option pricing theory and the risk management was published.

There are two different types of hedging: Static hedging and Dynamic hedging.

**Static hedging.** We buy necessary options for the hedging purposes and hold them during all the hedging horizon. We can (for example) apply this strategy if all options in our hedge portfolio are based on the same asset. For this method is important: the Prices of all our options have to equally change during all the hedging horizon.

**Dynamic hedging.** We permanently change the structure of hedge portfolio during all the hedging horizon. It is obvious that the dynamic hedging expect making many deals. Prices of all our options have to equally change during a minimum period. We need dynamic hedging if other instruments or too expensive or not possible from different reason.

We will consider the Static hedging in this chapter. We will use it for narrowing of the envelopes of the option price. Over the last time interesting and many significant results about the Static hedging have appeared. One of the first
theoretical work in this field was published by Bowie & Carr (1994), later are Derman, Ergener & Kani (1995) and Avellaneda, Levy & Paras (1995). There is still a very active research area. Some of the latest paper are from Andersen, Andreasen & Eliezer (2002), Carr & Wu (2002) and Meyer (2004). We will proceed close to Meyer’s paper [1] here.

3.1 The narrowing procedure for the envelopes with the static hedging

The envelopes for the option price \( V(S, t) \) may lie very wide from each other although the bounds for the values the volatility \( \sigma \) and the interest rate \( r \) are tight. We can narrow the envelopes with help of the static hedging. We assume that the related instruments are traded.

Meyer assumed only the validity of the Black-Scholes equation, and relying on the maximum principle, described the static hedging in the context of PDEs with the uncertain volatility.

Let us define the static hedging for an European option with an other European option having the same maturity date. We price in this chapter the European option with the strike price \( K \) and the expiration \( T \) today. We again set \( t = T - \tau \) where \( \tau \) is the calendar time. The option value \( V(S, t) \) satisfies the Black-Scholes equation

\[
\mathcal{L}(\sigma, r)V(S, t) = 0, \quad (3.1)
\]

the initial and boundary conditions characterize the option. As in previous chapter the value of \( \sigma \) and \( r \) are uncertain but bounded above and below by

\[
\sigma_0(S, t) \leq \sigma(S, t) \leq \sigma_1(S, t),
\]

\[
r_0(S, t) \leq r(S, t) \leq r_1(S, t),
\]

and "today" we have following bounds for \( V(S(T), T) \), which cannot be improved

\[
V_0(S(T), T) \leq V(S(T), T) \leq V_1(S(T), T),
\]

where \( V_0 \) and \( V_1 \) satisfy conditions (or equations)(2.17),(2.18).

We shall assume that \( N \) other European options on the same asset are freely traded. They have the same expiration and various pay-offs at \( t = 0 \). We know their values at the expiration. Let us denote their quoted price today by \( W_i(S(T), T) \), where \( i = 1, ..., N \).

We suppose like in [1] the option prices \( W_i(S, t) \), where \( i = 1, ..., N \) satisfy the Black-Scholes equation (3.1) for the same uncertain volatility and the same interest rate functions as the option \( V(S, t) \) which we wish to bound.
Let us constitute the following portfolio:

$$\pi(S, t, \overrightarrow{c}) = V(S, t) - \sum_{i=1}^{M} c_i W_i(S, t),$$

(3.2)

where $\overrightarrow{c}$ is the set $c_1, .., c_M$ of real numbers. We choose $\overrightarrow{c}$ once and it remain unchanged (i.e. static) for the all portfolio life time. The sum $\sum_{i=1}^{M} c_i W_i(S, t)$ is called a static hedge of $V(S, t)$.

Now we can apply Black-Scholes operator for the portfolio $\pi(S, t, \overrightarrow{c})$. We obtain the following expression

$$\mathcal{L}(\sigma, r)\pi(S, t, \overrightarrow{c}) = 0,$$

with the initial condition

$$\pi(S, 0, \overrightarrow{c}) = V(S, 0) - \sum_{i=1}^{M} c_i W_i(S, 0).$$

Furthermore the values for $\pi(S, t, \overrightarrow{c})$ at $S_0$ and $S_1$ are known.

Similar to the previous chapter we can bound portfolio $\pi(S, t, \overrightarrow{c})$ from below with the BSB equation (2.17) for all values of the constants $c_i$.

For all $\sigma$ and $r$ in their specified ranges the lower bound $\pi_0$ for $\pi(S(T), T, \overrightarrow{c})$ gives the lower bound on $V(S(T), T)$

$$V(S(T), T) = \pi(S(T), T, \overrightarrow{c}) + \sum_{i=1}^{M} c_i W_i(S(T), T).$$

In the same way we determine an upper bound $\pi_1$ for a portfolio $\pi$

$$\pi_1(S, t, \overrightarrow{d}) = V(S, t) - \sum_{j=1}^{N} d_j W_j(S, t),$$

(3.3)

which gives us an upper bound on $V(S, t)$.

For simplicity, we assume that the lower and upper bounds are independent on each other. The portfolios (3.2) and (3.3) may have different numbers and types of options.

Our goal is to define the coefficients $c_i$ and $d_i$. If we look on the seller position then to determine the lowest upper bound we have to find $c_i$ which minimizes the function

$$E_u(\overrightarrow{c}) = \pi_1(S(T), T, \overrightarrow{c}) + \sum_{i=1}^{N} c_i W_i(S(T), T).$$

(3.4)
Similarly, if we look on the buyer position then to determine the biggest lower bound from $V(S(T), T)$ we have to find $d_j$ which maximizes

$$E_i(d) = \pi_0(S(T), T, d) + \sum_{j=1}^{N} d_j W_j(S(T), T). \quad (3.5)$$

Let us define

$$\hat{V}_1(S(T), T) = \min_{c_1, \ldots, c_M} E_u(\overline{c}),$$
$$\hat{V}_0(S(T), T) = \min_{d_1, \ldots, d_N} E_l(\overline{d}).$$

It is easy to see, that

$$V_0(S(T), T) \leq \hat{V}_1(S(T), t) \leq E_u(\overline{0}) = V_1(S(T), T),$$
$$V_0(S(T), T) = E_l(\overline{0}) \leq \hat{V}_0(S(T), t) \leq V_1(S(T), T),$$

where $V_0$ and $V_1$ satisfy (2.7) and (2.17) at $t = T$.

The arguments remain unchanged if the options $W_i$ expire at different times.

### 3.2 Numerical examples for the static hedging

Now we represent some examples of the narrowing procedure for the envelopes with the static hedging. We repeat partly calculations given in [1]. Improve of the previous results. We values the European Call option with $T = 0.5, K = S = 100, r = 0.05$ and $0.2 \leq \sigma(S, t) \leq 0.3$.

**Example 3.2.1** We consider a European Put $W(S, t)$ with $K = 100, T = 0.5$ and $r = 0.05$. We know $W(100, 0.5) = 5.791$, which corresponds to an implied volatility of $\sigma = 0.25$. For the static hedging we construct the portfolio

$$\pi(S, t, c) = V(S, t) - cW(S, t).$$

The Black-Scholes equation hold true

$$\mathcal{L}(\sigma, r)\pi = 0 \quad (3.6)$$

where the initial conditions are

$$\pi(S, 0, c) = (S - K)^+ - c(K - S)^+ \quad (3.7)$$

and the boundary conditions defined by

$$\lim_{S \to \infty} \pi(S, t, c) = \lim_{S \to \infty} (S - Ke^{-rt}),$$
$$\pi(0, t, c) = -cKe^{-rt}. \quad (3.9)$$
Now we prove that the static hedging give us a very good improvement. If we apply the BSB equation to the above Call we obtain the following result

\[ 6.89 = V_0(100; 0.5) < V(100; 0.5) < V_1(100; 0.5) = 9.64. \]

There is too large spread between upper and lower envelopes. The static hedging give us zero spread in this case. We presented this result in Figure 3.2.1 for \( c = 1 \).

Let us consider more complicated example.

**Example 3.2.2** Let us consider two European Call options \( W_1 \) and \( W_2 \) with strikes \( K_1 = 90 \) and \( K_2 = 110 \). They are freely traded with option prices \( W_1(100, 0.5) = 14.42 \) and \( W_2(100, 0.5) = 4.22 \), respectively (an implied volatility for both of them is equal to \( \sigma = 0.25 \)).

We construct the portfolio

\[ \pi(S, t, c) = V(S, t) - c_1 W(S, t) - c_2 W(S, t). \] (3.10)

Now we solve

\[ \mathcal{L}(\sigma, r)\pi = 0, \]

with the initial conditions

\[ \pi(S, 0, c) = (S - 100)^+ - c_1(S - 90)^+ - c_2(S - 110)^+, \]

and the boundary conditions

\[ \pi(0, t, c) = 0, \]

\[ \pi(X, t, c) = (X - 100e^{-rt}) - c_1(X - 90e^{-rt}) - c_2(X - 110e^{-rt}). \]
The example gave us following results

\[ \hat{V}_0(S(T), T) = 7.76012 \quad \text{at} \quad (c_1, c_2) = (0.3, 1), \]

\[ \hat{V}_1(S(T), T) = 8.5237 \quad \text{at} \quad (c_1, c_2) = (1, 0.5). \]

In [1] Meyer obtain following results

\[ \hat{V}_0(S(T), T) = 7.760 \quad \text{at} \quad (c_1, c_2) = (0.6, 0.6), \]

\[ \hat{V}_1(S(T), T) = 8.667 \quad \text{at} \quad (c_1, c_2) = (0.7, 0.5). \]

In his paper the spread for the same case was equal to 1.038, in our work the spread is equal to 0.9225. We have some improvement about 11, 13\%.
Chapter 4

The numerical approach

4.1 The Finite Difference Method

Let us represent some citation from [5] to explain idea of the finite difference method.

"The idea underlying finite-difference methods is to replace the partial derivatives occurring in partial differential equation by approximations based on Taylor series expansions of functions near the point or points of interest.

For example, the partial derivative \( \frac{\partial V}{\partial t} \) may be defined to be the limiting difference

\[
\frac{\partial V}{\partial t}(S, t) = \lim_{\Delta t \to 0} \frac{V(S, t + \Delta t) - V(S, t)}{\Delta t}.
\]

If, instead of taking the limit \( \Delta t \to 0 \), we regard \( \Delta t \) as nonzero but small, we obtain the approximation

\[
\frac{\partial V}{\partial t}(S, t) \approx \frac{V(S, t + \Delta t) - V(S, t)}{\Delta t} + O(\Delta t).
\]

This is called a finite-difference approximation or finite difference of \( \partial V/\partial t \) because it involves small, but not infinitesimal, differences of the dependent variable \( V \). This particular finite-difference approximation is called a forward difference, since the differencing is in the forward \( t \) direction; only the values of \( V \) at \( t \) and \( t + \Delta t \) are used. As the \( O(\Delta t) \) term suggests, the smaller \( \Delta t \) is, the more accurate the approximation.

We also have

\[
\frac{\partial V}{\partial t}(S, t) = \lim_{\Delta t \to 0} \frac{V(S, t) - V(S, t - \Delta t)}{\Delta t},
\]

so that the approximation

\[
\frac{\partial V}{\partial t}(S, t) \approx \frac{V(S, t) - V(S, t - \Delta t)}{\Delta t} + O(\Delta t).
\]
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is likewise a finite-difference approximation for $\frac{\partial V}{\partial t}$.

We call this finite-difference approximation a backward difference.

We can also define central differences by noting that

$$\frac{\partial V}{\partial t}(S, t) = \lim_{\Delta t \to 0} \frac{V(S, t + \Delta t) - V(S, t - \Delta t)}{2\Delta t}.$$  

This gives rise to the approximation

$$\frac{\partial V}{\partial t}(S, t) \approx \lim_{\Delta t \to 0} \frac{V(S, t + \Delta t) - V(S, t - \Delta t)}{2\Delta t} + O((\Delta t)^2).$$  

Note that central differences are more accurate (for small $\Delta t$) than either forward or backward differences.

We can define finite-difference approximations for the $x$-partial derivative of $V$ in exactly the same way. For example, the central finite-difference approximation is easily seen to be

$$\frac{\partial V}{\partial t} \approx \frac{V(S, t + \Delta t/2) - V(S, t - \Delta t/2)}{\Delta t} + O((\Delta t)^2).$$  

For second partial derivatives, such as $\frac{\partial^2 V}{\partial S^2}$, we can define a symmetric finite-difference approximation as the forward difference of backward-difference approximations to the first derivative or as the backward-difference of forward-difference approximations to the first derivative. In either case we obtain the symmetric central-difference approximation

$$\frac{\partial^2 V}{\partial S^2}(S, t) \approx \frac{V(S + \Delta S, t) - 2V(S, t) + V(S - \Delta S, t)}{(\Delta t)^2} + O((\Delta t)^2).$$  

Although there are other approximation, this approximation to $\frac{\partial^2 V}{\partial V^2}$ is preferred, as its symmetry preserves the reflectional symmetry of the second partial derivative. It is left invariant by reflections of the form $x \mapsto -x$. It is also more accurate than other similar approximations."

4.1.1 Application of the explicit finite difference method to the values of the option $V(S, t)$

In paper of Ioffe & Ioffe [2] the finite difference method is applied to pricing barrier options. Authors considered the finite difference method for the usual Black-Scholes equation (2.1). We apply this method to the nonlinear BSB equation.

To implement a finite difference method we take a grid where the horizontal direction represents discrete values of the variable $S$ with constant step $\Delta S$ and
the vertical direction represents discrete points in time with constant step $\Delta t$. In any node $(i, j)$ we calculate the values of time $t$ and variable $S$ as

$$t_i = i \Delta T, \quad i = 0, ..., N_t - 1,$$

$$S_j = S_1 + i \Delta S, \quad j = 0, ..., N_S - 1,$$

where

$$\Delta S = \frac{S_2 - S_1}{N_S},$$

$$\Delta t = \frac{T}{N_t},$$

where $N_S$ is a number of nodes in horizontal direction, and $N_t$ in vertical direction.

Partial derivatives we approximate in the following way:

$$\frac{\partial V}{\partial t} = \frac{V(j, i + 1) - V(j, i)}{\Delta t},$$

$$\frac{\partial V}{\partial S} = \frac{V(j + 1, i) - V(j - 1, i)}{2 \Delta S},$$

$$\frac{\partial^2 V}{\partial S^2} = \frac{V(j + 1, i) + V(j - 1, i) - 2V(j, i)}{\Delta S^2},$$

For $\partial V/\partial t$ we use the backward approximation, because we have to do with a backward differential equation (2.7).

For $\partial V/\partial S$ we use central approximation, because this approximation is the most precesses one.

The expressions (2.8),(2.9) will take a form:

$$g_0(r) = \begin{cases} r_0(S, t), & S_j(V(j + 2, i) - V(j, i)) \geq 2V(j + 1, i), \\ r_1(S, t), & S_j(V(j + 2, i) - V(j, i)) < 2V(j + 1, i), \end{cases}$$

and, correspondingly

$$f_0(\sigma) = \begin{cases} \sigma_0(S, t), & V(j + 2, i) - V(j, i) \geq V(j + 1, i), \\ \sigma_1(S, t), & V(j + 2, i) - V(j, i) < V(j + 1, i), \end{cases}$$

The PDEs (2.7) is now reduced to an algebraic system

$$\frac{1}{2} f_0(\sigma)^2 S_j \frac{V(j + 1, i) + V(j - 1, i) - 2V(j, i)}{\Delta S^2} + g_0(r) \left( \frac{V(j + 1, i) - V(j - 1, i)}{2 \Delta S} S_j - V(j, i) \right) - \frac{V(j, i + 1) - V(j, i)}{\Delta t} = 0.$$
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Otherwise

\[ V(j + 1, i) \left( \frac{1}{2} \frac{f_0(\sigma)^2}{\Delta S^2} S_j^2 + \frac{g_0(r)}{2\Delta S} S_j \right) + V(j, i) \left( \frac{1}{\Delta t} - \frac{f_0(\sigma)^2}{\Delta S^2} S_j^2 - g_0(r) \right) \\
+ V(j - 1, i) \left( \frac{1}{2} \frac{f_0(\sigma)^2}{\Delta S^2} S_j^2 - \frac{g_0(r)}{2\Delta S} S_j \right) - \frac{V(j, i + 1)}{\Delta t} = 0. \]

We obtain the following finite difference equation to calculate values of the function \( V(S, t) \) in the grid nodes

\[ V(j, i + 1) = \Delta t \left( \frac{1}{2} \left( \frac{f_0(\sigma)}{\Delta S} \right)^2 S_j^2 - \frac{g_0(r)}{2\Delta S} S_j \right) V(j - 1, i) \\
+ \left( 1 - \left( \frac{f_0(\sigma)}{\Delta S} \right)^2 S_j^2 \Delta t - g_0(r) \Delta t \right) V(j, i) \]

\[ + \left( \frac{1}{2} \left( \frac{f_0(\sigma)}{\Delta S} \right)^2 S_j^2 + \frac{g_0(r)}{2\Delta S} S_j \right) \Delta t V(j + 1, i). \] (4.1)

The boundary conditions in Example 2.0.1 are

\[ V(0, i) = V(N_S - 1, i) = 0, \] (4.2)

and initial conditions are

\[ V(j, 0) = (S_j - K)^+ + (K - S_j)^+. \] (4.3)

The initial and boundary conditions can be similarly obtain for another examples.

4.1.2 Application of the explicit finite difference method to the delta function

The same transformations as in part 4.1.1 for equations (2.27),(2.29), (2.28),(2.25) give us:

\[ \delta_t = \frac{\delta(j, i + 1) - \delta(j, i)}{\Delta t}, \]
\[ \delta_S = \frac{\delta(j + 1, i) - \delta(j - 1, i)}{2\Delta S}, \]
\[ \delta_{SS} = \frac{\delta(j + 1, i) + \delta(j - 1, i) - 2\delta(j, i)}{\Delta S^2}, \]
\[ \tilde{g}_0(r) = \begin{cases} r_0(S, t), & \delta(j + 1, i) \geq \delta(j - 1, i), \\
 r_1(S, t), & \delta(j + 1, i) < \delta(j - 1, i), \end{cases} \]
\[ \tilde{k}_0(r) = \begin{cases} r_0(S, t), & \delta(j + 1, i) \geq \delta(j - 1, i), \\
 r_1(S, t), & \delta(j + 1, i) < \delta(j - 1, i), \end{cases} \]
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and

\[ \hat{f}_0(\sigma) = \begin{cases} \sigma_0(S, t), & \delta(j + 1, i) + \delta(j - 1, i) \geq 2\delta(j, i), \\ \sigma_1(S, t), & \delta(j + 1, i) + \delta(j - 1, i) < 2\delta(j, i). \end{cases} \]

The PDEs (2.23) is now reduced to an algebraic system of equations

\[ \frac{1}{2} \hat{f}_0(\sigma)^2 S_j^2 \frac{\delta(j + 1, i) + \delta(j - 1, i) - 2\delta(j, i)}{\Delta S^2} \]

\[ + \left( \hat{k}_0(\sigma)^2 + \hat{\gamma}_0(r) \right) \frac{\delta(j + 1, i) - \delta(j - 1, i)}{2\Delta S} \left( S_j - \frac{\delta(j, i + 1) - \delta(j, i)}{\Delta t} \right) = 0. \]

Otherwise

\[ \delta(j + 1, i) \left( \frac{1}{2} \hat{f}_0(\sigma)^2 S_j^2 + \left( \hat{k}_0(\sigma)^2 + \hat{\gamma}_0(r) \right) \frac{S_j}{2\Delta S} \right) \]

\[ + \delta(j - 1, i) \left( \frac{1}{2} \hat{f}_0(\sigma)^2 S_j^2 - \left( \hat{k}_0(\sigma)^2 + \hat{\gamma}_0(r) \right) \frac{S_j}{2\Delta S} \right) \]

\[ - \delta(j, i) \left( \hat{f}_0(\sigma)^2 \frac{S_j^2}{\Delta S^2} - \frac{1}{\Delta t} \right) - \delta(j, i + 1) \frac{1}{\Delta t} = 0. \]

We obtain the following finite difference equation for the function \( \delta(j, i + 1) \)

\[ \delta(j, i + 1) = \frac{\Delta t}{2} \frac{S_i}{\Delta S} \left( \hat{f}_0(\sigma)^2 \frac{S_i}{\Delta S} + \hat{k}_0(\sigma)^2 + \hat{\gamma}_0(r) \right) \delta(j + 1, i) \]

\[ + \frac{\Delta t}{2} \frac{S_i}{\Delta S} \left( \hat{f}_0(\sigma)^2 \frac{S_i}{\Delta S} - \hat{k}_0(\sigma)^2 - \hat{\gamma}_0(r) \right) \delta(j - 1, i) - \delta(j, i) \left( \hat{f}_0(\sigma)^2 \frac{S_i^2}{\Delta S^2} - 1 \right). \]  

(4.4)

The initial conditions in Example 2.2.1 can be rewritten in terms of the finite differences as

\[ \delta(j, 0) = \begin{cases} 1, & S > K, \\ 0, & S \leq K, \end{cases} \]

and boundary conditions are

\[ \delta_0(0, i) = \delta_0(N_S - 1, i) = 0. \]

This conditions done for Example 2.2.1 and can be easy repeated for another case.

Remark 4.1.1 The program for this calculations is presented in the Appendix.

4.2 Stability of the finite difference method

4.2.1 Stability of the finite difference method by valuating of the option \( V(S, t) \)

Now we consider stability of the scheme (4.1). In [3] M.Avellaneda and R.Buff studied the stability problem. The introduce the special inequalities to solve this
problem in case of BS equation. Now we recalculate the inequalities in our case. We have following conditions on $A_0$, $B_0$ and $C_0$

\[
A_0 = \Delta t \left( \frac{1}{2} \left( \frac{f_0(\sigma)}{\Delta S} \right)^2 S^2 - \frac{g_0(r)}{2\Delta S} S \right) > 0,
\]

\[
B_0 = 1 - \left( \frac{f_0(\sigma)}{\Delta S} \right)^2 S \Delta t - g_0(r) \Delta t > 0,
\]

\[
C_0 = \Delta t \left( \frac{1}{2} \left( \frac{f_0(\sigma)}{\Delta S} \right)^2 S^2 + \frac{g_0(r)}{2\Delta S} S \right) > 0,
\]

or the in other words $A_0 > 0$, $C_0 > 0$, $A_0 + C_0 < \frac{1}{2}$.

It is easy to see, that $C_0$ is positive for all $S$ because it is the sum of positive terms. Let us rewrite expression for $A_0$. We know that $\Delta t > 0$, $S > 0$. In this case we can divide the expression by $\Delta t$ and $S$

\[
\frac{1}{2} \left( \frac{f_0(\sigma)}{\Delta S} \right)^2 S - \frac{g_0(r)}{2\Delta S} > 0,
\]

We also know, that $\Delta S > 0$. Let us multiple this equation by 2 and $\Delta S$

\[
\frac{1}{2} \frac{f_0(\sigma)^2}{\Delta S} S > g_0(r),
\]

finally, for $\Delta S$ we obtain the expression:

\[
\Delta S < \frac{f_0(\sigma)^2 S}{g_0(r)}. \tag{4.5}
\]

For $A_0 + C_0$ we have the following inequality:

\[
\frac{\Delta t S^2 f_0^2(\sigma)}{2\Delta S^2} < \frac{1}{2},
\]

Otherwise

\[
\frac{\Delta t}{\Delta S^2} < \frac{1}{2f_0^2(\sigma)S^2}. \tag{4.6}
\]

We have to update (4.5)to prove it for all $S$ and $t$, where $S_1 < S < S_2$, and the expressions for $f_0$ (2.9),$g_0$ (2.8). We have the following expression:

\[
\Delta S < \min_{S,t} \frac{f_0(\sigma)^2 S}{g_0(r)}. \tag{4.7}
\]

For (4.6) we should minimize the function \( \frac{1}{2f_0^2(\sigma)S^2} \). Finally, we obtain

\[
\frac{\Delta t}{\Delta S^2} < \min_{S,t} \frac{1}{2f_0^2(\sigma)S^2} \tag{4.8}
\]
4.2.2 Stability of the finite difference method by the delta function valuation

Similarly, to study the stability of the (4.15), where

\[ A_0 = \Delta t \frac{S}{\Delta S} \left( \frac{1}{2} \hat{f}_0(\sigma) \frac{S}{\Delta S} - \hat{k}_0^2(\sigma) - \hat{g}_0(r) \right) > 0, \]

\[ B_0 = 1 - \frac{\hat{f}_0(\sigma)^2 S^2 \Delta t}{\Delta S^2} > 0, \]

\[ C_0 = \Delta t \frac{S}{\Delta S} \left( \frac{1}{2} \hat{f}_0(\sigma)^2 \frac{S}{\Delta S} + \hat{k}_0^2(\sigma) + \hat{g}_0(r) \right) > 0. \]

It is easy to see, that \( C_0 \) always positive. Let us rewrite the inequality for \( B_0 \) as

\[ \frac{\hat{f}_0(\sigma)^2 S^2 \Delta t}{\Delta S^2} < 1. \]

After evaluation

\[ \frac{\Delta t}{\Delta S^2} < \frac{1}{\hat{f}_0(\sigma)^2 S^2}. \] \hspace{1cm} (4.9)

Let us divide the inequality for \( A_0 \) by the positive number \( \frac{\Delta t}{\Delta S} S \) and obtain

\[ \frac{1}{2} \hat{f}_0(\sigma)^2 \frac{S}{\Delta S} > \hat{k}_0(\sigma)^2 + \hat{g}_0(r), \]

so that

\[ \Delta S < \frac{S \hat{f}_0(\sigma)^2}{2 \left( \hat{k}_0(\sigma)^2 + \hat{g}_0(r) \right)}. \] \hspace{1cm} (4.10)

We have to determine the general expression for \( \Delta S \) for all values of \( S \). We know, that \( S_1 \leq S \leq S_2 \) and the expressions for \( \hat{f}_0(\sigma) \), \( \hat{g}_0(r) \), \( \hat{k}_0(\sigma) \) are given in (2.27), (2.29), (2.28).

To prove (4.10) we approximate the value \( \Delta S \) for all \( S \) and \( t \)

\[ \min_{S,t} \frac{S \hat{f}_0(\sigma)^2}{2 \left( \hat{k}_0(\sigma)^2 + \hat{g}_0(r) \right)}, \]

i.e.

\[ \Delta S < \min_{S,t} \frac{S \hat{f}_0(\sigma)^2}{2 \left( \hat{k}_0(\sigma)^2 + \hat{g}_0(r) \right)}. \]

The same transformations as before we use to prove the inequality (4.9)

\[ \frac{\Delta t}{\Delta S^2} < \min_{S,t} \frac{1}{S \hat{f}_0(\sigma)^2}. \]
4.3 Implicit method for the option price $V(S,t)$

In this section we consider a semi implicit finite difference method. We use so called "frozen" procedure in our work. This procedure will be applied to the nonlinear part in BSB equation. Instead to use functions $g_0, g_1, f_0, f_1$ on current time step use it from previous time step ("froze" their values). This method is presented in [7].

The partial derivatives in this case we approximate in the following way

$$\frac{\partial V}{\partial t} = \frac{V(j, i) - V(j, i - 1)}{\Delta t},$$

$$\frac{\partial V}{\partial S} = \frac{V(j + 1, i) - V(j - 1, i)}{2\Delta S},$$

$$\frac{\partial^2 V}{\partial S^2} = \frac{V(j + 1, i) + V(j - 1, i) - 2V(j, i)}{\Delta S^2},$$

For $\partial V/\partial t$ we use the forward approximation, because we have to do with a backward differential equation (2.7). In the case of implicit finite difference method we should do our calculations in following direction from unknown data to known data.

The expressions (2.8),(2.9) will take a form

$$g_0(r) = \begin{cases} r_0(S, t), & S_j V(j + 2, i) - V(j, i) \geq 2V(j + 1, i), \\ r_1(S, t), & S_j V(j + 2, i) - V(j, i) < 2V(j + 1, i), \end{cases}$$

and, correspondingly,

$$f_0(\sigma) = \begin{cases} \sigma_0(S, t), & V(j + 2, i) - V(j, i) \geq V(j + 1, i), \\ \sigma_1(S, t), & V(j + 2, i) - V(j, i) < V(j + 1, i). \end{cases}$$

The PDEs (2.7) is now reduced to an algebraic system of equations

$$\frac{1}{2} f_0(\sigma)^2 S_j^2 \frac{V(j + 1, i) + V(j - 1, i) - 2V(j, i)}{\Delta S^2} + g_0(r) \left( \frac{V(j + 1, i) - V(j - 1, i)}{2\Delta S} S_j - V(j, i) \right) - \frac{V(j, i) - V(j, i - 1)}{\Delta t} = 0,$$

$$i = 1, ..., N_t - 1,$n $$j = 1, ..., N_s - 1.$n

Otherwise

$$V(j + 1, i) \left( \frac{1}{2} \frac{f_0(\sigma)^2}{\Delta S^2} S_j^2 + g_0(r) S_j \right) + V(j, i) \left( -\frac{1}{\Delta t} - \frac{f_0(\sigma)^2}{\Delta S^2} S_j^2 - g_0(r) \right) + V(j - 1, i) \left( \frac{1}{2} \frac{f_0(\sigma)^2}{\Delta S^2} S_j^2 - g_0(r) S_j \right) + \frac{V(j, i - 1)}{\Delta t} = 0.$$
We obtain the following finite difference equation to calculate values of the function \( V(S, t) \) in the grid nodes

\[
V(j, i - 1) = \Delta t \left( \frac{1}{2} \left( \frac{f_0(\sigma)}{\Delta S} \right)^2 S_j^2 - \frac{g_0(r)}{2\Delta S} S_j \right) V(j - 1, i)
+ \left( 1 + \left( \frac{f_0(\sigma)}{\Delta S} \right)^2 S_j^2 \Delta t + g_0(r) \Delta t \right) V(j, i)
+ \left( -\frac{f_0(\sigma)^2 S_j}{\Delta S} - g_0(r) \right) \frac{S_j \Delta t}{\Delta S} \Delta t V(j + 1, i).
\]

(4.11)

Let us denote

\[
A^i = \frac{1}{2} \left( \frac{f_0(\sigma)}{\Delta S} \right)^2 S_j^2 - \frac{g_0(r)}{2\Delta S} S_j,
B^i = 1 + \left( \frac{f_0(\sigma)}{\Delta S} \right)^2 S_j^2 \Delta t + g_0(r) \Delta t,
C^i = \left( -\frac{f_0(\sigma)^2 S_j}{\Delta S} - g_0(r) \right) \frac{S_j \Delta t}{\Delta S} \Delta t.
\]

We can rewrite (4.11) as a linear system

\[
\begin{bmatrix}
B^1 & C^1 & 0 & \cdots & 0 \\
A^2 & B^2 & C^2 & \cdots & 0 \\
0 & A^3 & B^3 & C^3 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & A^{M-1} & B^{M-1}
\end{bmatrix}
\begin{bmatrix}
V(1, i + 1) \\
V(2, i + 1) \\
V(3, i + 1) \\
\vdots \\
V(M - 1, i + 1)
\end{bmatrix}
= 
\begin{bmatrix}
V(1, i) - A^1 V(0, i) \\
V(2, i) \\
V(3, i) \\
\vdots \\
V(M, i) - C^{M-1} V(M, i)
\end{bmatrix}.
\]

Finally we obtain the following expression for \( V(j, i - 1) \)

\[
\begin{bmatrix}
V(1, i + 1) \\
\vdots \\
V(M - 1, i + 1)
\end{bmatrix}
= 
\begin{bmatrix}
B^1 & C^1 & 0 & \cdots & 0 \\
A^2 & B^2 & C^2 & \cdots & 0 \\
0 & A^3 & B^3 & C^3 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & A^{M-1} & B^{M-1}
\end{bmatrix}
\begin{bmatrix}
V(1, i) - A^1 V(0, i) \\
V(2, i) \\
V(3, i) \\
\vdots \\
V(M, i) - C^{M-1} V(M, i)
\end{bmatrix}.
\]

The boundary conditions in Example 2.0.1 are

\[
V(0, i) = V(N_S - 1, i) = 0,
\]

(4.12)

and initial conditions are

\[
V(j, 0) = (S_j - K)^+ + (K - S_j)^+.
\]

(4.13)

The initial and boundary conditions can be similarly obtain for another examples.
Chapter 4. The numerical approach

4.4 Implicit method for the function $\delta$

The same transformations as described in part 4.3 we calculate for equation

$$\frac{1}{2}\hat{f}_0(\sigma)^2S^2\delta_{0SS} + \left(\hat{k}_0(\sigma)^2 + \hat{g}_0(r)\right)S\delta_{0S} - \delta_{0t} = 0. \quad (4.14)$$

The partial derivatives in this case we approximate in the following way

$$\delta_t = \delta(j, i) - \delta(j - 1, i) \Delta t,$$
$$\delta_S = \frac{\delta(j + 1, i) - \delta(j - 1, i)}{2\Delta S},$$
$$\delta_{SS} = \frac{\delta(j + 1, i) + \delta(j - 1, i) - 2\delta(j, i)}{\Delta S^2},$$

the functions (2.27), (2.28), (2.29) are rewritten as

$$\hat{g}_0(r) = \begin{cases} r_0(S, t), & \delta(j + 1, i) \geq \delta(j - 1, i), \\ r_1(S, t), & \delta(j + 1, i) < \delta(j - 1, i), \end{cases}$$
$$\hat{k}_0(r) = \begin{cases} r_0(S, t), & \delta(j + 1, i) \geq \delta(j - 1, i), \\ r_1(S, t), & \delta(j + 1, i) < \delta(j - 1, i), \end{cases}$$

and

$$\hat{f}_0(\sigma) = \begin{cases} \sigma_0(S, t), & \delta(j + 1, i) + \delta(j - 1, i) \geq 2\delta(j, i), \\ \sigma_1(S, t), & \delta(j + 1, i) + \delta(j - 1, i) < 2\delta(j, i). \end{cases}$$

The PDEs (4.14) is now reduced to an algebraic system of equations

$$\frac{1}{2}\hat{f}_0(\sigma)^2S^2\frac{\delta(j + 1, i) + \delta(j - 1, i) - 2\delta(j, i)}{\Delta S^2}$$
$$\left(\hat{k}_0(\sigma)^2 + \hat{g}_0(r)\right)\frac{\delta(j + 1, i) - \delta(j - 1, i)}{2\Delta S}S_j - \frac{\delta(j, i) - \delta(j, i - 1)}{\Delta t} = 0,$$

$$i = 1, ..., N_t - 1,$$
$$j = 1, ..., N_S - 1.$$

Otherwise

$$\delta(j + 1, i)\left(\frac{1}{2}\hat{f}_0(\sigma)^2\frac{S_j^2}{\Delta S^2} + \left(\hat{k}_0(\sigma)^2 + \hat{g}_0(r)\right)\frac{S_j}{2\Delta S}\right)$$
$$+ \delta(j - 1, i)\left(\frac{1}{2}\hat{f}_0(\sigma)^2\frac{S_j^2}{\Delta S^2} - \left(\hat{k}_0(\sigma)^2 + \hat{g}_0(r)\right)\frac{S_j}{2\Delta S}\right)$$
$$+ \delta(j, i)\left(-\hat{f}_0(\sigma)^2\frac{S_j^2}{\Delta S^2} - \frac{1}{\Delta t}\right) + \delta(j, i - 1)\frac{1}{\Delta t} = 0,$$
We obtain the following finite difference equation for the function $\delta(j, i - 1)$:

$$
\delta(j, i - 1) = \frac{\Delta t}{2} S_j \Delta S \left( \tilde{f}_0(\sigma)^2 S_j \frac{\Delta S}{\Delta S} + \tilde{k}_0(\sigma)^2 + \tilde{g}_0(r) \right) \delta(j + 1, i) \\
+ \frac{\Delta t}{2} S_j \Delta S \left( \tilde{f}_0(\sigma)^2 S_j \frac{\Delta S}{\Delta S} - \tilde{k}_0(\sigma)^2 - \tilde{g}_0(r) \right) \delta(j - 1, i) \\
+ \delta(j, i) \left( -\tilde{f}_0(\sigma)^2 \Delta t \frac{S_j^2}{\Delta S} - 1 \right). 
$$

(4.15)

Let us denote

$$
A_i = \frac{\Delta t}{2} S_j \left( \tilde{f}_0(\sigma)^2 S_j \frac{\Delta S}{\Delta S} - \tilde{k}_0(\sigma)^2 - \tilde{g}_0(r) \right), \\
B_i = -\tilde{f}_0(\sigma)^2 \Delta t \frac{S_j^2}{\Delta S} - 1, \\
C_i = \frac{\Delta t}{2} S_j \left( \tilde{f}_0(\sigma)^2 S_j \frac{\Delta S}{\Delta S} + \tilde{k}_0(\sigma)^2 + \tilde{g}_0(r) \right).
$$

Following we can represent the system of difference equation corresponding to (4.14) in the form

$$
\begin{bmatrix}
B^1 & C^1 & 0 & \cdots & 0 \\
A^2 & B^2 & C^2 & \cdots & 0 \\
0 & A^3 & B^3 & C^3 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & A^{M-1} & B^{M-1}
\end{bmatrix}
\begin{bmatrix}
\delta(1, i + 1) \\
\delta(2, i + 1) \\
\delta(3, i + 1) \\
\vdots \\
V(M - 1, i + 1)
\end{bmatrix}
= 
\begin{bmatrix}
\delta(1, i) - A^1 \delta(0, i) \\
\delta(2, i) \\
\delta(3, i) \\
\vdots \\
\delta(M, i) - C^{M-1} \delta(M, i)
\end{bmatrix}
$$

Finally we reduce this system to the form

$$
\begin{bmatrix}
\delta(1, i + 1) \\
\delta(2, i + 1) \\
\delta(3, i + 1) \\
\vdots \\
V(M - 1, i + 1)
\end{bmatrix}
= 
\begin{bmatrix}
B^1 & C^1 & 0 & \cdots & 0 \\
A^2 & B^2 & C^2 & \cdots & 0 \\
0 & A^3 & B^3 & C^3 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & A^{M-1} & B^{M-1}
\end{bmatrix}
\begin{bmatrix}
\delta(1, i) - A^1 \delta(0, i) \\
\delta(2, i) \\
\delta(3, i) \\
\vdots \\
\delta(M, i) - C^{M-1} \delta(M, i)
\end{bmatrix}
$$

The initial conditions in Example 2.2.2 can be rewritten in terms of the finite differences as

$$
\delta(j, 0) = \begin{cases} 
1, & S > K, \\
0, & S \leq K,
\end{cases}
$$

and boundary conditions are

$$
\delta_0(0, i) = \delta_0(N_S - 1, i) = 0.
$$

This conditions done for Example 2.2.2 and can be easy repeated for another case.
Chapter 5

Conclusions

Nonlinear models appear often by the modelling of the volatility risk. This models demand methods using of advanced numerical and algorithmic techniques.

In our work we considered uncertain volatility model. To valuate the upper and lower bonds we used numerical methods to solve BSB equation. To reduce the volatility risk we used static hedging. We achieved improvements of the previous results, using the implicit finite difference method.

In Chapter 2 we considered the uncertain volatility model for the sensitivity parameter delta. We represented examples where the lower and upper bound of volatility $\sigma$, i.e. functions $\sigma_0$ and $\sigma_1$ are the functions dependent on time $t$.

In Chapter 3 we use the static hedging for narrowing procedure of the envelop of the option price. We improved spread between upper and lower bounds.

In Chapter 4 we represented the explicit and implicit finite difference methods to valuate the different types of European and American options. We studied the stability problem for the explicit method. Our algorithms can be applied to different types of European and American options, for instance, to straddles, butterflies, double barrier and others options.
Notation

\( r \) The risk-free interest rate, constant over time, in an environment with no liquidity constraints.

\( S \) A stock’s price.

\( t \) The current time.

\( T \) The expiration date of a Put option and a Call option.

\( K \) The strike price of the Put option and Call option.

\( C(S, t) \) The price of the Call option when the current stock price is \( S \) and the current time is \( t \).

\( P(S, t) \) The price of the Put option when the current stock price is \( S \) and the current time is \( t \).

\[ P(S, t) = C(S, t) - S + Ke^{-r(T-t)} \] Put-Call parity is the relationship between the price of a Put option and a Call option on a stock according to a standard BS model.
Glossary

The interest rate
The interest rate is the yearly price charged by a lender to a borrower in order for the borrower to obtain a loan. This is usually expressed as a percentage of the total amount loaned. Usually denoted by $r$.

Expiry
The last date an option can be traded or exercised. Denoted by $T$.

Expiration
The time when the option contract ceases to exist. Usually denoted by $T$.

Underlying
The asset that the parties agree to exchange in a derivative contract.

In derivatives, the security that must be delivered if the contract is exercised.

Stock
A Stock (also known as an equity or a share) is a portion of the ownership of a corporation. A share in a corporation gives the owner of the stock a stake in the company and its profits.

Option
A contract which gives the holder of the contract the right to buy or sell a commodity or financial asset for a given price before a specified date or only at the expiration date.

Call Options
A Call option is a contract that gives the bearer the right to buy a share at a given price at the ahead prescribed time. Usually these options expire after a certain date.

Put Option
A Put option is a security which conveys the right to sell a specified quantity of an underlying asset at or before a fixed date.
**Straddle**

Straddle is an options trading strategy of buying a Call option and a Put option on the same stock with the same strike price and the same expiration date. Such a strategy would result in a profitable position if the stock price is far enough from the strike price.

**Put-Call parity**

Put-Call parity is the relationship between the price of a Put option and a Call option on a stock according to a standard BS model. The relationship is derived from the fact that combinations of options can make portfolios that are equivalent to holding the stock through time $T$, and that must return exactly the same amount or an arbitrage would be available to traders.

**Delta**

Delta in context of the options theory: the rate of change of a financial derivative’s price with respect to change in the price of the underlying asset. Formally this is a partial derivative of the option price along asset price $S$.

A derivative is perfectly delta-hedged if it is in a portfolio with a delta equal to zero. Financial companies use delta-hedged portfolios.

**Portfolio**

The entire collection of financial assets held by an investor.

**European option**

An option that may be exercised only at the expiration date.

**American option**

An option that may be exercised at the expiration date or before it.

**Maximum principle**

Suppose $V(S, t)$ satisfies the inequality $\mathcal{L}V > 0$ in the rectangular region $D = (0, X) \times (0, T]$ then $V$ cannot have (a local) maximum at any interior point.

http://economics.about.com
http://glossary.itlocus.com
Bibliography


Chapter 6

Appendix

In this Chapter we provide some examples of programmes used for evaluation of the option prices in Chapters 2 and 3.

We will use following variables:

$T$ - maturity date;

$S_1, S_2$ - bound of asset price in calculations;

$K$ - strike price;

$\sigma_0, \sigma_1$ - bound of volatility;

$r_0, r_1$ - bound of interest rate;

$NS$ - a number of nodes in horizontal direction;

$Nt$ - a number of nodes in vertical direction;

$dt$ - time step;

$dS$ - asset price step;

$V_0, V_1$ - envelopes of the option price;

The sense of variables $f_0, f_1, g_0, g_1, A_0, B_0, C_0, A_1, B_1, C_1$ follows from considerations in previous chapters of our work.
Program 1 Calculation of envelopes $V_0, V_1$ and Black-Scholes Solution $V$ for the European Double Barrier Straddle (the explicit finite difference method).

```matlab
function BSBL(T, S1, S2, K, sigma0, sigma1, r0, r1, Nt, NS)
% an illustration we show the envelopes for a Double Barrier European Straddle $V(S,t)$.

display('Insert initial condition');
display('T is the time of expiry');
T = input('T=');
display('S1 < S < S2');
S1 = input('S1=');
S2 = input('S2=');
display('Input K, strike price');
K = input('K=');
display('Input volatility sigma. sigma0< sigma <sigma1');
sigma0 = input('sigma0=');
sigma1 = input('sigma1=');
display('Input the risk-free interest rate r. r0<r<r1');
r0 = input('r0=');
r1 = input('r1=');
Nt = input('Nt=');
NS = input('NS=');

dt = T/Nt;
dS = (S2-S1)/NS;

Payoff = @(S) max(0, S-K)+max(0, K-S);

U0 = zeros(Nt+1, NS+1);
U1 = zeros(Nt+1, NS+1);

for j = 1: NS-1,
    U0(1, j+1) = Payoff(S1+j*dS);
    U1(1, j+1) = Payoff(S1+j*dS);
end

for i = 1:Nt,
    ti = (i-1)*dt;
    for j = 1: NS-1,
        Sj = S1+(j-1)*dS;
        if U0(i, j+2) + U0(i, j) > 2*U0(i, j+1)
            f0 = sigma0;
        else
            f0 = sigma1;
        end
        if U1(i, j+2) + U1(i, j) < 2*U1(i, j+1)
            f1 = sigma0;
        else
            f1 = sigma1;
        end
        if Sj*(U0(i, j+2) - U0(i, j)) > 2*U0(i, j+1)
            g0 = r0;
        else
            g0 = r1;
end
```

if Sj*(Vl(i,j+2)-Vl(i,j)) < 2*Vl(i,j+1)
gl=r0;
else
gl=r1;
end

A0=dt*Sj*(f0.2*Sj/dS-g0);
B0=1-dt*((f0*Sj/dS)^2-g0);
C0=dt*Sj*(f0^2*Sj/dS+g0);
A1=dt*Sj*(f1.2*Sj/dS-g1);
B1=1-dt*(((f1*Sj/dS)^2-g1));
C1=dt*Sj*(f1^2*Sj/dS+g1);

V0(i+1,j+1)=A0*V0(i,j)+B0*V0(i,j+1)+C0*V0(i,j+2);
Vl(i+1,j+1)=A1*Vl(i,j)+B1*Vl(i,j+1)+C1*Vl(i,j+2);
end
end
S=S1:dS:S2;
VW= zeros (1,NS+1);
for j=1:(NS+1)
    V0(j)=V0(Ne+1,j);
end
Vl= zeros (1,NS+1);
for j=1:(NS+1)
    Vl(j)=Vl(Ne+1,j);
end

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

Plot of the Black-Scholes Solution V(S,t)
V= zeros (Nt+1,NS+1);
eps=0.055;
sigma=0.1;
for j=1: NS-1,
    U(i,j+1)=Payoff(S1+j*dS);
end
for i=1:Nt,
    ti=(i-1)*dt;
    for j=1: (NS-1),
        Sj = S1+(j-1)*dS;
        f0=sigma;
        g0=eps;
        A=dt*Sj*(f0.2*Sj/dS-g0);
        B=1-dt*(((f0*Sj/dS)^2-g0));
        C=dt*Sj*(f0^2*Sj/dS+g0);
        U(i+1,j+1)=A*U(i,j)+B*U(i,j+1)+C*U(i,j+2);
    end
end
S=S1:dS:S2;
VW= zeros (1,NS+1);
for j=1:(NS+1)
    WV(j)=V(Ne+1,j);
end
plot(S,VW,'-x',S,VW,':b',S,VW,'r-','LineWidth',2);
Program 2 Application of the implicit finite difference method to the valuation of the option price \( V(S,t) \).

\[
\begin{align*}
T &= 0.5; \\
S_1 &= 40; \\
S_2 &= 120; \\
K &= 100; \\
\sigma_0 &= 0.2; \\
\sigma_1 &= 0.3; \\
r_0 &= 0.05; \\
r_1 &= 0.05; \\
r &= 0.05; \\
\gamma &= 0.25; \\
N_t &= 100; \\
N_3 &= 60; \\
d_t &= T/N_t; \\
d_3 &= (S_2 - S_1)/N_3; \\
Payoff &= \Theta(S) \max(0, K - S) + \max(0, S - K); \\
\end{align*}
\]

*-------------------------- initialization --------------------------*

\[
\begin{align*}
V_0 &= \text{zeros}(N_t+1, N_3+1); \\
V_1 &= \text{zeros}(N_t+1, N_3+1); \\
V &= \text{zeros}(N_t+1, N_3+1); \\
\end{align*}
\]

for \( j = 1: N_3 - 1, \)

\[
\begin{align*}
V_0(j+1) &= \text{Payoff}(S_1 + j \cdot d_3); \\
V_1(j+1) &= \text{Payoff}(S_1 + j \cdot d_3); \\
V(j+1) &= \text{Payoff}(S_1 + j \cdot d_3); \\
\end{align*}
\]

end

*-------------------------- main part --------------------------*

for \( i = 1: N_t, \)

\[
\begin{align*}
M_0 &= \text{zeros}(N_3 - 1, N_3 - 1); \\
M_1 &= \text{zeros}(N_3 - 1, N_3 - 1); \\
f_0 &= \text{zeros}(N_3 - 1, 1); \\
f_1 &= \text{zeros}(N_3 - 1, 1); \\
M &= \text{zeros}(N_3 - 1, N_3 - 1); \\
f &= \text{zeros}(N_3 - 1, 1); \\
\end{align*}
\]

\( t_i = (i - 1) \cdot d_t; \)
for j=1:(WS-1),
    S_j = S_{l+(j-l)*dS};
    if U0(i,j+2)+U0(i,j) >= 2*U0(i,j+1)
        f_0 = sigma0;
    else
        f_0 = sigma1;
    end
    if U1(i,j+2)+U1(i,j) < 2*U1(i,j+1)
        f_1 = sigma0;
    else
        f_1 = sigma1;
    end
    g_0 = x0;
    g_1 = x1;

    A0 = dS*S_j/(2*dS)*(_f_0*2*S_j/dS+g_0);
    B0 = 1+dS/((f_0*S_j/dS)^2+g_0);
    C0 = dS*S_j/(2*dS)*(_f_0*2*S_j/dS-g_0);
    A1 = dS*S_j/(2*dS)*(_f_1*2*S_j/dS+g_1);
    B1 = 1+dS/((f_1*S_j/dS)^2+g_1);
    C1 = dS*S_j/(2*dS)*(_f_1*2*S_j/dS-g_1);
    A = dS*S_j/(2*dS)*(-sigma*2*S_j/dS+x);
    B = 1+dS/((sigma*S_j/dS)^2+x);
    C = dS*S_j/(2*dS)*(-sigma*2*S_j/dS-x);

    if j=1
        M0(1,1:2) = [B0 C0];
        M(1,1:2) = [B1 C1];
        M1(1,1:2) = [B1 C1];
        ff0(1,1) = -U0(i,j)*A0;
        ff(1,1) = -V(i,j)*A;
        ff1(1,1) = -U1(i,j)*A1;
    elseif j=WS-1
        M0(WS-1,(WS-2):(WS-1)) = [A0 B0];
        M((WS-1,(WS-2):(WS-1)) = [A B];
        M1((WS-1,(WS-2):(WS-1)) = [A B];
        ff0(WS-1,1) = -C0*U0(i,WS-1);
        ff1(WS-1,1) = -U1(i,WS-1)*C1;
        ff((WS-1,1) = -C*V(i,WS-1);
    else
        M0(j,(j-1):(j+1)) = [A0 B0 C0];
        M(j,(j-1):(j+1)) = [A B C];
        M1(j,(j-1):(j+1)) = [A B C];
    end
end
ff0 = ff0 + (U0(i,2:WS))';
ff1 = ff1 + (U1(i,2:WS))';
ff = ff + (U(i,2:WS))';
R0 = inv(M0)*ff0;
U0(i+1,2:WS) = R0';
R1 = inv(M1)*ff1;
U1(i+1,2:WS) = R1';
R = inv(M)*ff;
V(i+1,2:WS) = R';

end
S=S1; dS=S2;
U00 = U0(Nt+1,:);
U11 = U1(Nt+1,:);
V = V(Nt+1,:);
plot(S,U01,'-g',S,U00,'-b',S,U11,'-r','LineWidth',2);
**Program 3** Application of the explicit finite difference method to the valuation of the delta function $\delta$ for $V(S,t)$.

```plaintext
format long
T=0.1;
S1=0;
S2=120;
K=100;
sigma0=0.2;
sigmal=0.4;
r0=0.04;
rl=0.06;
sigma=0.3;
r=0.05;

Nt=80;
N3=100;

dt=T/Nt;
d3=(S2-S1)/N3;

* --------------------------initialization-------------------

D0=zeros(Nt+1,N3+1);
Dl=zeros(Nt+1,N3+1);
D=zeros(Nt+1,N3+1);
for j=1:N3-1,
    if S1+(j-1)*d3 > K
        D0(1,j+1)=1;
        Dl(1,j+1)=1;
        D(1,j+1)=1;
    end
end
end
```
Main part

for i=1:Nt,
   ti = (i-1)*dt;
   for j=1:(NS-1),
      \( S_j = S_{j+1} + dS \);
      if \( D0(i,j+2) >= D0(i,j) \)
         \( g0 = r0; \)
         \( k0 = sigma0; \)
      else
         \( g0 = r1; \)
         \( k0 = sigma1; \)
      end
      if \( Dl(i,j+2) <= Dl(i,j) \)
         \( gl = r0; \)
         \( kl = sigma0; \)
      else
         \( gl = r1; \)
         \( kl = sigma1; \)
      end
      if \( D0(i,j+2) + D0(i,j) >= 2 \cdot D0(i,j+1) \)
         \( f0 = sigma0; \)
      else
         \( f0 = sigma1; \)
      end
      if \( Dl(i,j+2) + Dl(i,j) <= 2 \cdot Dl(i,j+1) \)
         \( fl = sigma0; \)
      else
         \( fl = sigma1; \)
      end

      \[
      C0 = dt \cdot S_j / dS / 2 \cdot (f0 \cdot S_j / dS + k0 \cdot g0);
      B0 = 1 - (f0 \cdot S_j / dS) \cdot 2 \cdot dt;
      A0 = dt \cdot S_j / dS / 2 \cdot (f0 \cdot S_j / dS - k0 - g0);
      C1 = dt \cdot S_j / dS / 2 \cdot (fl \cdot S_j / dS + k1 \cdot gl);
      B1 = 1 - (fl \cdot S_j / dS) \cdot 2 \cdot dt;
      A1 = dt \cdot S_j / dS / 2 \cdot (fl \cdot S_j / dS - k1 - gl);
      C = dt \cdot S_j / dS / 2 \cdot (sigma^2 \cdot S_j / dS + sigma \cdot 2 \cdot r);
      B = 1 - (sigma \cdot S_j / dS)^2 \cdot dt;
      A = dt \cdot S_j / dS / 2 \cdot (sigma^2 \cdot S_j / dS - sigma^2 \cdot r);
      \]
      \[
      D0(i+1,j+1) = C0 \cdot D0(i,j+2) + B0 \cdot D0(i,j+1) + A0 \cdot D0(i,j);
      Dl(i+1,j+1) = C1 \cdot Dl(i,j+2) + B1 \cdot Dl(i,j+1) + A1 \cdot Dl(i,j);
      D(i+1,j+1) = C \cdot D(i,j+2) + B \cdot D(i,j+1) + A \cdot D(i,j);
      \]
   end
end

plot(3,DD0,'--k',3,DD1,'-k',3,DD,'k-');
hold on
Program 4 Application of the implicit finite difference method to the valuation of the delta function of Up-and-Out European Call (Example 2.2.1).

```c
format long
T=0.1;
S1=0;
S2=120;
K=100;
sigma0=0.2;
eta=1=0.4;
sigma0=0.3;
\tau=0.05;
\nu0=0.04;
\nu1=0.06;

Nt=50;
NS=50;

dt=T/Nt;
\delta(S)=(S2-S1)/NS;

#-------- initialization ------------
D0=seos(NS-1,NS-1);
D1=seos(NS-1,NS+1);
D=seos(NS-1,NS+1);
for j=1:NS-1,
    if D1+(j-1)*\delta S > K
        D0(1,j+1)=1;
        D1(1,j+1)=1;
        D(1,j+1)=1;
    end
end

#-------- main part ------------
for i=1:Nt,
    H0 = seos(NS-1,NS-1);
    H1 = seos(NS-1,NS-1);
    M = seos(NS-1,NS-1);
    \delta f0 = seos(NS-1,1);
    \delta fl = seos(NS-1,1);
    f0 = seos(NS-1,1);
    c1 = (i-1)*dt;
    for j=1:NS-1,
        Dj = D1((j-1)*\delta S;
        if D0(1,j+2)==D0(i,j)
            g0=0;
            k0=\nu0;
        else
            g0=\nu1;
            k0=\nu1;
        end
        if D1(i,j+2)==D1(i,j)
            gl=0;
            kl=\nu0;
        else
            gl=\nu1;
            kl=\nu1;
        end
        if D0(1,j+2)*D0(1,j)\neq D0(1,j+1)
            f0=\sigma0;
        else
            f0=\sigma0;
        end
    end
end
```
if D1(i,j+2)+D1(i,j)<=(2*D1(i,j+1)
    f1=sigma0;
else
    f1=sigma1;
end

C0=(1)*d0*3j/d3/2*(f0^2+3j/d3+k0^2+g0);
B0=1+(f0*3j/d3)^2*d0;
A0=(1)*d0*3j/d3/2*(f0^2*3j/d3-k0^2-g0);

C1=(-1)*d0*3j/d3/2*(f1^2*3j/d3+k1^2+g1);
B1=1+(f1*3j/d3)^2*d0;
A1=(1)*d0*3j/d3/2*(f1^2*3j/d3-k1^2-g1);

C=(-1)*d0*3j/d3/2*(sigma^2*3j/d3+sigma^2+z);
B=1+(sigma*3j/d3)^2*d0;
A=(-1)*d0*3j/d3/2*(sigma^2*3j/d3-sigma^2-z);

if j=1
    M (1,1,2) = [B0 C0];
    M0(1,1,2) = [B0 C0];
    M1(1,1,2) = [B1 C1];
    f00(1,1) = -D0(i,1)*A0;
    f11(1,1) = -D1(i,1)*A1;
    f0 (1,1) = -D (i,1)*A;
elseif j=NS-1
    M (NS-1,NS-2) = [A B];
    M0(NS-1,NS-2) = [A0 B0];
    M1(NS-1,NS-2) = [A1 B1];
    f00(NS-1,1) = -C0*D0(i,NS+1);
    f11(NS-1,1) = -D1(i,NS+1)*C1;
    f0 (NS-1,1) = -D(i,NS+1)*C;
else
    M0(j,(j-1):(j+1)) = [A0 B0 C0];
    M1(j,(j-1):(j+1)) = [A1 B1 C1];
    M (j,(j-1):(j+1)) = [A B C];
end

f0 = f00 + (D0(i,2:NS))^2;
ff = f11 + (D1(i,2:NS))';
ff = f0 + (D(i,2:NS))';

R0 = inv(M0)*f00;
D0(i+1,2:NS) = R0';
D1(i+1,2:NS) = f1';
D(i+1,2:NS) = R';

*----------------- plot ---------------------

S=S1:dS:S2;
WW0 = D0(W0+1,:);
WU1 = D1(W0+1,:);
W = D(W0+1,:);

plot(S,WU1,'-g',S,WW0,'-b',S,WW,'-r');
hold on
Program 5 Application of the static hedging procedure to simple European Call.

\begin{verbatim}
T=0.5;
S1=0;
S2=300;
K=100;
sigma0=0.2;
sigma1=0.0;
r0=0.05;
r1=0.05;
U=5.791;
d=0;
uk=0;

Nt=100;
NS=42;

dt=T/Nt;
dS=(S2-S1)/NS;

Payoff = @*(S,d) max(0,S-K) - d*max(0,K-S);

CC0=zeros(21,1);
CC1=zeros(21,1);

for d=-0.5:0.1:1.5,
    uk=uk+1;

* initialisation-----------------------------------------------

U0=zeros(Nt+1,NS+1);
U1=zeros(Nt+1,NS+1);

for i=1:Nt-1,
    ti = (i-1)*dt;
    o=-r0*ti;
    U0(i,1)=-d*K*exp(o);
    U1(i,1)=-d*K*exp(o);
end

for j=1:NS-1,
    U0(1,j+1)=Payoff(S1+j*dS,d);
    U1(1,j+1)=Payoff(S1+j*dS,d);
End
\end{verbatim}
for i=1:Nv,
    ti = (i-1)*dt;
    for j=1:(NS-1),
        Sj = Sl+(j-1)*dS;
        if U0(i,j+2)+U0(i,j) >= 2* U0(i,j+1)
            f0=sigma0;
        else
            f0=sigma1;
        end
    end
if U1(i,j+2)+U1(i,j) < 2* U1(i,j+1)
    f1=sigma0;
else
    f1=sigma1;
end
f_0 = f0;
f_1 = f1;
g_0 = x0;
g_1 = x1;
A0=dt*Sj/(2*dS)*(f_0^2+Sj/dS-g_0);
B0=1-dt*((f_0*Sj/dS)^2-g_0);
C0=dt*Sj/(2*dS)*(f_0^2+Sj/dS+g_0);
A1=dt*Sj/(2*dS)*(f_1^2+Sj/dS-g_1);
B1=1-dt*((f_1*Sj/dS)^2-g_1);
C1=dt*Sj/(2*dS)*(f_1^2+Sj/dS+g_1);
U0(i+1,j+1)=A0*U0(i,j)+B0*U0(i,j+1)+C0*U0(i,j+2);
U1(i+1,j+1)=A1*U1(i,j)+B1*U1(i,j+1)+C1*U1(i,j+2);
end

S=S1; dS=S2;
U0=zero(l,NS+1);
for j=1:(NS+1)
    U0(j)=U0(Nv+1,j);
    if (Sl+(j-1)*dS) == K
        CC0(uk,1)=U0(j)+d*W;
    end
end
U1=zero(l,NS+1);
for j=1:(NS+1)
    U1(j)=U1(Nv+1,j);
    if (Sl+(j-1)*dS) == K
        CC1(uk,1)=U1(j)+d*W;
    end
end
E=0.5:0.1:1.5;
plot(E,CC0,'-b',E,CC1,'-x');